On the field with one element

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1. Let $\mathbf{G}$ be a Chevalley group scheme over $\mathbf{Z}$, and $\mathbf{F}_q$ the finite field with q elements. It has been noticed for some time that, when q tends to 1, the cardinality of the group of points of $\mathbf{G}$ in $\mathbf{F}_q$ behaves as follows:

$$\operatorname{card} \mathbf{G}(\mathbf{F}_q) \underset{q \to 1}{\sim} (q-1)^r \times \operatorname{card} \mathbf{W},$$

where r is the rank of $\mathbf{G}$ and $\mathbf{W}$ is its Weyl group. In 1957, Tits [1] proposed to think of $\mathbf{W}$ as the group of points of $\mathbf{G}$ in "the field of characteristic one":

$$\mathbf{G}(\mathbf{F}_1) := \mathbf{W}. \tag{1}$$

He also argued that the finite geometries attached to each group $\mathbf{G}$ have a limit when q goes to 1, namely the finite geometry attached to the Coxeter group $\mathbf{W}$.

In 1993, Manin [2] wrote some lectures on zeta functions where he mentions the field with one element. He proposes to develop algebraic geometry over $\mathbf{F}_1$, and predicts that varieties over that field have simple zeta functions. For instance

$$\zeta_{\mathbf{P}^N_{\mathbf{F}_1}}(s) = s(s-1)(s-2)\cdots(s-N). \tag{2}$$

He also notes that the equation (1) for $\mathbf{G} = \mathbf{SL_N}$ leads to the fact that the higher K-theory of $\mathbf{F}_1$ must be the homotopy groups of spheres. Indeed, by the Barratt-Priddy-Quillen theorem, we get

$$K_m(\mathbf{F}_1) = \pi_m BGL(\mathbf{F}_1)^+ := \pi_m B\Sigma_\infty^+ = \pi_m^s. \tag{3}$$

Later on Smirnov and Kapranov-Smirnov (unpublished preprints) studied the question further. Among other things, they developped linear algebra over $\mathbf{F}_1$ (a vector space being a pointed finite set), and obtained in this way a description of the Gauss reciprocity law similar to the description given earlier by Arbarello, De Concini and Kac of the Weil reciprocity law on curves by means of determinants of vector spaces.

2. The analogy between number fields and function fields finds a basic limitation with the lack of a ground field. One says that Spec($\mathbf{Z}$) (with a point at infinity added, as is familiar in Arakelov theory) is like a (complete) curve; but over which field? In particular, one would dream of having an object like

$$\operatorname{Spec}(\mathbf{Z}) \times_{\operatorname{Spec}(\mathbf{F}_1)} \operatorname{Spec}(\mathbf{Z}),$$

since Weil's proof of the Riemann hypothesis for a curve over a finite field makes use of the product of two copies of this curve. I have nothing to say about that question. The mere fact that the Riemann zeta function has infinitely many zeroes indicates that $\mathrm{Spec}(\mathbf{Z})$ cannot be viewed as an object of finite type over $\mathbf{F}_1$, hence $\mathrm{Spec}(\mathbf{Z}) \times_{\mathrm{Spec}(\mathbf{F}_1)} \mathrm{Spec}(\mathbf{Z})$ cannot be an honest algebraic variety over $\mathbf{Z}$.

On the other hand, if X is a variety of finite type over $\mathbf{F}_1$, its base change $X_{\mathbf{Z}} = X \otimes_{\mathbf{F}_1} \mathbf{Z}$ to $\mathbf{Z}$ must be a variety of finite type over $\mathbf{Z}$. So there is hope to describe such objects X using usual algebraic geometry, and we can ask:

Question: Which varieties over $\mathbf{Z}$ are obtained by base change from $\mathbf{F}_1$ to $\mathbf{Z}$?

For instance, when X is smooth over $\mathbf{F}_1$, $X_{\mathbf{Z}}$ will be smooth over $\mathbf{Z}$, which is already a strong constraint.

