Classification of Complex Algebraic Surfaces

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Abstract

In this note we present the classical Enriques’ classification theorem for complex algebraic surfaces. We’ll recall basic facts about the theory of complex surfaces (structure theorems for birational maps), and discuss (using a modern (=Mori) approach) some important results like the Castelnuovo’s rationality criterion and the classification of minimal ruled surfaces. Finally, after the description of some fundamental examples (K3, Enriques surfaces, bi-elliptic,...), we’ll sketch the proof of the Enriques’ classification theorem.

1 Introduction

Objects that we’d like to classify: smooth algebraic surfaces /\( \mathbb{C} \).

Start recalling: Classification of curves

Birational (= biregular) rough classification. Use the genus \( g = \dim H^0(\Omega^1_C) \) to divide curves into families:

- \( g = 0 \Leftrightarrow C \) is rational \( \Leftrightarrow C \cong \mathbb{P}^1 \);
- \( g = 1 \Leftrightarrow C \) is an elliptic curve, i.e. \( \mathbb{C}/\Gamma \);
- \( g \geq 2 \Leftrightarrow C \) is of ‘general type’ (\( K_C \) ample).

Rough because if \( g \geq 1 \) we have moduli.

Q: Can we classify complex algebraic surfaces (birational classes) in a similar way? Yes, it’s possible (Enriques ∼1914)!

Preview of Enriques’ Classification (after Kodaira), preliminary version: \( g \mapsto \kappa(X) \) Kodaira dimension.

Four possibilities (for ‘minimal’ \( X \)):

- \( \kappa(X) = -\infty \): ruled (i.e. bir to \( \mathbb{P}^1 \times C \));
- \( \kappa(X) = 0 \): four subtypes;
- \( \kappa(X) = 1 \): elliptic;
- \( \kappa(X) = 2 \): general type.
Complete proofs (and much more material!) can be founded in the books listed in the bibliography.

This is our plan:

1. Review of birational maps between surfaces and birational invariants;
2. Hard Dichotomy Theorem and classification of minimal ruled surfaces;
3. Castelnuovo’s rationality criterion and classification of Del Pezzo surfaces;
4. Examples and definition in $\kappa(X) \geq 0$;
5. The Enriques’ Classification.

In the appendix we have listed the main formulas-facts we need for the classification.

2 Structure of birational maps

Let $f : X \rightarrow Y$ be a rational map. Recall: $f$ is a well-defined morphism away from a cod 2 subset. In particular if $X$ is a surface the locus of indeterminacy consists of isolated points.

**Theorem 2.1** (Castelnuovo’s elimination of indeterminacy). Given $f : X \rightarrow Y$ rational map. We have a commutative diagram

$$
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & \searrow & \\
\rightarrow & \\
\end{array}
$$

where the vertical arrows are morphisms (the first one being a sequence of blow-ups).

**Proof** W.l.o.g. let $f$ be given by a linear system $|D|$ with no fixed part (hence $D.D \geq 0$). Consider $|D_1| := |\pi^*D - kE|$ linear system on the blow-up at a point of indeterminacy ($E$ exceptional divisor). Taking $k$ suff. big $D_1$ will not have fixed part and $D_1.D_1 = D.D - k$. Repeat the procedure on possible indeterminacy points on the exceptional divisor. The procedure must end, since $D_{n+1}.D_{n+1} = D_n.D_n - k_n$ and $D_n.D_n \geq 0$ for all $n$.

Conversely:

**Theorem 2.2.** Given $f : X \rightarrow Y$ birational morphism. We have a commutative diagram

$$
\begin{array}{ccc}
Y \\
\nearrow \downarrow \\
X & \rightarrow & Y \\
\end{array}
$$

where the diagonal arrow is an isomorphism, and the vertical one a sequence of blow-ups.
PROOF Sketch: Take \( p \) to be a point of indeterminacy of \( f^{-1} \), and let \( \pi \) be the blow-up map at \( p \). Define the rational map \( f_1 := \pi^{-1} \circ f \). It can be proved that \( f_1 \) is actually a morphism. Now repeat the argument for \( f_1, f_2, \ldots \). Since the number of irreducible curves contracted by \( f_k \) are clearly strictly less than the curves contracted by \( f_{k-1} \), the process must terminate.

