Very stable extensions on arithmetic surfaces

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Abstract Given a line bundle \( L \) on a smooth projective curve over the complex numbers, we show that a general extension \( E \) of \( L \) by the trivial line bundle is very stable: line bundles contained in \( E \) have degree much less than half the degree of \( E \). From this result we deduce new inequalities for the successive minima of the euclidean lattice \( H^1(X, L^{-1}) \), where \( L \) is an hermitian line bundle on the arithmetic surface \( X \).

Keywords Projective curve · Semi-stable bundle · Secant variety · Arithmetic surface · Successive minima

Mathematics Subject Classification (2000) MSC 14H60 · MSC 14G40

1 Introduction

Let \( X \) be an arithmetic surface and \( \tilde{N} \) an hermitian line bundle on \( X \). The lattice

\[ A = H^1(X, N^{-1}) \]

is equipped with the \( L^2 \)-metric. In this paper we keep on studying the successive minima of this euclidean lattice; see [2], [3] and [4] for previous results. When the degree of \( \tilde{N} \) is large enough we get a lower bound for the \( k \)-th minimum of \( A \), when \( k > \frac{\deg(N)}{2} + g \), where \( g \) is the generic genus of \( X \); cf. Theorem 2 for a precise statement.

As in \textit{op. cit.}, we get this inequality by considering the extension

\[ 0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow N \rightarrow 0 \]

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defined by a class $e \in \Lambda$. If $a \geq 0$ is an integer, we say that $e$ is $a$-stable when the restriction of $E$ to the geometric generic fiber $C$ of $X$ does not contain any line bundle $L$ with

$$\deg(L) > \frac{\deg(E) - a}{2}.$$ 

The main ingredient in the proof of Theorem 2 is the assertion that any $V \subset H^1(C, N^{-1})$ contains a class $e$ which is $a$-stable when $\dim(V)$ is large enough (Theorem 1). This is proved by induction, the case $a = 0$ being Proposition 2 in [4].

The paper is organized as follows. In Section 1 we introduce the notion of $a$-stability for a rank two vector bundle on $C$. The Lemma 1 relates $a$-stability and semi-stability when $E$ is an extension of line bundles. In Lemma 2 we introduce secant varieties. Sections 1.4 to 1.9 are then devoted to the proof of Theorem 1. In Section 2 we let $\bar{N}$ be an hermitian line bundle on some arithmetic surface $X$. Proposition 2 gives a lower bound for the $L^2$- norm of $e \in A$ if its restriction to $C$ is $a$-stable. Theorem 2 follows by arguments similar to those in [2], [3] and [4].

I thank Y. Miyaoka for suggesting to look at very stable bundles, and C. Voisin for her comments on a first draft of this article.

2 Very stable extensions on curves

2.1

Let $k$ be an algebraically closed field of characteristic zero and $C$ a smooth projective curve of genus $g$ over $k$. Let $a \geq 0$ be an integer. A rank two vector bundle $E$ over $C$ is said to be $a$-stable when, for every line bundle $L$ contained in $E$, the following inequality holds:

$$\deg(L) \leq \frac{\deg(E) - a}{2}.$$ 

So, $E$ is semi-stable (resp. stable) iff it is 0-stable (resp. 1-stable).

2.2

Let $M$ and $L$ be two line bundles on $C$ and

$$0 \to L \to E \to M \to 0$$

an extension of $M$ by $L$. Let $A$ be an effective line bundle of degree $a$ on $C$ and $s : \mathcal{O}_C \to A$ a non trivial global section of $A$ on $C$. If $A^{-1}$ is the dual
of $A$ and $MA^{-1}$ its tensor product with $M$, the section $s$ defines an injective morphism

$$i : MA^{-1} \rightarrow M.$$  

If we pull-back the extension $E$ by $i$ we get a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & L & \rightarrow & E & \rightarrow & M & \rightarrow & 0 \\
\| & & \| & & \| & & \uparrow i \\
0 & \rightarrow & L & \rightarrow & E' & \rightarrow & MA^{-1} & \rightarrow & 0
\end{array}
$$

for some rank two vector bundle $E'$ on $C$.

