Preface

The present book contains fifteen contributions on various topics related to Number Theory, Physics and Geometry. It presents, together with a forthcoming second volume, most of the courses and seminars delivered at the meeting entitled " Frontiers in Number Theory, Physics and Geometry", which took place at the Centre de Physique des Houches in the french Alps March 9-21, 2003.

The relation between mathematics and physics has a long history. Let us mention only ordinary differential equations and mechanics, partial differential equations in solid and fluid mechanics or electrodynamics, group theory is essential in crystallography, elasticity or quantum mechanics...

The role of number theory and of more abstract parts of mathematics such as topological, differential and algebraic geometry in physics has become prominent more recently. Diverse instances of this trend appear in the works of such scientists as V. Arnold, M. Atiyah, M. Berry, F. Dyson, L. Faddeev, D. Hejhal, C. Itzykson, V. Kac, Y. Manin, J. Moser, W. Nahm, A. Polyakov, D. Ruelle, A. Selberg, C. Siegel, S. Smale, E. Witten and many others.

In 1989 a first meeting took place at the Centre de Physique des Houches. The triggering idea was due at that time to the late Claude Itzykson (1938-1995). The meeting gathered physicists and mathematicians, and was the occasion of long and passionate discussions.

The seminars were published in a book entitled "Number Theory and Physics", J.-M. Luck, P. Moussa, and M. Waldschmidt editors, Springer Proceedings in Physics, Vol. 47, 1990. The lectures were published as a second book entitled "From Number Theory to Physics", with C. Itzykson joining the editorial team, Springer (2nd edition 1995).

Ten years later the evolution of the interface between theoretical physics and mathematics prompted M. Waldschmidt, P. Cartier and B. Julia to renew the experience. However the emphasis was somewhat shifted to include in particular selected chapters at the interface of physics and geometry, random matrices or various zeta- and L- functions. Once the project of the new meeting entitled " Frontiers in Number Theory, Physics and Geometry" received support from the European Union the High level scientific conference was organized in Les Houches.

The Scientific Committee for the meeting "Frontiers in Number Theory, Physics and Geometry", was composed of the following scientists: Frits Beukers, Jean-Benoît Bost, Pierre Cartier, Predrag Cvitanovic, Michel Duflo, Giovanni Gallavotti, Patricio Leboeuf, Werner Nahm, Ivan Todorov, Claire Voisin, Michel Waldschmidt, Jean-Christophe Yoccoz, and Jean-Bernard Zuber. The Organizing Committee included:

Bernard Julia (LPTENS, Paris scientific coordinator),
Pierre Moussa (SPhT CEA-Saclay), and
Pierre Vanhove (CERN and SPhT CEA-Saclay).

During two weeks, five lectures or seminars were given every day to about seventy-five participants. The topics belonged to three main domains:

1. Dynamical Systems, Number theory, and Random matrices,
with lectures by E. Bogomolny on Quantum and arithmetical chaos, J. Conrey on L-functions and random matrix theory, J.-C. Yoccoz on Interval exchange maps, and A. Zorich on Flat surfaces;

2. Polylogarithms and Perturbative Physics,
with lectures by P. Cartier on Polylogarithms and motivic aspects, W. Nahm on Physics and dilogarithms, and D. Zagier on Polylogarithms;

3. Symmetries and Non-pertubative Physics, with lectures by
A. Connes on Galoisian symmetries, zeta function and renormalization,
R. Dijkgraaf on String duality and automorphic forms,
P. Di Vecchia on Gauge theory and D-branes,
E. Frenkel on Vertex algebras, algebraic curves and Langlands program,
G. Moore on String theory and number theory,
C. Soulé on Arithmetic groups.

In addition seminars were given by participants many of whom could have given full sets of lectures had time been available. They were: Z. Bern, A. Bondal, P. Candelas, J. Conway, P. Cvitanovic, H. Gangl, G. Gentile, D. Kreimer, J. Lagarias, M. Marcolli, J. Marklof, S. Marmi, J. McKay, B. Pioline, M. Pollicott, H. Then, E. Vasserot, A. Vershik, D. Voiculescu, A. Voros, S. Weinzierl, K. Wendland, A. Zabrodin.

We have chosen to reorganize the written contributions in two parts according to their subject. These naturally lead to two different volumes. The present volume is the first one, let us now briefly describe its contents.