In conclusion, combining 2.1 and 2.2:

**Theorem 2.3** (Structure of birational maps). Given \( f : X \rightarrow Y \) birational map. Then exists \( Z \) s.t.

\[
\begin{array}{c}
Z \\
X \rightarrow Y
\end{array}
\]

where the diagonal arrows are sequence of iso and blow-ups.

Basic classical example:

\[
\begin{array}{c}
\text{Bl}_{2} \mathbb{P}^2 \\
\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2
\end{array}
\]

Here the birational map is the projection from the Segre embedding of \( \mathbb{P}^1 \times \mathbb{P}^1 \) to a plane in \( \mathbb{P}^3 \), the first diagonal arrow is the blow-up at the projection point, the second one the blow-up of the plane in two points (up to iso).

The exceptional divisor of a blow-up is a \((-1)\)-curve, i.e. smooth rational curve of self-intersection \(-1\). A natural question, in relation to the previous structure theorem, is then the following: are \((-1)\)-curves all exceptional divisors of a blow-up? The answer is yes!

**Theorem 2.4** (Castelnuovo’s contraction criterion). Every \((-1)\)-curve can be contracted (i.e. it is the exceptional divisor of a blow-up of a smooth surface).

PROOF Sketch. Main idea: find the blow-up morphism by constructing a linear system \( |L| \) that gives an isomorphism away from the \((-1)\)-curve and contract to a point the \((-1)\)-curve. Then prove that the image is actually smooth.

Who is \( |L| \)? \( L := H + kE \) where \( H \) is very ample with no higher cohomology, \( E \) the \((-1)\)-curve, \( k := H.E > 0 \). Since \( |L| \supseteq |H| + kE \), we need only to prove that there is no base locus on \( E \). By considering the long exact sequence in cohomology induced by the short exact sequence of sheaf given by restricting \( L \) to \( E \), since \( H^0(O_E(L)) = H^0(\mathbb{P}^1, O_{P_1}) = \mathbb{C} \), to construct non vanishing sections is sufficient to prove that \( H^1(X, O(L - E)) = 0 \) (which is the case). The hard part of the criterion is then the study of the smoothness of the image.

This criterion suggests the following definition:

**Definition 2.1.** \( X \) is called minimal if it does not contain any \((-1)\)-curve.

Thus, combining the previous theorems and remembering that the blow-up process increase the rank of the Neron-Severi group \( NS(X) \) by one (dually contraction decreases), we find
Proposition 2.1. Every surface is birational to a, maybe not unique (see previous example), minimal model.

3 Birational Invariants

Let describe the birational invariants important for the classification:

- $q(X) = \dim H^0(\Omega^1)$, irregularity;
- $p_g(X) = P_1(X) = \dim H^0(K_X)$, geometric genus or first plurigenus;
- $P_n(X) = \dim H^0(nK_X)$, $n$-plurigenus.

Why are they birational invariant? Easy: use the birational map $f$ for pulling-back holomorphic forms $\omega$. Since the possible pole of $f^*\omega$ is a divisor and the map $f$ is a morphism away from a discrete set of points, $f^*\omega$ must extend to a holomorphic form. To conclude, make the same reasoning for the inverse of $f$.

Note that also the holomorphic Euler characteristic is a birational invariant $\chi(\mathcal{O}_X) = 1 - q + p_g$.

Note that $b_2(X)$ ($h^{1,1}(X)$), $c(X)$ and $K_X.K_X$ are not birational invariant.

Here the Hodge diamond of a complex algebraic surface (Hodge+Serre duality):

$$
\begin{array}{cccc}
 & & q & \\
 & p_g & & q \\
1 & h^{1,1} & & p_g \\
 & q & & q \\
 & & 1 & \\
\end{array}
$$

Note that $h^{1,1}$ is the unique not birational invariant of the Hodge diamond (it increases by one under blow-up).

Using the plurigenera $P_n$ is possible to define the important birational invariant called Kodaira dimension:

- $\kappa(X) = -\infty \Leftrightarrow P_n(X) = 0$ for all $n$;
- $\kappa(X) = 0 \Leftrightarrow P_n(X) = 0, 1$ but for at least one $n$, $P_n = 1$;
- $\kappa(X) = k \Leftrightarrow P_n(X) \sim n^k$ for $n >> 1$.