**Lemma 1.** If $E$ is a-stable, $E'$ is semi-stable.

**Proof.** The morphism $E' \rightarrow E$ is injective, therefore any line bundle $N$ contained in $E'$ is also contained in $E$. Hence

$$\deg(N) \leq \frac{\deg(E) - a}{2} = \frac{\deg(E')}{2}$$

and $E'$ is semi-stable.

2.3

Let $N$ be a line bundle of degree $n \geq 3$ on $C$. Each cohomology class

$$e \in H^1(C, N^{-1}) = \text{Ext}(N, \mathcal{O}_C)$$

classifies an extension

$$0 \rightarrow \mathcal{O}_C \rightarrow E \rightarrow N \rightarrow 0$$

of $N$ by the trivial line bundle. We say that $e$ is a-stable (resp. semi-stable) if $E$ is a-stable (resp. semi-stable).

Let

$$\mathbb{P} = \mathbb{P}(H^1(C, N^{-1}))$$

be the projective space of lines in $H^1(C, N^{-1})$. If $\omega$ is the sheaf of differentials on $C$, Serre duality implies that $H^1(C, N^{-1}) \simeq H^0(C, \omega \otimes N)^*$ and we get a canonical immersion $C \hookrightarrow \mathbb{P}$. If $D$ is an effective divisor on $C$ we let $\langle D \rangle \subset \mathbb{P}$ be the linear span of $D$, and $|D|$ be the support of $D$. For every integer $d \geq 0$ we consider the secant variety

$$\Sigma_d = \bigcup_{\deg(D) = d} \langle D \rangle .$$

**Lemma 2.** The extension class $e$ is a-stable iff its image $\bar{e}$ in $\mathbb{P}$ does not belong to $\Sigma_d$ when $d < \frac{n+a}{2}$.

2.4

We keep the notation of the previous paragraph.

**Theorem 1.** Assume that \( n \geq a + 3 \) and let \( V \subset H^1(C, N^{-1}) \) be a \( k \)-vector space of dimension
\[
\dim(V) \geq \frac{n + a}{2} + g. \tag{1}
\]
Then there exists a class \( e \in V \) which is \( a \)-stable.

In view of Lemma 2, Theorem 1 can be rephrased as follows. Let \( \delta = (n + a)/2 \). Assume that \( n \geq \delta + 2 \). When \( d < \delta \) the secant variety \( \Sigma_d \) does not contain any linear subspace \( \mathbb{P}(V) \) with \( \dim(V) \geq \delta + g \).

2.5

To prove Theorem 1 we can assume that \( n + a \) is even. Indeed, if \( n + a \) is odd the condition (1) is equivalent to
\[
\dim(V) \geq \frac{n + a + 1}{2} + g,
\]
and, if \( e \) is \((a + 1)\)-stable, it is also \( a \)-stable.

When \( n + a \) is even, we proceed by induction on \( a \). When \( a = 0 \) (and \( n \) is even) Theorem 1 is Proposition 2 in [4].

Assume Theorem 1 has been proved for \( a - 1 \). If \( P \in C(k) \) is a point on \( C \) we let
\[
X_P = \bigcup_{\deg(D) < \frac{n + a}{2}} \langle D \rangle,
\]
and we consider a linear subspace \( V \subset H^1(C, N^{-1}) \) of dimension at least \( \frac{n+a}{2} + g \). Assume that \( P \) does not lie in the projective space \( \mathbb{P}(V) \subset \mathbb{P} \).