3. Our starting point for a tentative definition of varieties over $\mathbf{F}_1$ is the short definition of schemes which says that a scheme is a covariant functor from rings to sets which is locally representable by a ring. In this language, the base change from a field k to a k-algebra Λ is described by the following easy lemma. Given a variety X over k (resp. a k-algebra A), let $X_\Lambda = X \otimes_k \Lambda$ (resp. $A_\Lambda = A \otimes_k \Lambda$).

Lemma 1:
i) There is a canonical inclusion $X(A) \subset X_\Lambda(A_\Lambda)$.
ii) Given any variety S over Λ and any natural transformation of functors (from k-algebras to sets)

$$\phi : (A \mapsto X(A)) \longrightarrow (A \mapsto S(A_\Lambda))$$

there exists a unique algebraic morphism

$$\widetilde{\phi} : X_\Lambda \mapsto S$$

inducing ϕ on each set $X(A)$.

We would like to have a similar situation with $k = \mathbf{F}_1$ and $\Lambda = \mathbf{Z}$. But nobody is telling us which are the Λ-algebras coming from k.

4. **A definition:**
4.1. For any integer $n \geq 1$, a finite field $\mathbf{F}_q$ has a finite extension $\mathbf{F}_{q^n}$ of degree n, obtained by adjoining roots of unity. As Kapranov and Smirnov suggest, $\mathbf{F}_1$ should have an extension $\mathbf{F}_{1^n}$ of degree n, and we decide that

$$\mathbf{F}_{1^n} \otimes_{\mathbf{F}_1} \mathbf{Z} = \mathbf{Z}[T]/(T^n - 1).$$

In other words, $\mathbf{F}_{1^n} \otimes_{\mathbf{F}_1} \mathbf{Z}$ is the ring R_n of functions on the affine group scheme of n-th roots of unity.

Let $\mathcal{R}$ be the full subcategory of Rings with objects the rings $R_n, n \geq 1$, and their finite tensor products. Let also $\mathcal{R}'$ be the full subcategory of Rings with objects the tensor products of rings in $\mathcal{R}$ with the rings $\mathbf{Z}[1/N], N \geq 1$.

Definition: A *variety over* $\mathbf{F}_1$ is a covariant functor X from $\mathcal{R}$ to finite sets, equipped with natural inclusions $X(R) \subset X_{\mathbf{Z}}(R)$, $R \in \mathrm{Ob}\mathcal{R}$, where $X_{\mathbf{Z}}$ is a variety over $\mathbf{Z}$, and such that the following property (U) holds: for any variety S over $\mathbf{Z}$ and any "good" transformation of functors (from $\mathcal{R}$ to sets)

$$\phi : (R \mapsto X(R)) \longrightarrow (R \mapsto S(R))$$

there exists a unique algebraic map

$$\widetilde{\phi} : X_{\mathbf{Z}} \mapsto S$$

inducing ϕ on each set $X(R), R \in \mathrm{Ob}\mathcal{R}$.

4.2. Here are three possibilities for what "good' could mean in the previous definition:

(G0) Let $X(\mathbf{C})$ be the union of the subsets $\sigma(X(R)) \subset X_{\mathbf{Z}}(\mathbf{C})$, where R runs over all rings in $\mathcal{R}$ and σ over all ring morphisms from R to $\mathbf{C}$. There is a continuous map $\phi_{\mathbf{C}}$ from the topological closure of $X(\mathbf{C})$ (in $X_{\mathbf{Z}}(\mathbf{C})$) to $S(\mathbf{C})$ such that, for all R and σ as above,

$$\sigma \circ \phi = \phi_{\mathbf{C}} \circ \sigma \,.$$

(G1) The same as (G0), but now $\phi_{\mathbf{C}}$ is holomorphic on the holomorphic hull of $X(\mathbf{C})$ in $X_{\mathbf{Z}}(\mathbf{C})$.

(G2) The functor X extends to a functor from $\mathcal{R}'$ to sets, and ϕ extends to a transformation of functors on $\mathcal{R}'$. Then (G0) holds and, furthermore, the same statement is true when $\mathbf{C}$ (resp. $\mathcal{R}$) gets replaced by any finite extension of a p-adic field (resp. by $\mathcal{R}'$).

