Interfaces in two dimensions

Some open questions

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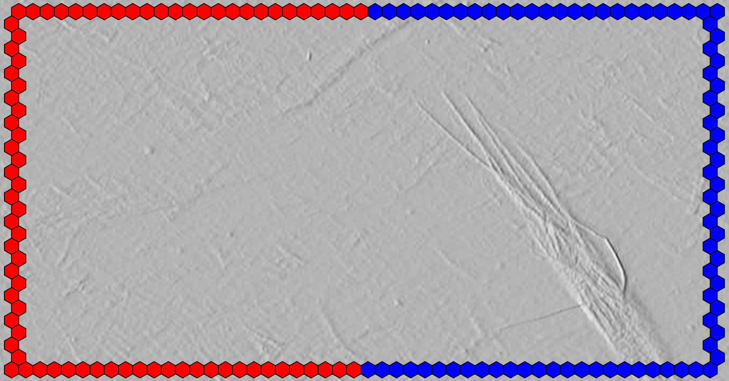
Summary

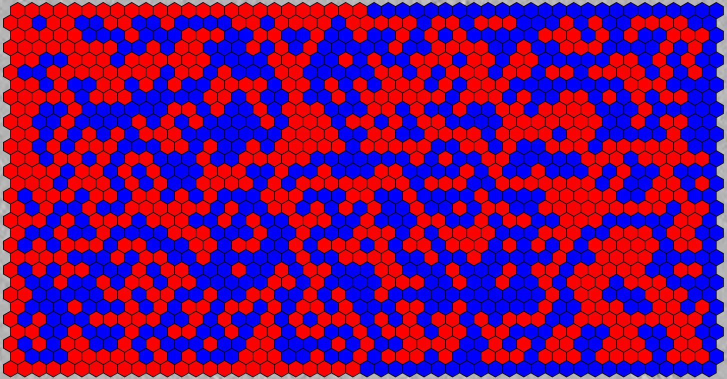
1. What we know and what we don't
2. Illustrate on the Ising model and percolation
   - with a crash course on Loewner equations
3. Conclusions and other directions

2d lattice domains with interfaces



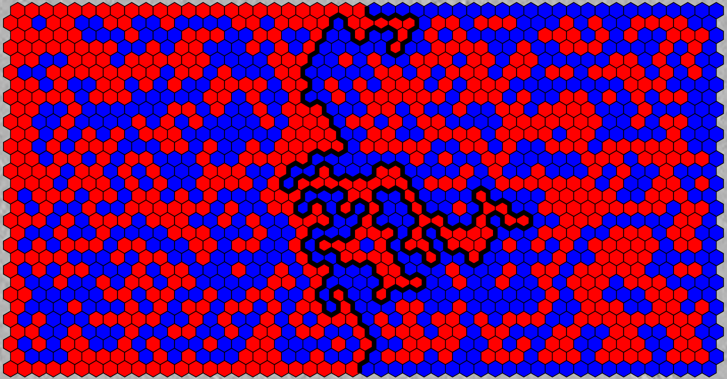
- Take a domain of the hexagonal lattice, and fix boundary colors ...

2d lattice domains with interfaces



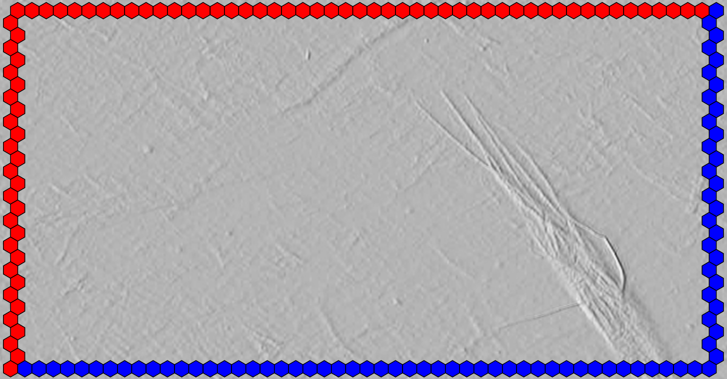
- ... such the for any coloring of the inner hexagons ...

2d lattice domains with interfaces



- ... there is an interface separating blue and red.