**Lemma 3.** The intersection \( X_P \cap \mathbb{P}(V) \) is a proper closed subset of \( \mathbb{P}(V) \).

2.6

To prove Lemma 3, let \( N^{-1}P \) be the tensor product of \( N^{-1} \) with the line bundle \( \mathcal{O}(P) \) and
\[
\pi : H^1(C, N^{-1}) \to H^1(C, N^{-1}P)
\]
the corestriction morphism. Let
\[
\mathbb{P}' = \mathbb{P}(H^1(C, N^{-1}P))
\]
and let
\[
p : \mathbb{P} - \{P\} \to \mathbb{P}'
\]
be the linear projection defined by $\pi$. Since $P$ is not in $\mathbb{P}(V)$, we have $\pi(V) = V'$, where $V'$ has the same dimension as $V$, and $p$ induces an isomorphism
\[ \mathbb{P}(V) \cong p(V') . \]

If $D$ is a divisor on $C$ such that $P \in |D|$, $p((D))$ is the linear span $\langle D - P \rangle'$ of $D - P$ in $\mathbb{P}'$. The secant variety
\[ \Sigma = \bigcup_{\deg(D) < \frac{n + a}{2} - 1} \langle D \rangle' \]
is a closed subset of $\mathbb{P}'$, hence its inverse image
\[ X_P - \{ P \} = p^{-1}(\Sigma) \]
is a closed subset of $\mathbb{P} - \{ P \}$.

If $\mathbb{P}(V)$ was contained in $X_P$, $\mathbb{P}(V')$ would be contained in $\Sigma$. But
\[ \dim(V') = \dim(V) \geq \frac{n + a}{2} + g > \frac{(n - 1) + (a - 1)}{2} + g \]
hence, by the induction hypothesis, $\mathbb{P}(V')$ contains a point $\bar{e}'$ such that $e'$ is $(a - 1)$-stable. Since
\[ \frac{n + a}{2} - 1 = \frac{(n - 1) + (a - 1)}{2} , \]
$\bar{e}'$ does not lie in $\Sigma$ (Lemma 2). This proves Lemma 3.

### 2.7

To prove Theorem 1 we can assume that $\dim(V) = \frac{n + a}{2} + g$. Since $H^1(C, N^{-1})$ has dimension $n + g - 1$ and $n \geq 3$, $V$ is a proper subspace of $H^1(C, N^{-1})$, and $\mathbb{P}(V)$ does not contain $C$. Let $P_1, \ldots, P_a$ be a distinct points of $C \setminus \mathbb{P}(V)$ and $A$ the divisor
\[ A = P_1 + \cdots + P_a . \]

From Lemma 3 we conclude that
\[ U = \mathbb{P}(V) - \bigcup_{\deg(D) < \frac{n + a}{2}} \langle D \rangle \]
is a nonempty open subset of $\mathbb{P}(V)$. Let $N^{-1}A^{-1}$ be the tensor product of $N^{-1}$ with $\mathcal{O}(-A)$ and
\[ \pi : H^1(C, N^{-1}A^{-1}) \to H^1(C, N^{-1}) \]
the corestriction map. Let $\mathbb{P}' = \mathbb{P}(H^1(C, N^{-1}A^{-1}))$ and
\[ p : \mathbb{P}' - \langle A \rangle' \to \mathbb{P} \]
the projection induced by $\pi$.

By Proposition 1 below, applied to $NA$ instead of $N$ and to $W = \pi^{-1}(V)$, there exists a non trivial class $e \in V$ such that $\bar{e} \in U$ and each $e' \in H^1(C, N^{-1} A^{-1})$ such that $\pi(e') = e$ is semi-stable. Assume $\bar{e}$ lies in $\langle D \rangle$, for some effective divisor $D$ on $C$. Then, either $\deg(D) \geq \frac{n+a}{2}$ or $|A| \cap |D| = \emptyset$ and $\deg(D) < \frac{n+a}{2}$.

In the latter case, since

$$\deg(NA \omega) = (2g - 2) + n + a > 2g - 2 + \deg(A) + \deg(D),$$

we have

$$\langle A \rangle \cap \langle D \rangle = \langle A \cap D \rangle = \emptyset$$

([1] p. 434) and there exists $\bar{e}' \in (D)'$ such that $p(\bar{e}') = \bar{e}$. Since $e'$ is semi-stable and $\deg(NA) = n + a$, Lemma 2 implies that

$$\deg(D) \geq \frac{n + a}{2}.$$

Applying Lemma 2 again, we conclude that $e$ is $a$-stable.