It can be proved that, provided we have at least one non vanishing plurigenera,

$$\kappa(X) = \max \{ \dim_{\mathbb{C}} \bar{\phi}_{[nK_X]}(X) \},$$

where $\bar{\phi}_{[nK_X]}(X)$ is the (closed) meromorphic image of $X$ given by the pluricanonical linear system. Thus $\kappa(X) \leq \dim X = 2$.

Examples (use the fact that the Kodaira dim is additive in products):
• $\kappa(X) = -\infty$, e.g. $P^2$, $C_g \times \mathbb{P}^1$, complete intersections in $\mathbb{P}^n$ of degree $< n + 1$;
• $\kappa(X) = 0$, e.g. abelian torus, complete intersections in $\mathbb{P}^n$ of degree $n + 1$;
• $\kappa(X) = 1$, e.g. $C_g \times \mathbb{C}/\Gamma$, $g > 0$;
• $\kappa(X) = 2$, e.g. $C_{g_1} \times C_{g_2}$, $g_1, g_2 > 0$, complete intersections in $\mathbb{P}^n$ of degree $> n + 1$.

4 Hard Dichotomy theorem and classification of minimal ruled surfaces.

Recall that a divisor is called nef if $D.C \geq 0$ for all integral (reduced and irreducible) curves (from now one simply ‘curves’). In particular if we define

$$NE(X) := \{\text{cone generated by effective divisors}\}/\equiv \subseteq NS(X) \otimes_\mathbb{Z} \mathbb{R},$$

$D$ nef iff is $D \geq 0$ on $NE(X)$.

Why is the nef notion so important for the classification of surfaces? Let us start with a simple fact

**Proposition 4.1.** Let $C$ be a curve on $X$. Then

$C$ is a $(-1)$-curve $\iff K_X.C < 0$, $C.C < 0$.

To prove this fact, just play around with the genus formula using the fact that $g_a(C) = 0$ iff $C$ is smooth and iso to $\mathbb{P}^1$.

In particular we see immediately:

**Corollary 4.1.** If $K_X$ is nef then $X$ is minimal.

The converse is obviously not true (e.g. $\mathbb{P}^2$). But we have that

**Proposition 4.2.** Let $X$ be minimal and $\kappa(X) \geq 0$ then $K_X$ is nef.

**Proof** Suppose that $K_X$ is not nef, i.e. that exists a curve s.t. $K_X.C < 0$. By the genus formula we find that $C.C \geq -1$. Since $\kappa(X) \geq 0$, there exist an effective pluricanonical divisor (if $nK_X$ is trivial then $K_X$ is obviously nef). By effectiveness and since $D.C < 0$, $D = aC + R$ for $a > 0$. Thus $0 > D.C = aC.C + R.C \geq aC.C$, in particular $C.C = -1$. But, using the previous proposition, $C$ must be a $(-1)$-curve. Hence $X$ is not minimal. Absurd

Important corollary of the previous proposition.

**Corollary 4.2.** If $\kappa(X) \geq 0$ the minimal model is unique.

**Proof** It is easy to check that, in general,

$$K_X.C \leq K_X^n C,$$
where $\overline{C}$ is the strict transform of the blow-up $\overline{X}$. Now let $f : X \dashrightarrow Y$ a birational map between two minimal model in $\kappa \geq 0$. In particular, by the previous proposition 4.2, $X$ and $Y$ have nef canonical. I claim that $f$ is a morphism. Suppose that the indeterminacy locus is not empty and take the minimum number of blow-ups to resolve the indeterminacy. By minimality the last exceptional divisor $E$ comes from a curve at the previous step (otherwise is just an ordinary contraction contradicting minimality). Since $K_X \cdot E = -1$, by the starting remark, we would find $K_X \cdot \pi(E) \leq -1$ contradicting nefness ($\pi$ is a composition of blow-ups). Identical argument for $f^{-1}$. Thus $f$ is an isomorphism, i.e. the minimal model is unique. 

Observe that actually we have proved more: a birational map with nef image must be a morphism.

A picture of what we have proved:

The big fundamental theorem of MMP for surfaces is the following:

**Theorem 4.1 (Hard Dichotomy).** Let $X$ be a minimal model. Then are equivalent

1. $\kappa(X) = -\infty$;
2. $K_X$ is not nef;
3. $X$ is ruled (i.e. birational to $C \times \mathbb{P}^1$).