5. **Examples**:

5.0. When $X_{\mathbf{Z}} = \mathrm{Spec}(\mathbf{Z})$ and $X(R) = X_{\mathbf{Z}}(R)$, we say that $X = \mathrm{Spec}(\mathbf{F}_1)$.

5.1. When $X_{\mathbf{Z}}(R) = R^*$ and $X(R)$ is the set $\mu(R)$ of roots of unity in R, we say that $X = \mathbf{G}_m/\mathbf{F}_1$.

Lemma 2: The property (U) holds when S is affine and ϕ satisfies (G0).

Sketch of proof: We prove the existence of $\widetilde{\phi}$ when S is the affine line (looking at coordinates reduces the proof to that case). For any $n \geq 1$ the map ϕ sends the element $T \in \mu(R_n)$ to a polynomial $\sum_{i=0}^{n-1} a_i(n) T^i$ in R_n. On the other hand, the topological closure of $X(\mathbf{C})$ is the circle S^1. The continuous map $\phi_{\mathbf{C}}$ from the circle to $\mathbf{C}$ can be written as a Laurent series

$$\phi_{\mathbf{C}}(z) = \sum_{-\infty}^{+\infty} \alpha_i z^i \,.$$

As n goes to infinity the coefficient

$$a_i(n) = \sum_{\zeta^n = 1} \phi_{\mathbf{C}}(\zeta) \zeta^{-i}/n$$

tends to

$$\alpha_i = \int_{S^1} \phi_{\mathbf{C}}(z) z^{-i} d\theta \,.$$

Since all the $a_i(n)$'s are integers, the sequence $a_i(n), n \geq 1$, must be stationary, hence $\phi_{\mathbf{C}}$ is a Laurent polynomial $\widetilde{\phi} \in \mathbf{Z}[T, T^{-1}] = Hom(\mathbf{G}_m, \mathbf{A_1})$. q.e.d.

5.2. When, for all $R \in \mathcal{R}$, $X_{\mathbf{Z}}(R) = R$ and $X(R)$ is the set $\mu(R) \cup \{0\}$, we say that $X = \mathbf{A_1}/\mathbf{F}_1$. When $R \in Ob(\mathcal{R}')$, if $\Sigma = Hom(R, \mathbf{C})$ has cardinality N, we let $X(R)$ be the set of elements $x \in R$ such that

$$\sum_{\sigma \in \Sigma} |\sigma(x)|^2 \leq N.$$

Property (U) holds if S is affine and ϕ satisfies (G1) or (G2).

5.3. More generally, if $X_{\mathbf{Z}}$ is a smooth toric variety over $\mathbf{Z}$, we may write it as a disjoint union of products of copies of $\mathbf{A_1}$ with copies of $\mathbf{G}_m$. By imposing that the corresponding coordinates lie in $\mu(R) \cup \{0\}$, we get a subset $X(R) \subset X_{\mathbf{Z}}(R)$, which satisfies (U) when S is affine and ϕ satisfies (G1) or (G2).

5.4. Let $E \simeq \mathbf{Z}^N$ be a lattice equipped with an hermitian scalar product h on $E \otimes \mathbf{C}$, and $||.||$ the corresponding norm. It is customary in Arakelov theory to view the pair $\bar{E} = (E, h)$ as a bundle on the curve $\mathrm{Spec}(\mathbf{Z}) \cup \{\infty\}$, the global sections of which are the elements of E of norm less than one.

If we define $X(R)$ as the set of elements $x \in E \otimes R$ such that

$$\sum_{\sigma \in \Sigma} ||\sigma(x)||^2 \leq N,$$

one can again show that property (U) holds (for some variety $X_{\mathbf{Z}}$) if S is affine and ϕ satisfies (G1) or (G2).