2.8

Let $N$ be a line bundle of even positive degree $n$ on $C$. Let

$$K \subset W \subset H^1(C, N^{-1})$$

be linear subspaces. We assume that $V = W/K$ is not zero and we let $U \subset \mathbb{P}(V)$ be a nonempty open subset. Let $\pi : W \to V$ be the projection and $a = \dim(K)$.

**Proposition 1.** If $\dim(V) \geq \frac{n}{2} + g$ there exists $\varepsilon \in V$ such that $\bar{e} \in U$ and any $e \in W$ such that $\pi(e) = \varepsilon$ is semi-stable.

2.9

To prove Proposition 1, we first note, as in [4] p. 288, that there exist two line bundles $L$ and $M$ on $C$ such that $LM = \omega$ and $ML^{-1} = N$. Any class $e \in H^1(C, N^{-1})$ defines an extension

$$0 \to L \to E \to M \to 0$$

and a boundary map

$$\partial_e : H^0(C, M) \to H^1(C, L).$$

The bundle $E$ is semi-stable iff $\partial_e$ is an isomorphism. We now adapt to our situation the argument of C. Voisin in [4] 2.2. Let

$$\mu : H^0(C, M)^{\otimes 2} \to W^*$$
be the composite of the cup-product with the projection

\[ H^0(C, M^2) = H^1(C, N^{-1})^* \to W^*. \]

Any vector \( e \in W \) defines, via \( \mu \), a quadric \( q_e \) in the projective space \( \mathbb{P}(H^0(C, M)) \). The boundary map \( \partial_e \) is an isomorphism iff \( q_e \) is non singular.

Arguing by contradiction, we assume that, for every \( \varepsilon \in V \) such that \( \bar{\varepsilon} \in U \), there exists \( e \in W \) such that \( \pi(e) = \varepsilon \) and \( q_e \) is singular. When \( r \geq 1 \) is a positive integer, we let \( U_r \subset U \) be the set of those \( \bar{\varepsilon} \) such that there exist \( e \in W \) with \( \pi(e) = \varepsilon \) and the singular locus of \( q_e \) has dimension \( r \). We have

\[ U = \bigcup_{r \geq 1} U_r \]

and each set \( U_r \) is constructible. Therefore there exists \( r_0 \) such that \( U_{r_0} \) contains a dense open subset of \( \mathbb{P}(V) \). Consider the Zariski closure \( B \subset \mathbb{P}(H^0(C, M)) \) of the union of the singular loci of the quadrics with singular locus of dimension \( r_0 \), and let \( b \) be the dimension of \( B \).

Let \( \sigma \in H^0(C, M) \) be a representative of a generic point \( \bar{\sigma} \in B \). We claim that the map

\[ \mu_\sigma : H^0(C, M) \to W^* \]

sending \( \tau \) to \( \mu(\sigma \otimes \tau) \) has rank at most \( a + b \). Indeed \( q \in W \) is singular at \( \tau \) iff it lies in the subspace \( Q_\tau \subset W \) orthogonal to the image of \( \mu_\tau \). The union of all the vector spaces \( Q_\tau, \tau \in B \), maps onto \( U_{r_0} \). Therefore the dimension of \( Q_\sigma \) is at least \( \dim(V) - b \) and the rank of \( \mu_\sigma \) is at most \( \dim(W) - (\dim(V) - b) = a + b \), as claimed.

It follows that the kernel \( H_\sigma \) of \( \mu_\sigma \) has dimension \( c \geq m - a - b \), where \( m = \dim H^0(C, M) \). Arguing as in op. cit., p. 290, we find that the subspace \( W^\perp \subset H^0(C, M^2) \) orthogonal to \( W \) has dimension at least

\[ b + c \geq m - a. \]

Therefore, since \( H^1(C, N^{-1}) \) has dimension \( n + g - 1 \), \( W \) has dimension at most \( n + a + g - m - 1 \). By Riemann-Roch and the fact that \( 2\deg(M) = 2g - 2 + n \), we know that

\[ n - m + g \leq \frac{n}{2} + g. \]