2d lattice domains with interfaces



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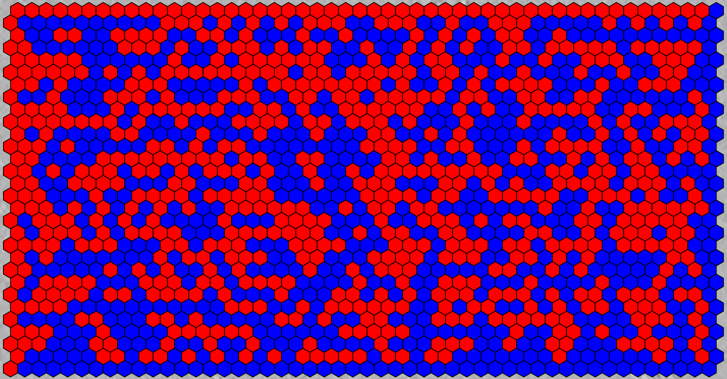
2d lattice domains with interfaces



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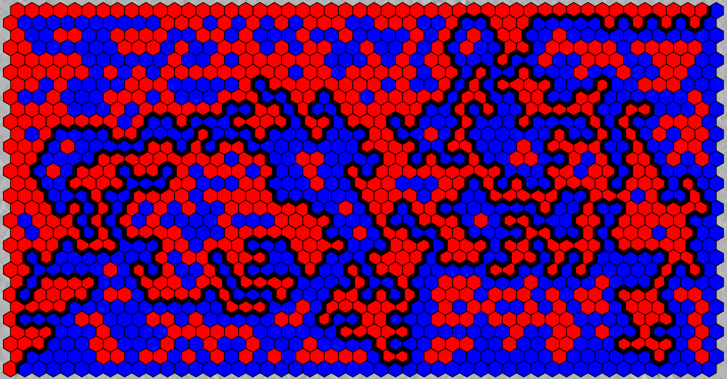
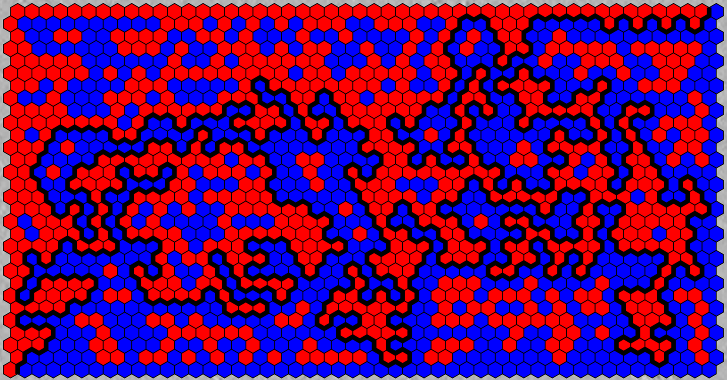
2d lattice domains with interfaces



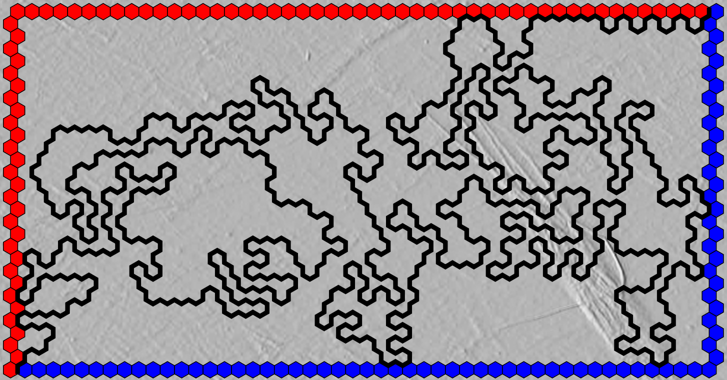
- ... there is an interface separating blue and red.

Interfaces in statistical mechanics models



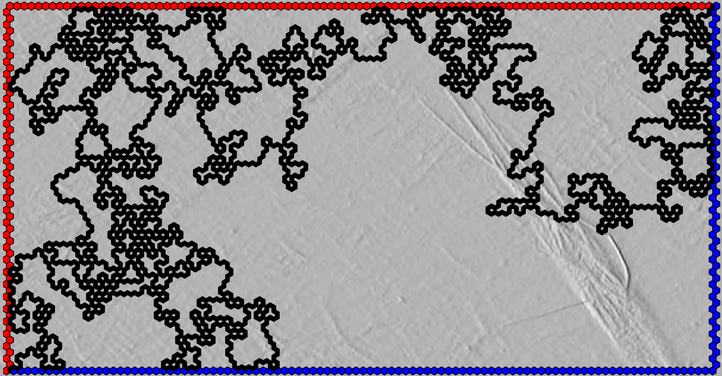
- Statistical mechanics assigns a weight to each coloring of the inner hexagons
- This weight may depend on parameter(s) p (temperature, fugacity, ...)
- The correlation length ℓ depends on p