Classically (Enriques-Kodaira) $1 \Rightarrow 3$ ($3 \Rightarrow 1$ trivial). Maybe under the weak assumption $P_{12} = 0$.... Hard arrows: $1 \Rightarrow 2$ (we have already proved in 4.2 $2 \Rightarrow 1$). This will be (unfortunately) a by-product of the Enriques' Classification theorem. Now we discuss $2 \Rightarrow 3$. 

6
The crucial ingredient is the *Rationality Theorem*. Fix a very ample line bundle $H$ on $X$ and assume $K_X$ not nef. Define the *nef threshold* to be

$$nt_X(H) := \sup\{t \in \mathbb{R} \mid H_t = H + tK_X \text{ is nef}\},$$

See picture:

![Diagram](image)

**Theorem 4.2.** Under the previous hypothesis, $nt_X(H) \in \mathbb{Q}$.

For a proof (quite easy, see Matsuki).

The idea is to use the (nef) line bundle $L := qH + pK_X$, where $\xi = nt_X(H)$ to construct an 'extremal contraction'. Recall that geometrically ruled means that there exists a morphism onto a curve with fibers smooth rational curves.

**Theorem 4.3.** ($2 \Rightarrow 3$), i.e. every minimal surface $X$ with $K_X$ not nef is $\mathbb{P}^2$ or geometrically ruled (i.e. they are Mori fiber spaces).

**Proof Sketch:** Look at the half line spanned by $-K_X$. Two cases:

- All the ample class are on this line (hence $Pic$ has rk 1). In particular $-K_X$ is ample (thus $q = p_g = 0$). Since $b_2 = 1$, the Noether formula implies $K_X.K_X = 9$. Take an ample generator $H$ of Picard. Then, by K. vanishing and RR, $H = -3K_X$, and $h^0(X) = 3$. This define a morphism to $\mathbb{P}^2$, which actually is an isomorphism (by $H.H = 1$ and structure theorems of birational maps).

- Take an (v) ample class $H$ and, applying the rationality theorem to find a nef line bundle $L := qH + pK_X$. In particular $L.L \ge 0$. We claim that $L.L = 0$. If $L.L > 0$, then for every irreducible curve $L.D > 0$ (if zero the Hodge index theorem and the fact that, by def of $L$, $K_X.D < 0$ and $D.D < 0$ implies $D$ is a (-1)-curve contradicting minimality). But the, by NM criterion $L$ is ample. And this is not possible since it is on the boundary of the nef cone. Thus $L.L = 0$. Assume for simplicity that $L$ has no fixed part. By aRR inequality is possible to prove that $h^0(kL) \sim k$. 7
Hence we can consider the morphism defined by that linear system. It has no base locus since $L.L = 0$. Moreover, by def, $K_X.L < 0$. By the genus formula the irreducible member of the linear system (hyperplan section of the image) are smooth rational curves. The general case can be handed using the Stein factorisation.

Now we discuss briefly the ruled surfaces. A priori is not obvious that a geometric ruled surfaces is actually ruled. However this is the case (Enriques-Noether theorem).

We have the following important results:

**Theorem 4.4** (Classification of ruled surface). Let $X$ be a minimal ruled surface ($\neq \mathbb{P}^2$). Then $X$ is

- a Hirzebruch surface $F_n := \mathbb{P}(O \oplus O(n))$ for $n \neq 1$, if $q = 0$.
- a projectivization of a rank 2 vector bundle over a curve of genus $g = q$.

In particular the theory of minimal ruled surface is essentially the same thing that the classification of rank 2 vector bundle.

### 5 Castelnuovo’s rationality criterion and classification of Del Pezzo surfaces

Let us prove some nice lemmas that will be useful to prove the famous Castelnuovo’s rationality criterion. We start with the following

**Theorem 5.1** (Kodaira lemma). Let $X$ with $K_X$ nef. Then

$$K_X.K_X > 0 \Leftrightarrow \kappa(X) = 2$$

**Proof** ($\Rightarrow$). By the RR inequality

$$h^0(kK_X) + h^0(-(k-1)K_X) \geq \frac{1}{2}k(k-1)K_X.K_X + \chi(O_X),$$

Thus to conclude is sufficient to prove that $h^0(-(k-1)K_X) = 0$ for $k > 1$. But if exist an effective divisor $D$ in $|-(k-1)K_X|$, then for some curve one has $K_X.C = -(k-1)D.C < 0$, contradicting nefness (note that $-(k-1)K_X$ can not be trivial, by the hypothesis on the self intersection).