Since \( \dim(V) = \dim(W) - a \), we get

\[ \dim(V) \leq \frac{n}{2} + g - 1, \]

contradicting our hypothesis.
3 Arithmetic surfaces

3.1

Let $F$ be a number field, $\mathcal{O}_F$ its ring of integers and $S = \text{Spec}(\mathcal{O}_F)$. Consider a proper flat curve $X$ over $S$ such that $X$ is regular and the generic fiber $X_F$ is geometrically irreducible of genus $g$. Let

$$\text{deg} : \text{Pic}(X) \to \mathbb{Z}$$

be the morphism which sends the class of a line bundle over $X$ to the degree of its restriction to $X_F$.

Let $\bar{N} = (N, h)$ be an hermitian line bundle on $X$. The cohomology group

$$A = H^1(X, N^{-1})$$

is a finitely generated module over $\mathcal{O}_F$. It can be endowed as follows with an hermitian norm. For every complex embedding $\sigma : F \to \mathbb{C}$, we let $X_\sigma = X \otimes \mathcal{O}_F \mathbb{C}$ be the corresponding complex curve. The cohomology group

$$A_\sigma = A \otimes \mathbb{C} = H^1(X_\sigma, N_{\mathbb{C}}^{-1})$$

is canonically isomorphic to the complex vector space $\mathcal{H}^{01}(X_\sigma, N_{\mathbb{C}}^{-1})$ of harmonic differential forms of type $(0, 1)$ with coefficients in the restriction $N_{\mathbb{C}}^{-1}$ of $N^{-1}$ to $X_\sigma$. Given $\alpha \in \mathcal{H}^{01}(X_\sigma, N_{\mathbb{C}}^{-1})$ we let $\alpha^*$ be its transposed conjugate (the definition of which involves $h$), and we define

$$\|\alpha\|_{L^2}^2 = \frac{i}{2\pi} \int_{X_\sigma} \alpha^* \alpha.$$ 

Given $e \in A$ we let

$$\|e\| = \sup_{\sigma} \|\sigma(e)\|_{L^2},$$

where $\sigma$ runs over all complex embeddings of $F$.

Let $a \geq 0$ be an integer and $n$ the degree of $N$. We assume that $n \geq a + 3$. Let $\bar{A}$ be an hermitian line bundle on $X$ of degree $\text{deg}(A) = a$, and $s : \mathcal{O}_X \to \bar{A}$ a non zero global section of $A$. Define

$$\|s\|_{\sup} = \sup_{x \in X(\mathbb{C})} \|s(x)\|,$$

where $X(\mathbb{C}) = \bigsqcup_\sigma X_\sigma$ is the set of complex points of $X$.

Any class $e \in A$ defines an extension

$$0 \to \mathcal{O}_X \to E \to N \to 0$$

on $X$. If $\bar{F}$ is a fixed algebraic closure of $F$, we let $E_{\bar{F}}$ be the restriction of $E$ to $X_{\bar{F}} \otimes \bar{F}$. Denote by $r = [F : \mathbb{Q}]$ the absolute degree of $F$. 
Proposition 2. Assume $E_F$ is a-stable. Then the following inequality holds
\[ \log \| e \| \geq \frac{(\bar{N} - \bar{A})^2}{2(n-a)r} - \log \| s \|_{\text{sup}} - 1, \]
where $(\bar{N} - \bar{A})^2 \in \mathbb{R}$ denotes the arithmetic self-intersection of the first Chern class $\hat{c}_1(\bar{N} \bar{A}^{-1}) \in \hat{C}H^1(X)$.

3.2

To prove Proposition 2 we consider the extension
\[ 0 \to \mathcal{O}_X \to E' \to NA^{-1} \to 0 \]
obtained by pulling back $e \in H^1(X, N^{-1})$ to $e' \in H^1(X, N^{-1}A)$. Since the restriction of $E'$ to $X_F$ is semi-stable (Lemma 1) we have
\[ \log \| e' \| \geq \frac{(\bar{N} - \bar{A})^2}{2(n-a)r} - 1 \quad (2) \]
(see [2] or [4] pp. 294-295). So we are left with comparing $\| e \|$ and $\| e' \|$.