Interfaces in statistical mechanics models



- This induces a weight on interfaces ...
- ... that may depend on p

Interfaces in statistical mechanics models



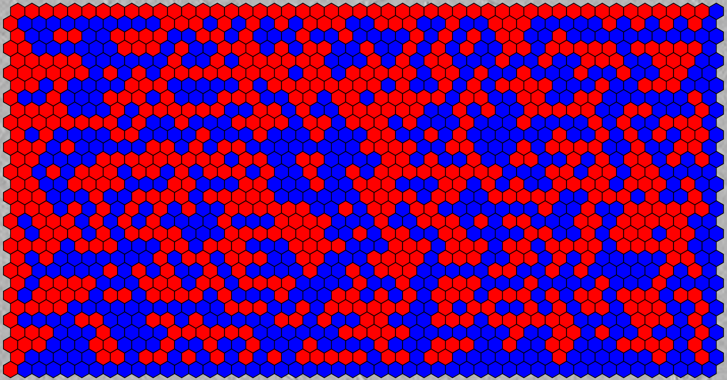
- Study the interface measure:
  - when macrospice size L and shape are fixed ...
  - ... but lattice mesh a goes to 0.

Interfaces in statistical mechanics models

1. Non critical case: $p \neq p_c$ fixed (so that ℓ/a is finite) and $a \to 0$
   Much has been known for quite some time
2. Critical case: $p = p_c$ fixed (so that $\ell \sim L$) and $a \to 0$
   Much is known since Schramm's breakthrough in 2000
3. Scaling region: $p \to p_c$ and $a \to 0$, tuning to have ℓ as a new macroscopic lentgh scale in the system
   Terra almost **incognita**

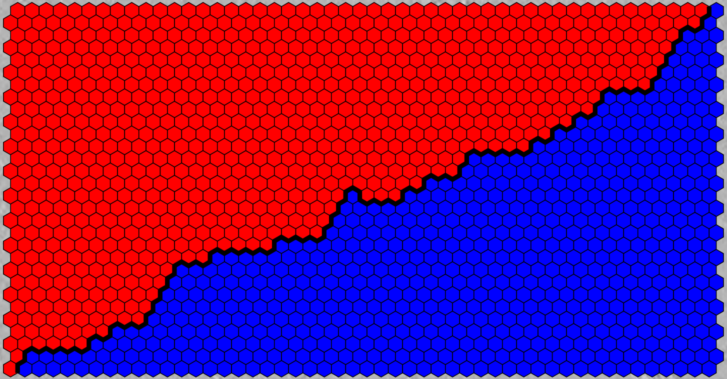
Interfaces in statistical mechanics models

Illustrate on the Ising model and on percolation.



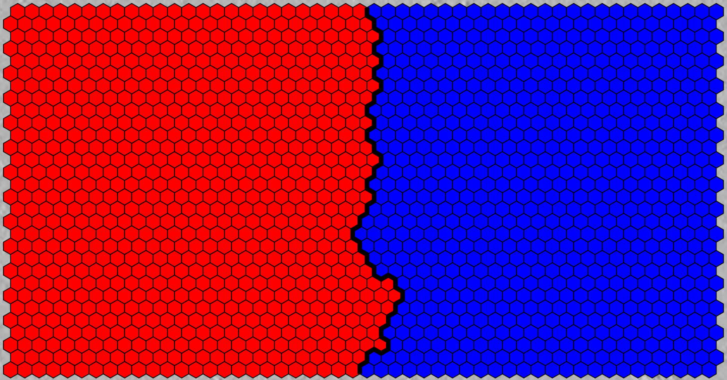
- Ising : each bond separating red and blue costs a unit of energy (the parameter is T, the temperature).
- Percolation : all hexagons are independent and an hexagon is red with probability p