($\Leftarrow$) Apply K-V vanishing ($K_X$ is big and nef).

**Lemma 5.1.** Let $X$ with $K_X$ nef. Then

$$p_g = 0 \Rightarrow q \leq 1.$$

**Proof** Rewrite the Noether formula: $1 \leq b_2 = 10 - 8q - K_X.K_X$, thus $q \leq 1$. Combining the two results (and looking at the RR inequality):
Corollary 5.1. Let $X$ be a surface with $K_X$ nef and $\kappa(X) = 2$. Then $P_2(X) > 0$.

With similar arguments is possible to prove:

Proposition 5.1. Let $K_X$ be nef and $\kappa(X) \leq 1$, then one of the following hold

- $K_X . K_X = 0, q = 0$. Moreover $P_2(X) > 0$. (RR inequality)
- $K_X . K_X = 0, p_g > 0, (q \geq 0)$.
- $K_X . K_X = 0, p_g = 0, q = 1$ (and $b_2 = 2$).

As an application of the previous calculations and theorems we find

Theorem 5.2 (Castelnuovo’s Rationality Criterion). $X$ is rational iff $q(X) = P_2(X) = 0$

Proof One direction is obvious. For the other direction, take a minimal model $Y$ for $X$. First suppose that $K_Y$ is not nef. Then or is $\mathbb{P}^2$ (hence rational) or geometrically ruled (in particularity ruled by Enriques-Noether). Hence $Y$ bir $\mathbb{C} \times \mathbb{P}^1$. Since $q = 0$, we must have $C = \mathbb{P}^1$ (hence $Y$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$ which is bir to $\mathbb{P}^2$, i.e. rational). Now assume $K_Y$ nef. If $K_Y . K_Y > 0$, we have seen that $P_2 > 0$ (impossible). Thus the only possibility is that $K_Y . K_Y = 0$, but then, by prop 5.1, $q(Y)$ or $P_2(X)$ must be different from zero contradicting our assumption.

Another nice application of the classification of minimal ruled surfaces is the classification of smooth Fano ($-K_X$ ample) surfaces (Del Pezzo).

Theorem 5.3 (Classification of Del Pezzo surfaces). Let $X$ be a Del Pezzo surface. Then $X$ is one of the following:

- $\mathbb{P}^2$;
- $\mathbb{P}^1 \times \mathbb{P}^1$;
- Blow-up of $\mathbb{P}^2$ in $k \leq 8$ points in "very general" position.

Proof It is easy to see (using blow-up formulas and NM criterion of ampleness) that ampleness is preserved under blow-up. Since $q(X) = P_n(X) = 0$ for all $n$ (by Kod vanishing), a minimal model must be or $\mathbb{P}^2$ or an Hirzebruch surface $\mathbb{F}_n$. Since it must be ample the only possibility are $\mathbb{P}^2$ or $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ (the others, containing $-n$ curves, can’t be Fano by genus formula). Since the blow up at one point of $\mathbb{P}^1 \times \mathbb{P}^1$ is (up to iso) the blow up at two points of the projective plane, $X$, if it is not $\mathbb{P}^1 \times \mathbb{P}^1$, it must be a blow-up of $\mathbb{P}^2$ at some points (blow-up on exceptional divisors is not permitted, otherwise we introduce $(-2)$ curves, violating Fano condition). It is easy to observe that the number of blow up points must be less or equal than $8 (K_X . K_X = 9 - k)$, and that the points must be in general position (otherwise again $(-2)$-curves). It is not hard to work out which other configuration of points must be excluded (use NM criterion).
6 Examples and definitions in $\kappa(X) \geq 0$

In this section we define and describe the fundamental examples of surfaces (not ruled).

6.1 Elliptic fibrations

Definition 6.1. An elliptic fibration is a morphism $f : X \to C$ onto a smooth curve with general fiber a smooth elliptic curve.

Observe that exist examples of elliptic fibrations in $\kappa(X) = \{-\infty, 0, 1\}$, just take product of an elliptic curve with a curve of genus $g = \{0, 1, 2\}$ respectively. There exists a classification of the possible singular fibers of an elliptic fibration due to Kodaira.