We have a commutative diagram:
\[
\begin{array}{cccccc}
0 & \to & \mathcal{O}_X & \to & E & \to & N & \to & 0 \\
\| & | & \| & | & \| & | & \| & | & \\
0 & \to & \mathcal{O}_X & \to & E' & \to & NA^{-1} & \to & 0.
\end{array}
\]

Any $C^\infty$ splitting $E_{C} \to \mathbb{C}$ of the top extension defines, by restriction, a $C^\infty$ splitting $E'_{\mathbb{C}} \to \mathbb{C}$. The Cauchy-Riemann operators $\bar{\partial}_E$ and $\bar{\partial}_{E'}$ can then be written as matrices
\[ \bar{\partial}_E = \begin{pmatrix} \bar{\partial}_C & \alpha \\ 0 & \bar{\partial}_N \end{pmatrix} \]
and
\[ \bar{\partial}_{E'} = \begin{pmatrix} \bar{\partial}_C & \alpha' \\ 0 & \bar{\partial}_{NA^{-1}} \end{pmatrix}, \]
where $\alpha$ is a linear map $C^\infty(N_C) \to A^{01}(\mathbb{C})$, and $\alpha' : C^\infty(NA_C^{-1}) \to A^{01}(\mathbb{C})$ is the restriction of $\alpha$ to $NA_C^{-1}$.

For any $\sigma : F \to \mathbb{C}$, choose a local chart $z$ of $X_{\sigma}$ and local trivializations of $N_C$ and $A_C$. We have
\[ \alpha = \varphi \, d\bar{z}, \]
where $\varphi$ is a smooth function and
\[ \alpha' = \varphi u \, d\bar{z}, \]
where \( u \) is the local section of \( A \) defined by \( s \). The transposed conjugates are

\[
\alpha^* = \frac{\bar{\varphi}}{h_N(1,1)} \, dz
\]

and

\[
\alpha'^* = \frac{h_A(1,1) \, \bar{\varphi} \, \bar{u} \, dz}{h_N(1,1)},
\]

where \( h_N(1,1) \) (resp. \( h_A(1,1) \)) is the squared norm of the local generator of \( N \) (resp. \( A \)). It follows that

\[
\alpha'^* \alpha' = h_A(1,1) \, \bar{u} \, \alpha^* \alpha = \| s \|^2 \, \alpha^* \alpha,
\]

and

\[
\| \alpha' \|^2_{L^2} = \frac{i}{2\pi} \int_{X_\sigma} \alpha'^* \alpha' \leq \| s \|^2 \, \sup \| \alpha \|^2_{L^2}.
\]

Assume that the splitting \( E_C \to \mathbb{C} \) has been chosen such that \( \alpha \) is harmonic. Then we get

\[
\| \alpha' \|^2_{L^2} \leq \| s \| \sup \| \sigma(e) \|^2_{L^2}.
\]

Since \( \| \sigma(e') \|^2_{L^2} \) is the smallest value of \( \| \alpha' \|^2_{L^2} \) when \( \alpha' \) runs over all representatives of \( e' \) in \( A^{[1]}(X_\sigma, N^{-1} A_C) \), we get

\[
\| \sigma(e') \|^2_{L^2} \leq \| s \| \sup \| \sigma(e) \|^2_{L^2}
\]

hence

\[
\| e' \| \leq \| s \| \sup \| e \|.
\]

This inequality and (2) imply Proposition 2.