Ising model: $T = 0$



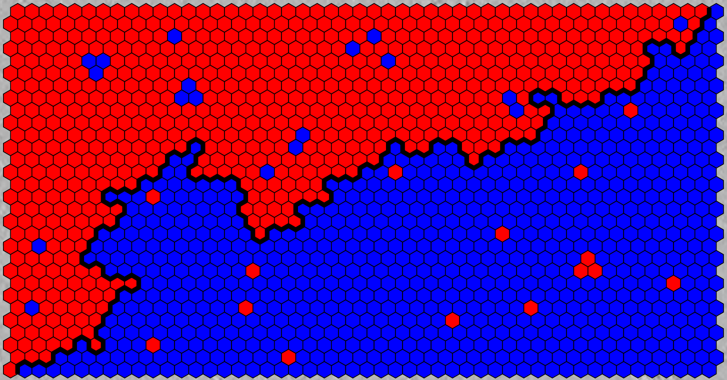
- At T=0 and finite *a*: only configurations of minimal energy count:
  - No island + Interface of minimal length
- Equivalence with a ballot problem

Ising model: $T = 0$



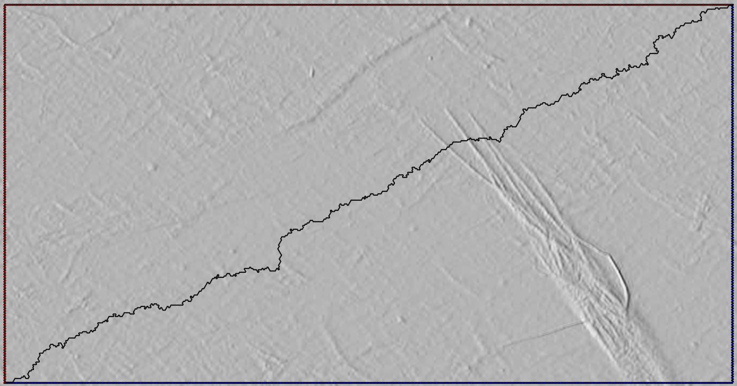
- At T=0 and mesh $a \to 0^+$:
  - Straight line
- Rescale transverse direction by $a^{-1/2}$
  - Brownian bridge

Ising model: $T < T_c$



- Renormalizes to the $T = 0$ limit when $a \to 0^+$

Ising model: $T < T_c$



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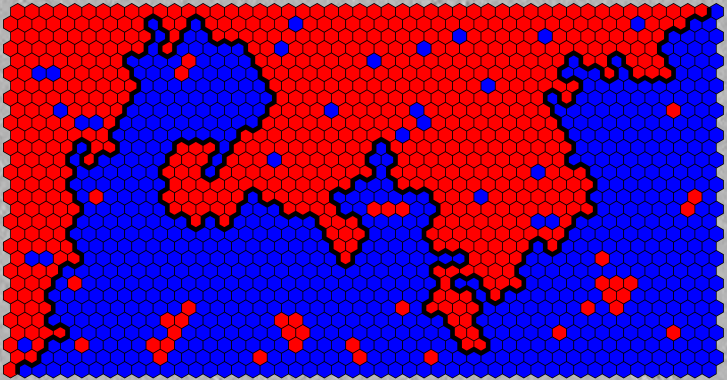
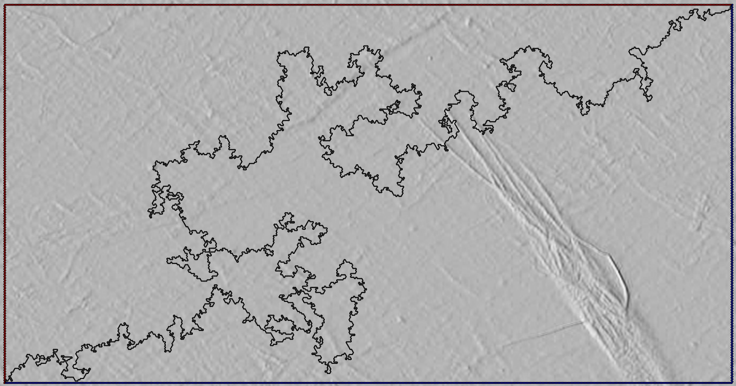


- Emergence of conformal invariance
- Converges to a Schramm-Loewner sample when $a \to 0^+$

