Interfaces in two dimensions
Some open questions

Michel Bauer

Institut de physique théorique de Saclay
et
Laboratoire de physique théorique de l’École normale supérieure

IHES, Bures sur Yvette, March 18, 2010
Summary

1. What we know and what we don’t
2. Illustrate on the Ising model and percolation ▶ with a crash course on Loewner equations
3. Conclusions and other directions
Take a domain of the hexagonal lattice, and fix boundary colors ...
2d lattice domains with interfaces

... such the for any coloring of the inner hexagons ...
2d lattice domains with interfaces

... there is an interface separating blue and red.
2d lattice domains with interfaces

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2d lattice domains with interfaces

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Interfaces in statistical mechanics models

- Statistical mechanics assigns a weight to each coloring of the inner hexagons.
- This weight may depend on parameter(s) $p$ (temperature, fugacity, ...).
- The correlation length $\ell$ depends on $p$. 


This induces a weight on interfaces ...
... that may depend on $\rho$
Study the interface measure:
- when macrospice size $L$ and shape are fixed ...
- ... but lattice mesh $a$ goes to 0.
Interfaces in statistical mechanics models

1. Non critical case: $p \neq p_c$ fixed (so that $\ell/a$ is finite) and $a \to 0$
   Much has been known for quite some time
2. Critical case: $p = p_c$ fixed (so that $\ell \sim L$) and $a \to 0$
   Much is known since Schramm’s breakthrough in 2000
3. Scaling region: $p \to p_c$ and $a \to 0$, tuning to have $\ell$ as a new macroscopic length scale in the system
   Terra almost incognita
Interfaces in statistical mechanics models

Illustrate on the Ising model and on percolation.

- **Ising**: each bond separating red and blue costs a unit of energy (the parameter is $T$, the temperature).
- **Percolation**: all hexagons are independent and an hexagon is red with probability $p$. 
Ising model: $T = 0$

- At $T=0$ and finite $a$: only configurations of minimal energy count:
  - No island + Interface of minimal length
- Equivalence with a ballot problem
Ising model: $T = 0$

- At $T=0$ and mesh $a \to 0^+$:
  - Straight line
  - Rescale transverse direction by $a^{-1/2}$
    - Brownian bridge
Ising model: $T < T_c$

- Renormalizes to the $T = 0$ limit when $a \to 0^+$
Ising model: $T < T_c$

- Renormalizes to the $T = 0$ limit when $a \to 0^+$
Ising model: $T = T_c$

- Emergence of conformal invariance
- Converges to a Schramm-Loewner sample when $a \to 0^+$
Ising model: $T = T_c$

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SLE (1): Riemann’ theorem

All domains (= proper open subsets of \( \mathbb{C} \) with no hole) are conformally equivalent.

Better: all domains with three boundary marked points are conformally equivalent.
Using Riemann’s theorem, map $\mathbb{H} \setminus \gamma[0,t]$, the upper-half plane minus a piece of (parameterized) curve, conformally to $\mathbb{H}$.

With appropriate time parameterization and normalization $g_t(z) \sim z + 2t/z + O(1/z^2)$ at large $z$, $g_t$ satisfies the Loewner differential equation:

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \xi_t},$$

where $\xi_t \equiv g_t(\gamma_t)$ is a real function.

**Illuminating exercice**: Work out $g_t$ for $\xi_t \equiv 0.$
SLE (3): Random curves

- Every curve from 0 to \( \infty \) has its \( \xi \).
- Conversely, “every” \( \xi \) goes with a curve.

So a probability measure on curves is the same as a probability measure on \( \xi \)'s.

- What makes critical systems special?
- What are the consequences on the distribution of \( \xi \)?

Answer is given by Schramm’s theorem.
Under the hypothesis of conformal invariance, a general domain is understood once $\mathbb{H}$ is.

Under an additional property, suggested by nearest-neighbour (short-range) interactions on the lattice

$$\xi_B = \sqrt{\kappa} B$$ is proportional to a Brownian motion.

$\kappa=3$ for the above Ising model interface

Interface properties derived from

- Conformal geometry
- Itô stochastic calculus

Lawler Schramm Werner collaboration

Relationship between SLE and CFT

- Representation theory of the Virasoro algebra and statistical mechanics martingales (MB, Denis Bernard)
Ising model: $T > T_c$

- On the hexagonal lattice (non universal)
  - Renormalizes to the $T = +\infty$ limit when $a \to 0^+$
Ising model: $T = +\infty$

- On the hexagonal lattice (non universal):
  - $T = +\infty$ Ising $\leftrightarrow$ symmetric percolation
  - Continuum limit is $\text{SLE}_{\kappa=6}$.
    - Much harder than $T = 0$. 
Ising model: critical region

- $T < T_c$, $|T - T_c| \sim ma$,
  - $a \to 0^+$, $m \sim \ell^{-1}$
- Conformal invariance is gone
 Explicit characterization of the process $\xi_t$
  - Not done, but most likely doable ... because of the relationship with free massive fermions
  - No other explicit property understood or computed at the moment (contrast with SLE)
Explicit characterization of the process $\xi_t$ done for other free models:

- Gaussian massive free field level lines
- Loop-erased random walks (free massive symplectic fermions)
  - MB, Denis Bernard, Luigi Cantini, Kalle Kytölä
Non-symmetric percolation: critical region

$|p - p_c| \sim a^{3/4}$,

- Fractal dimension remains $7/4$ but ...
- ... the off-critical measure is singular with respect to the critical one (Pierre Nolin, Wendelin Werner)

A single sample is enough to decide whether percolation is critical or not
Intuitive argument:

- In an $N$ sites region,
  - Color fluctuation around mean is $\sim N^{1/2} \ldots$
  - $\ldots$ but systematic asymmetry is $\sim Na^{3/4}$.
  - As soon as $Na^{3/4} \gg N^{1/2}$ one makes the difference.
The critical interface is bounded by \( N \sim a^{-7/4} \) hexagons so \( Na^{3/4} \sim a^{-1} \gg N^{1/2} \sim a^{-7/8} \).

These scalings are indeed to crucial ones but the real math argument is tougher!
Conclusions

1. The scaling region deserves more attention:
   - Apart from the Nolin-Werner percolation result and the free-field descriptions of $\xi_t$ nothing is known

2. Open questions at the critical point,
   - Prove convergence to SLE
   - Matching between continuum description and discrete models, a dilemma already present in CFT (too many SLE’s without discrete counterpart)
   - Global versus growth description of interfaces (conformal welding and relationship with quantum gravity or ...)
   - SLE and ground states in disordered systems ?
   - ...

3. ...