3.3

We keep the notation of §2.1 and we consider the (logarithms of the) successive minima of the euclidean lattice \((A, \| \cdot \|)\). When \( k \leq rk(A) \), \( \mu_k \) is the infimum of all real numbers \( \mu \) such that there exists \( k \) elements \( e_1, \ldots, e_k \) in \( A \) which are linearly independent in \( A \otimes \mathbb{F} \) and such that

\[
\| e_i \| \leq \exp(\mu) \quad \text{for all} \quad i = 1, \ldots, k.
\]

**Theorem 2.** Assume that

\[
\frac{n + a}{2} + g \leq k < n + g - 1.
\]

Then

\[
\mu_k \geq \frac{(\bar{N} - \bar{A})^2}{2(n - a) r} - \log \| s \| \sup - C,
\]

where \( C = 1 + \log(d(n, a) k) \), and \( d(n, a) \) is bounded as in (3) below.
3.4

To prove Theorem 2 we let
\[ V \subset H^1(X_F, N^{-1}) = \Lambda \otimes \bar{F} \]
be the linear space spanned by \( e_1, \ldots, e_k \). Since \( k < n + g - 1 \), \( V \) is a proper subspace of \( \Lambda \otimes \bar{F} \). From Theorem 1 we know that there exists \( e \in V \) such that the corresponding extension \( E \) of \( \mathcal{O}_C \) on \( C = X_F \) is \( a \)-stable. Moreover, \( E \) is a stable extension when \( e \) does not belong to \( \mathbb{P}(V) \cap H(n, a) \), where \( H(n, a) \) is a hypersurface defined as follows. When \( n + a \) is odd we let \( H(n, a) = H(n, a + 1) \). When \( n + a \) is even, \( H(n, a) \) is defined by induction on \( a \). We choose \( A = P_1 + \ldots + P_k \) as in 1.7. The class \( \bar{e} \) is \( a \)-stable when it satisfies the following two conditions. First, for any \( P \in |A| \), the projection of \( \bar{e} \) into \( \mathbb{P}(H^1(C, N^{-1}P)) \) should not lie in \( H(n - 1, a - 1) \). Second, let \( L \) and \( M \) be line bundles on \( C \) such that \( LM = \omega \) and \( ML^{-1} = NA \); then, any class \( e' \in \mathbb{P}(H^1(C, N^{-1}A^{-1})) \) which maps to \( e \in \mathbb{P}(H^1(C, N^{-1})) \) should be such that the boundary map
\[ \partial_{e'} : H^0(C, M) \to H^1(C, L) \]
is an isomorphism. Let \( m \) be the dimension of \( H^0(C, M) \), \( \sigma_1, \ldots, \sigma_m \) a basis of \( H^0(C, M) \), and \( \tau_1, \ldots, \tau_m \) a basis of \( H^1(C, L) \). Then \( \partial_{e'} \) is injective as soon as it satisfies the inequality
\[ (\partial_{e'}(\sigma_1) \wedge \ldots \wedge \partial_{e'}(\sigma_m), \tau_1 \wedge \ldots \wedge \tau_m) \neq 0, \]
which is of degree \( m \leq \frac{n + a}{2} \) in \( e' \). It follows from the proof of Theorem 1 that \( \bar{e} \) is \( a \)-stable as soon as it satisfies these two conditions, which is the case when \( \bar{e} \notin H(n, a) \), where \( H(n, a) \) is an hypersurface of degree \( d(n, a) \) with
\[ d(n, a) \leq \frac{n + a}{2} + a(n - 1, a - 1) \]
and
\[ d(n, 0) \leq \frac{n}{2}. \]

Therefore we get
\[ d(n, a) \leq p + a(p - 1) + a(a - 1)(p - 2) + a(a - 1)(a - 2)(p - 3) + \ldots + a!(p - a), \quad \text{when } n + a = 2p \text{ or } 2p - 1. \] (3)

Therefore, as in [3] Prop. 5, there exist \( k \) integers \( n_1, \ldots, n_k \), with \( |n_i| \leq d(n, a) \) for all \( i \), such that
\[ e = n_1 e_1 + \ldots + n_k e_k \]
does not lie in \( H(n, a) \). The extension \( E \) defined by \( e \) on \( X \) is then \( a \)-stable, and Proposition 2 implies that
\[ \log ||e|| \geq \frac{(N - A)^2}{2(n - a) r} - \log ||s||_{sup} - 1. \]

Since
\[ ||e|| \leq k d(n, a) \exp(\mu_k), \]
Theorem 2 follows.
References