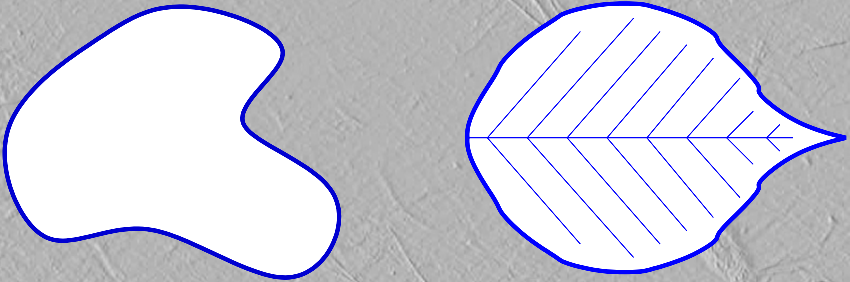
Ising model: $T = T_c$



- Emergence of conformal invariance
- Converges to a Schramm-Loewner sample when $a \to 0^+$

SLE (1): Riemann' theorem

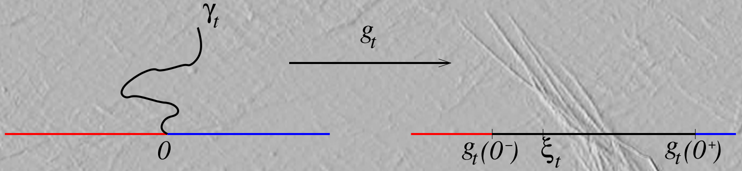
All domains (= proper open subsets of $\mathbb{C}$ with no hole) are conformally equivalent.



Better: all domains wit three boundary marked points are are conformally equivalent.

SLE (3): Loewner evolutions

Using Riemann's theorem, map $\mathbb{H}\backslash\gamma_{[0,t]}$, the upper-half plane minus a piece of (parameterized) curve, conformally to $\mathbb{H}$.



With appropriate time parameterization and normalization $g_t(z) \sim z + 2t/z + O(1/z^2)$ at large z, g_t satisfies the Loewner differential equation:

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \xi_t},$$

where $\xi_t \equiv g_t(\gamma_t)$ is a real function.
Illuminating exercice: Work out g_t for $\xi_t \equiv 0$.

SLE (3): Random curves

- Every curve from 0 to ∞ has its $\xi_\bullet$.
- Conversely, "every" $\xi_\bullet$ goes with a curve.

So a probability measure on curves
is the same as
a probability measure on $\xi_\bullet' s$.

- What makes critical systems special?
- What are the consequences on the distribution of $\xi_\bullet$?

Answer is given by Schramm's theorem

SLE (4): Stochastic/Schramm Loewner evolutions

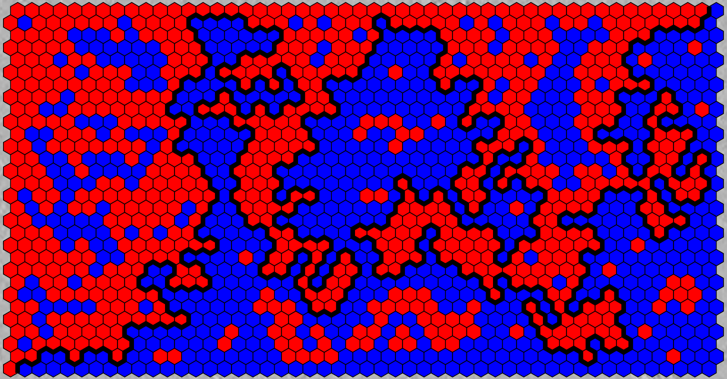
- Under the hypothesis of conformal invariance, a general domain is understood once $\mathbb{H}$ is.
- Under an additional property, suggested by nearest-neighbour (short-range) interactions on the lattice

  $\xi_\bullet = \sqrt{\kappa} B_\bullet$ **is proportional to a Brownian motion.**

  κ=3 for the above Ising model interface
- Interface properties derived from
  - Conformal geometry
  - Itô stochastic calculus