5.5. When $X_{\mathbf{Z}} = \mathbf{G}$ is a Chevalley group as in §1, it seems natural to define $X(R)$ as the set of elements $g \in \mathbf{G}(R)$ which are mapped to the standard maximal compact subgroup of $\mathbf{G}(\mathbf{C})$ by all $\sigma \in \Sigma$. The group $\mathbf{G}(\mathbf{F}_1)$ will then be an extension of $\mathbf{W}$ by a finite abelian elementary 2-group. I have not checked if property (U) holds.

6. **Zeta functions:**

Let X be a variety over $\mathbf{F}_1$. For any $n \geq 1$, let $q = 2n + 1$, and

$$N(q) := \mathrm{card}\ X(R_n).$$

In all cases considered in §5, $N(q)$ happens to be a polynomial in q with integral coefficients (this is easy to show, but somewhat surprising in case 5.4). We may then define

$$Z(q, T) = \exp(\sum_{k \geq 1} N(q^k) T^k / k).$$

Replacing T by q^{-s} and letting q go to 1, we get

$$Z(q, q^{-s}) \underset{q \to 1}{\sim} (q-1)^{\chi} \zeta_X(s),$$

where χ is the Euler-Poincaré characteristic of $X_{\mathbf{Z}}(\mathbf{C})$ and

$$\zeta_X(s) = P(s)/Q(s),$$

where P and Q are polynomials with both integral coefficients and integral roots.

For instance, when $X = \mathbf{P}^N_{\mathbf{F}_1}$ (defined as in 5.3, i.e. $X(R)$ consists of points in $\mathbf{P}^N(R)$ having homogeneous coordinates in $\mu(R) \cup \{0\}$) the formula (2) holds.

7. Motives:

7.1. Let T be a split torus over $\mathbf{C}$, M a T-toric variety, and $r > 1$ an integer. The endomorphism which maps $t \in T$ to its r-th power extends to an endomorphism Φ_r of M. Totaro noticed in a recent paper [3] that the subspace of the rational cohomology of M where Φ_r acts by multiplication by r^i provides a canonical splitting of the weight filtration (of degree i).

In terms of mixed motives (still a conjectural notion), M gives rise to (several) extensions of Tate motives (because of the stratification mentioned in §5.3), and their classes $e \in \mathrm{Ext}(\mathbf{Z}(i), \mathbf{Z}(i+n))$ are killed by the greatest common divisor of $r^{i+n} - r^i$, $r > 1$, i.e. (essentially) the denominator of B_n/n, where B_n is the n-th Bernoulli number.

7.2. On the other hand, Beilinson proposed the formula

$$K_{2n-1}(\mathbf{Z}) = \mathrm{Ext}_{\mathcal{M}/\mathbf{Z}}(\mathbf{Z}, \mathbf{Z}(n))$$

describing the K-theory of $\mathbf{Z}$ as extensions of Tate motives over $\mathrm{Spec}(\mathbf{Z})$. Because of (3) one might then speculate that

$$\pi^s_{2n-1} = \mathrm{Ext}_{\mathcal{M}/\mathbf{F}_1}(\mathbf{Z}, \mathbf{Z}(n)).$$

If the toric variety M is defined and smooth over $\mathbf{Z}$, the example of §5.3 indicates that the extension classes e above should lie in the image of the morphism $\pi^s_{2n-1} \to K_{2n-1}(\mathbf{Z})$. It is a theorem of Quillen and Mitchell that this image is (up to 2-torsion) the cyclic group $\mathrm{Im}(J)_{2n-1}$, the order of which is the denominator of B_n/n. This is quite consistent with Totaro's remark.

It would be interesting to check if this bound on e is sharp, i.e. if all elements in $\mathrm{Im}(J)_{2n-1}$ can be interpreted in terms of toric varieties.

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[2] Manin, Y.: Lectures on zeta functions and motives (according to Deninger and Kurokawa). Columbia University Number Theory Seminar (New York, 1992). Astérisque No. 228 (1995), 4, 121–163.

[3] Totaro, B. : Chow groups, Chow cohomology, and linear varieties, preprint (1998).

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