6.2 Abelian surfaces

Definition 6.2. An abelian surface is just a two dimensional complex torus which is algebraic.

Note that obviously we have $q(X) = 2$, $p_g(X) = 1$, and $\kappa(X) = 0$.

6.3 K3 surfaces

Definition 6.3. A K3 is a surface with $q(X) = 0$ and $K_X = \mathcal{O}_X$.

Note in particular that $p_g(X) = 1$ and, by Noether formula, $e(X) = 24$.

A concrete example of K3 is given by a quartic hypersurface in $\mathbb{P}^3$ (apply Lefshetz thm). Moreover every K3 is diffeomorphic to a quartic. Another important example is the Kummer’s construction: desingularize $T^4/-1$ at the 16 singular points.

6.4 Enriques surfaces

Definition 6.4. An Enriques surface is a surface with $q(X) = p_g(X) = 0$ and $K^2_X = \mathcal{O}_X$.

Their discovery is important from an historical point of view: for a long time was believed that $q(X) = p_g(X) = 0$ was sufficient to conclude that the surface must be rational (Castelnuovo).

Proposition 6.1. Let $X$ be a K3 with a fixed point-free involution $i$. Then $X/i$ is an Enriques surface.

PROOF Consider the natural $2 : 1$ quotient covering map: $\pi : X \to X/i$. Since $\pi_*\pi^*K_{X/i} = 2K_{X/i}$ and $\pi^*K_{X/i} = K_X = \mathcal{O}_X$, we find $2K_{X/i} = \mathcal{O}_{X/i}$. □

An explicit example: consider the K3 given by the intersection of three quadrics in $\mathbb{P}^5$ of the form $P_i(x, y, z) + Q_i(t, u, v) = 0$ and restrict the involution of the ambient space $(x, y, z, t, u, v) \mapsto (x, y, z, -t, -u, -v)$. If $P_i$ (resp. $Q_i$)
doesn’t have common point in \( \mathbb{P}^2_{x,y,z} \) (fixed set point of the involution) the quotient is an Enriques surface.

It is possible also to prove the following:

**Proposition 6.2.** Every Enriques surface is the quotient of a K3 by an involution.

**Proof** Define \( Y := \{ s \in K_X \mid s^2 = 1 \} \). This is a unramified cover of \( X \) of degree 2. Thus, since the intersection form and the euler characteristic are changed by the degree of the cover, by Noether’s formula \( \chi(\mathcal{O}_Y) = 2 \chi(\mathcal{O}_X) = 2 \), since the Kodaira dimension is unchanged under etale covering, the only possibility is that \( q(Y) = 0 \) and \( p_g(Y) = 1 \) (use the Enriques classification to prove that the canonical is trivial, or the argument that we’ll see later). \( \Box \)

### 6.5 Bi-elliptic surfaces

**Definition 6.5.** A bi-elliptic surface is a surface \( X \) of the form \( (E \times F)/G \) where \( E, F \) are elliptic curves and \( G \) a finite group of translations of \( E \), acting on \( F \) so that \( F/G = \mathbb{P}^1 \).

Example: Take \( G = \mathbb{Z}_2 \) acting on \( F \) by involution.

It is possible to prove that (we omit) \( p_g = 0, \, q = 1 \) and \( P_n \leq 1 \) (formulas using ramification divisors). Therefore \( \kappa(X) = 0 \).

There exists a complete classification of elliptic surfaces, due to Bagnera De-Franchis, dividing the bi-elliptic surface in seven families according to the possible different \( G \).

### 6.6 Surfaces of general type

**Definition 6.6.** By definition a surface is called of general type if \( \kappa(X) = 2 \).

The majority of surface are in this family (e.g. take complete intersection of sufficiently big degree).

A note on the so called "geography": We have already seen that for minimal surfaces of general type \( K^2 \geq 1 \). Moreover, it is possible to show that \( e(X) > 0 \). Thus we have, by Noether’s formula, \( \chi(\mathcal{O}_X) \geq 1 \) and \( K^2 \leq 12 \chi(\mathcal{O}_X) \). The last inequality can be improved (Bogomolov-Miyaoka-Yau) inequality: \( K^2 \leq 9 \chi(\mathcal{O}_X) \) (optimal). Another important inequality is the Noether’s inequality \( K^2 \geq 2 \chi - 6 \).