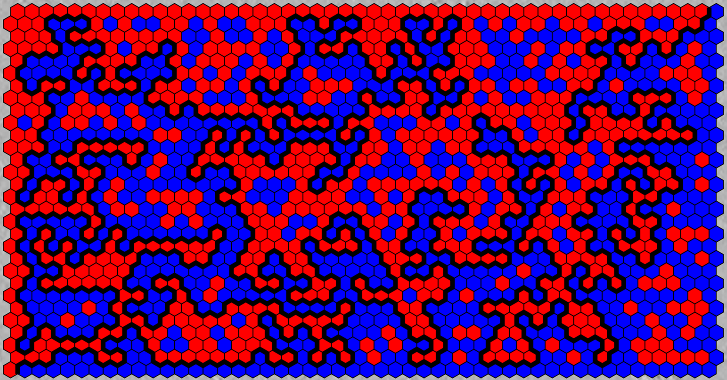
  Lawler Schramm Werner collaboration
- Relationship between SLE and CFT
  - Representation theory of the Virasoro algebra and statistical mechanics martingales (MB, Denis Bernard)

Ising model: $T > T_c$



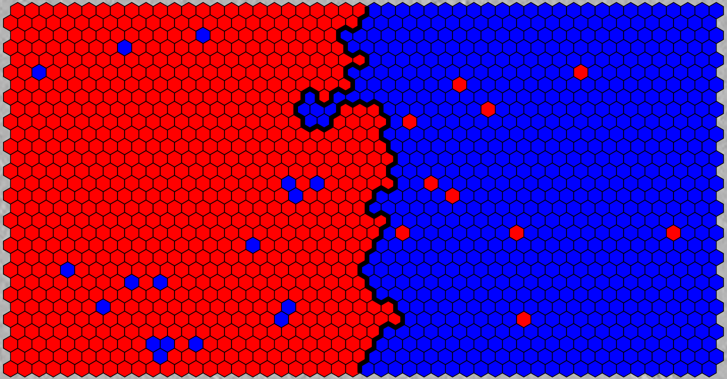
- On the hexagonal lattice (non universal)
  - Renormalizes to the $T = +\infty$ limit when $a \to 0^+$

Ising model: $T = +\infty$



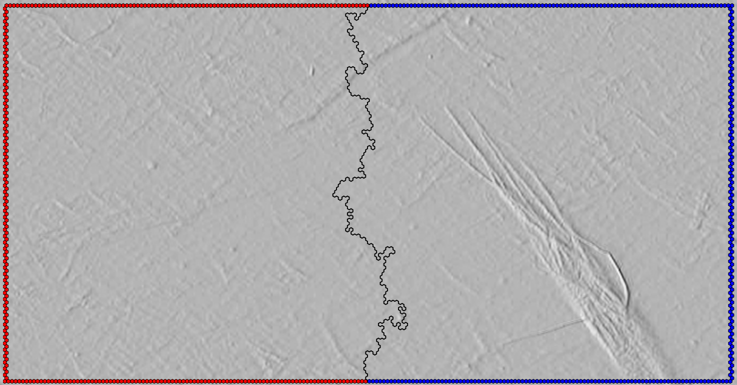
- On the hexagonal lattice (non universal):
  - $T = +\infty$ Ising $\leftrightarrow$ symmetric percolation
- Continuum limit is $\mathrm{SLE}_{\kappa=6}$.
  - Much harder than $T = 0$.

Ising model: critical region



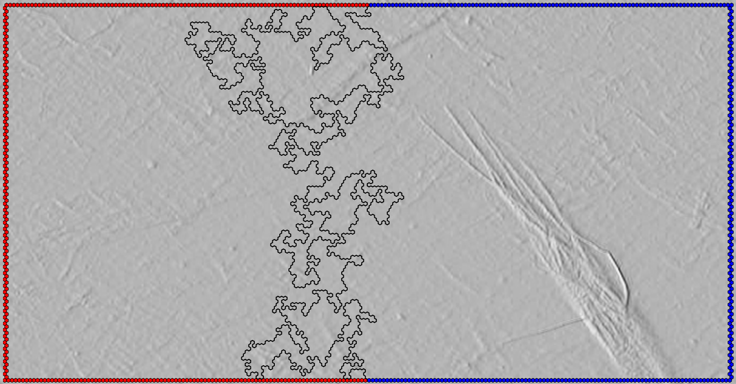
- $T < T_c$, $|T - T_c| \sim ma$,
  - $a \to 0^+$, $m \sim \ell^{-1}$
- Conformal invariance is gone