This volume is itself composed of three parts including each lectures and seminars covering one theme. In the first part, we present the contributions on the theme "Random matrices : from Physics to Number Theory". It begins with lectures by E. Bogomolny, which review three selected topics of quantum chaos, namely trace formulas with or without chaos, the two-point spectral correlation function of Riemann zeta function zeroes, and the two-point spectral correlation functions of the Laplace-Beltrami operator for modular

domains leading to arithmetic chaos. The lectures can serve as a non-formal introduction to mathematical methods of quantum chaos. A general introduction to arithmetic groups will appear in the second volume. There are then lectures by J. Conrey who examines relations between random-matrix theory and families of arithmetic L-functions (mostly in characteristics zero), that is Dirichlet series satisfying functional equations similar to those obeyed by the Riemann zeta-function. The relevant L-functions are those associated with cusp-forms. The moments of L-functions are related to correlation functions of eigenvalues of random matrices.

Then follow a number of seminar presentations: by J. Marklof on some energy level statistics in relation with almost modular functions; by H. Then on arithmetic quantum chaos in a particular three-dimensional hyperbolic domain, in relation to Maass waveforms. Next P. Wiegmann and A. Zabrodin study the large N expansion for normal and complex matrix ensembles. D. Voiculescu reviews symmetries of free probability models. Finally A. Vershik presents some random (resp. universal) graphs and metric spaces.

In the second part "Zeta functions: a transverse tool", the theme is zeta-functions and their applications.

First the lectures by A. Connes were written up in collaboration with M. Marcolli and have been divided into two parts. The lectures contain the most up-to date research work by the authors, including a lot of original material as well as the basic material in this exciting subject.

The second one will appear in the second volume as it relates to renormalization of quantum field theories. In their first chapter they introduce the noncommutative space of commensurability classes of Q-lattices and the arithmetic properties of KMS states in the corresponding quantum statistical mechanical system. In the 1-dimensional case this space gives the spectral realization of zeroes of zeta-functions. They give a description of the multiple phase transitions and arithmetic spontaneous symmetry breaking in the case of Q-lattices of dimension two. The system at zero temperature settles onto a classical Shimura variety, which parametrizes the pure phases of the system. The noncommutative space has an arithmetic structure provided by a rational subalgebra closely related to the modular Hecke algebra. The action of the symmetry group involves the formalism of superselection sectors and the full noncommutative system at positive temperature. It acts on values of the ground states at the rational elements via the Galois group of the modular field.

Then we report seminars given by A. Voros on zeta functions built on Riemann zeroes; by J. Lagarias on Hilbert spaces of entire functions and Dirichlet L-functions; and by M. Pollicott on Dynamical zeta functions and closed orbits for geodesic and hyperbolic flows.

In the third part " Dynamical systems: interval exchanges, flat surfaces and small divisors", are gathered all the other contributions on dynamical systems.

The lectures by A. Zorich provide an extensive self-contained introduction to the geometry of Flat surfaces which allows a description of flows on compact Riemann surfaces of arbitrary genus. The course by J.-C. Yoccoz analyzes Interval exchange maps such as the first return maps of these flows. Ergodic properties of maps are connected with ergodic properties of flows. This leads to a generalization to surfaces of higher genus of the irrational flows on the two dimensional torus. The adaptation of a continued fraction like algorithm to this situation is a prerequisite to extension of small divisors techniques to higher genus cases.

Finally we conclude this volume with seminars given by G. Gentile on Brjuno numbers and dynamical systems and by S. Marmi on Real and Complex Brjuno functions. In both talks either perturbation of irrational rotations or twist maps are considered, with fine details on arithmetic conditions (Brjuno condition and Brjuno numbers) for stability of trajectories under perturbations of parameters, and on the size of stability domains in the parametric space (Brjuno functions).

The following institutions are most gratefully acknowledged for their generous financial support to the meeting:

Département Sciences Physiques et Mathématiques and the Service de Formation permanente of the Centre National de la Recherche Scientifique; École Normale Supérieure de Paris; Département des Sciences de la matière du Commissariat à l'Énergie Atomique; Institut des Hautes Etudes Scientifiques; National Science Foundation; Ministère de la Recherche et de la Technologie and Ministère des Affaires Étrangères; The International association of mathematical physics and most especially the Commission of the European Communities.

Three European excellence networks helped also in various ways. Let us start with the most closely involved "Mathematical aspects of Quantum chaos", but the other two were "Superstrings" and "Quantum structure of spacetime and the geometric nature of fundamental interactions".

On the practical side we thank CERN Theory division for allowing us to use their computers for the webpage and registration process. We are also grateful to Marcelle Martin, Thierry Paul and the staff of les Houches for their patient help. We had the privilege to have two distinguished participants: Cécile de Witt-Morette (founder of the Les Houches School) and the late Bryce de Witt whose communicative and critical enthusiasm were greatly appreciated.

Paris July 2005
The editors,