In conclusion, minimal surfaces of general type must be contained in the region of the \((\chi, K^2)\) plane:

- \( \chi \geq 1 \);
- \( K^2 \geq 1 \);
- \( K^2 \leq 9 \chi \);
- \( K^2 \geq 2 \chi - 6 \).
7 The Enriques’ classification

Theorem 7.1 (Enriques’s classification). Every smooth complex surfaces is birational to

<table>
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<tr>
<th>( \kappa )</th>
<th>( q )</th>
<th>( p_g )</th>
<th>Minimal Model</th>
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<tr>
<td>(-\infty)</td>
<td>0</td>
<td>0</td>
<td>( \mathbb{P}^2 ), ( \mathbb{F}_n ) ( n \neq 1 )</td>
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<tr>
<td>(-\infty)</td>
<td>( &gt; 0 )</td>
<td>0</td>
<td>( \mathbb{P}^1 )-bundle on ( C_q )</td>
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<tr>
<td>1</td>
<td>( \geq 0 )</td>
<td>( \geq 0 )</td>
<td>Elliptic fibrations (no ( \kappa = 0, 1 ))</td>
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<tr>
<td>2</td>
<td>( \geq 0 )</td>
<td>( \geq 0 )</td>
<td>General type</td>
</tr>
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In the following we’ll sketch the proof of the classification.

We assume the following (hard!) step in the classification:

Theorem 7.2. Let \( X \) be a surface with \( K_X \) nef, \( K_X.K_X = 0 \), \( p_g = 1 \), \( q = 0 \). Then \( \kappa(X) = 0 \) (\( \Leftrightarrow \) is bi-elliptic) or \( \kappa(X) = 1 \)

The proof use a sophisticated analysis of the Albanese map.

From this it follows (final step of the Hard Dichotomy):

Proposition 7.1. Let \( X \) be minimal. Then \( \kappa(X) = -\infty \Rightarrow K_X \) not nef.

Proof If it false, it should exist a (minimal) surface with \( \kappa = -\infty \) and \( K_X \) nef. Now the fact that surfaces of that type do not exist follows immediately from 5.1 and 7.2.

In order to complete the classification is now sufficient to study the case \( \kappa = 1 \), proving that the minimal model are elliptic fibration, and the case \( \kappa = 0 \).

Proposition 7.2. Let \( X \) be a minimal surface with \( \kappa(X) = 1 \). Then \( X \) is an elliptic fibration.

Proof By the hypothesis on the Kodaira dimension we can assume that, for \( n >> 1 \), there exist an effective divisors in \( |nK| \). Let us denote with \( D \) the moving part and with \( D_f \) the fixed part of the linear system. Then \( D.D = D.K = 0 \). In fact, \( 0 = nK.K = D_f.K + D.K \) and \( K \) nef implies \( D.K = 0 = D_f.K \). Moreover \( 0 = nD.K = D.D + D.D_f \) implies \( D.D = 0 = D.D_f \). Take the Kodaira map (is a morphism) given by \( |nK| \) (the image being a curve), and observe that the map contract \( D \) to a point. By the genus formula \( D \) (irr. and red.) must have genus 1.

Now we look at the case \( \kappa = 0 \).

Lemma 7.1. Let \( X \) be a minimal surface with \( \kappa(X) = 0 \). Then \( \chi(\mathcal{O}_X) \geq 0 \).

Proof This is an elementary computation using the Noether formula. Since \( K.K = 0 \) (Kod lemma),
\[ 12\chi = e = 2 - 2b_1 + b_2 = 2 - 4q + b_2. \]

Substituting \( q = 1 - \chi + p_g \), and using the fact that \( p_g \leq 1 \) (hyp on Kod dim) and \( b_2 \geq 0 \) (actually > 0), we find

\[ 8\chi = -2 - 4p_g + b_2 \geq -6. \]

Since \( \chi \) is an integer number, we must have \( \chi \geq 0 \).

Since, by the condition on the Kodaira dimension, \( p_g \leq 1 \), we are left with only five possible combination of irregularity and geometric genus:

A priori it could exist a surface of zero Kodaira dimension with \( q = p_g = 1 \) (??? in the figure).