Ising model: critical region



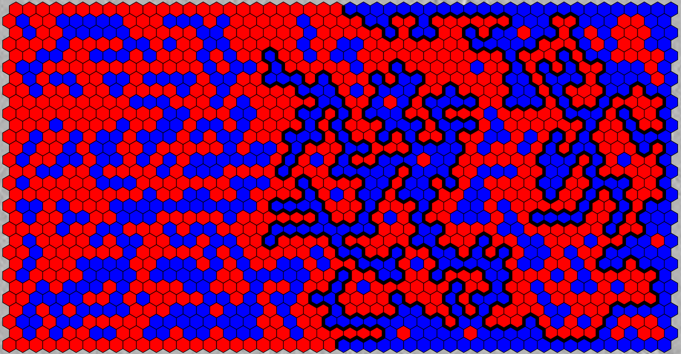
- Explicit characterization of the process ξ_t
  - Not done, but most likely doable ... because of the relationship with free massive fermions
- No other explicit property understood or computed at the moment (contrast with SLE)

Ising model: critical region



- Explicit characterization of the process ξ_t done for other free models:
  - Gaussian massive free field level lines
  - Loop-erased random walks (free massive symplectic fermions)
    - MB, Denis Bernard, Luigi Cantini, Kalle Kytölä

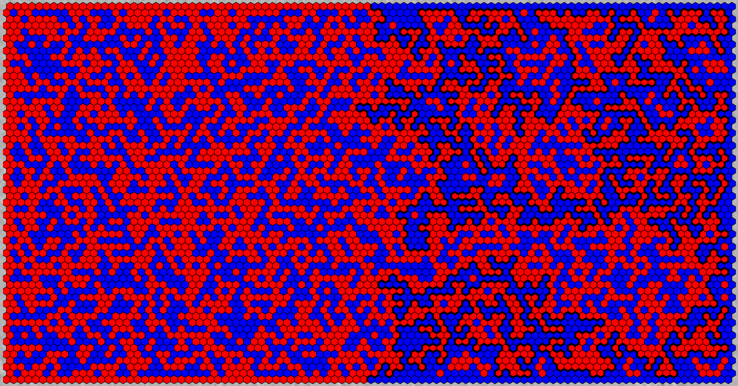
Non-symmetric percolation: critical region



- $|p - p_c| \sim a^{3/4}$,
  - Fractal dimension remains $7/4$ but ...
  - ... the off-critical measure is singular with respect to the critical one (Pierre Nolin, Wendelin Werner)

A single sample is enough to decide whether percolation is critical or not

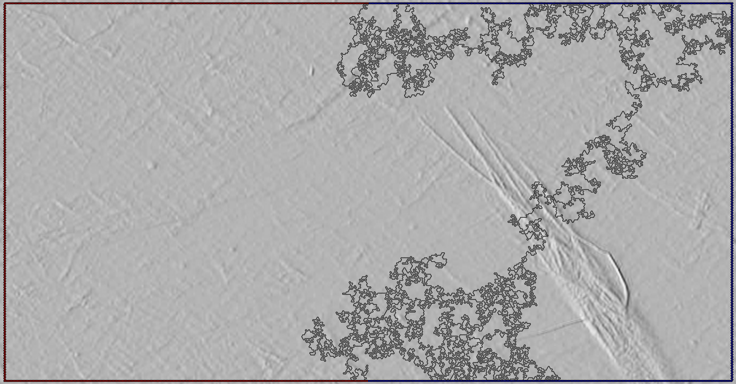
Non-symmetric percolation: critical region



Intuitive argument :

- In an N sites region,
  - Color fluctuation around mean is $\sim N^{1/2}$...
  - ... but systematic asymmetry is $\sim Na^{3/4}$.
  - As soon as $Na^{3/4} \gg N^{1/2}$ one makes the difference.

Non-symmetric percolation: critical region



- The critical interface is bounded by $N \sim a^{-7/4}$ hexagons so $Na^{3/4} \sim a^{-1} \gg N^{1/2} \sim a^{-7/8}$.

These scalings are indeed to crucial ones but the real math argument is tougher!

Conclusions

1. The scaling region deserves more attention:
   - Apart from the Nolin-Werner percolation result and the free-field descriptions of ξ_t **nothing** is known
2. Open questions at the critical point,
   - Prove convergence to SLE
   - Matching between continuum description and discrete models, a dilemma already present in CFT (too many SLE's without discrete counterpart)
   - Global versus growth description of interfaces (conformal welding and relationship with quantum gravity or ...)
   - SLE and ground states in disordered systems ?
   - ...
3. ...