**Proposition 7.3.** It does not exist a minimal surface of \( \kappa = 0 \), with \( q = p_g = 1 \).

**Proof** Since \( q = 1 \), it must exists a non effective divisor \( L \) such that \( L \sim 0 \) and \( 2L \sim 0 \) (just take a torsion element in the \( q = 1 \) dim torus \( \text{Pic}^0(X) \)). Applying the RR inequality, since \( L \) is numerically trivial and \( \chi = 1 \), we find

\[ h^0(L) + h^0(K - L) \geq 1. \]

Since \( h^0(L) = 0 \) (non effectiveness), it must exist \( G \in |K - L| \), hence \( 2G \in |2(K - L)| = |2K| \) by the hypothesis on \( L \). Moreover, since \( p_g = 1 \), also \( P_2 = 1 \) (by the hypothesis on \( \kappa \) it must be less or equal to 1). In particular it must exist \( D \in |K| \) such that \( 2D \in |2K| \). But since \( P_2 = 1 \), we have \( 2D = 2G \), hence \( D = G \). But

\[ K \sim D = G \sim K - L, \]

and therefore \( L \sim 0 \). Absurd.

Now we discuss briefly the last three cases.

**Proposition 7.4.** Let \( X \) be a minimal surface with \( \kappa(X) = 0 \) and \( q = p_g = 0 \). Then \( X \) is an Enriques surface (i.e. \( 2K_X \) is trivial).
Proof Main idea: by 5.1 we know that $P_2 = 1$. Thus to conclude is sufficient to show $h^0(-2K) \geq 1$. By the RR inequality (remember that in this case $K.K = 0$):

$$P_3 + h^0(-2K) \geq 1.$$  

Thus, to conclude, I claim that $P_3 = 0$. If not, let $D_2 \in |2K|$ and $D_3 \in |3K|$. Since $P_6 \leq 1$ it must be $3D_2 = 2D_3$ and (since we can divide) it must exist an effective divisor $D$ such that $D_2 = 2D$ and $D_3 = 3D$. But $D \in |K|$. Absurd ($p_g = 0$).

Similarly:

**Proposition 7.5.** Let $X$ be a minimal surface with $\kappa(X) = 0$ and $q = 0$ and $p_g = 1$. Then $X$ is a K3 surface (i.e. $K_X$ is trivial).

**Proof** By the RR inequality $P_2 + h^0(-K) \geq 2$. Hence $h^0(-K) \geq 1$. Thus $K$ must be trivial. \hfill \square

Finally (the proof is quite technical, we omit it):

**Proposition 7.6.** Let $X$ be a minimal surface with $\kappa(X) = 0$ and $q = 2$ and $p_g = 1$. Then $X$ is Abelian.

**Proof** Sketch: Prove that the Albanese map is onto (thus the image is a $q = 2$ dim Abelian surface). Use the fact that $b_2 = 6$, to prove that the albanese map gives an iso of $H_2$. Strength the result to prove that $X$ is itself a torus by proving that the Albanese map is a finite unramified covering. \hfill \square

**Appendix**

Important formulas and facts:

- $2g_a(C) - 2 = K_X.C + C.C$, Genus Formula ($C$ irreducible, maybe singular, curve);
- $\chi(D) = \chi(\mathcal{O}_X) + \frac{1}{2} D.(D - K_X)$, Riemann-Roch;
- $h^i(D) = h^{n-i}(K_X - D)$, Serre Duality;
- $h^0(D) + h^0(K_X - D) \geq \chi(\mathcal{O}_X) + \frac{1}{2} D(D - K_X)$, RR inequality;
- $L$ big and nef (in part. ample), then $h^i(K_X + L) = 0$ for $i > 0$, Kawamata-Viehweg (Kodaira Vanishing).
- $\chi(\mathcal{O}_X) = \frac{1}{12}(K_X.K_X + e(X))$ Noether’s formula (HRR).
- Let $D$ nef, then $D.D \geq 0$.
- $L$ is ample iff $L.L > 0$ and $L.C > 0$ for all curves $C$ (Nakai-Moishezon criterion).
- $L$ is ample iff $L > 0$ on $\overline{NE(X)}\setminus\{0\}$ (Kleiman criterion).
- If $D.D > 0$ and $D.C = 0$, then $C.C \leq 0$ with equality iff $C \equiv 0$ (Hodge index theorem).
Bibliography


