Distance in random planar maps

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based on joint work with E. Guitter, earlier with P. Di Francesco

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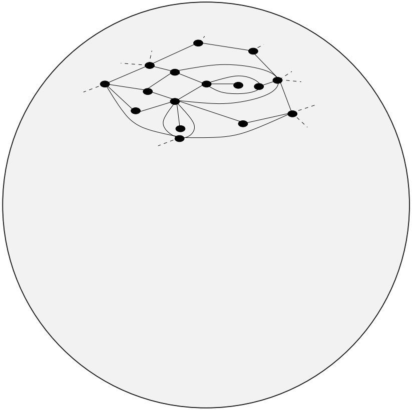
Rencontre IPhT-IHÉS, 18 mars 2010

Introduction

Planar map: connected (multi)graph embedded in the sphere and considered up to continuous deformation.

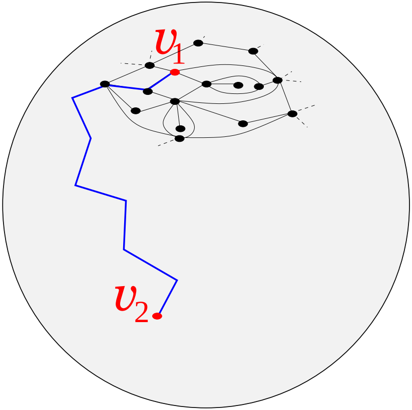


(aka planar diagrams, fatgraphs, dessins d'enfants...)

Quadrangulation: every face has degree 4

Introduction

Geodesic (or graph) distance: minimal number of consecutive edges connecting two given vertices.



Random map: any "reasonable" probability distribution over {maps of given "size"}, e.g uniform distribution over {quadrangulations with n faces}.

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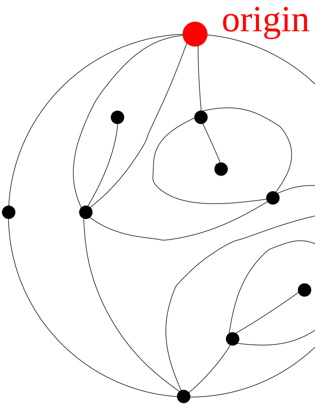
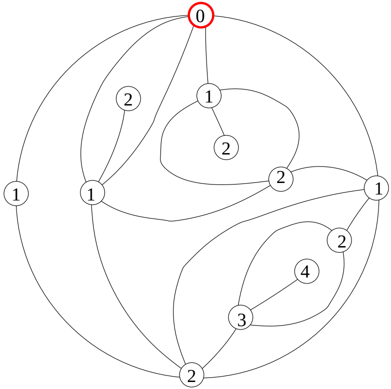
Our approach: study metric properties of large random maps using bijections with trees and integrability. Here is a flavor.

The Schaeffer bijection



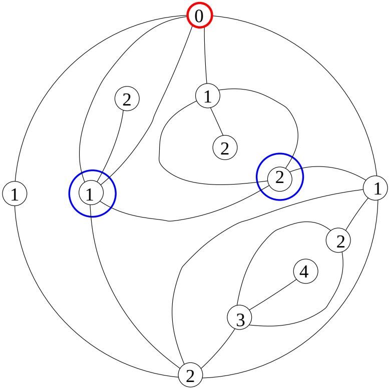
Start with a pointed planar quadrangulation (marked vertex: origin).

The Schaeffer bijection



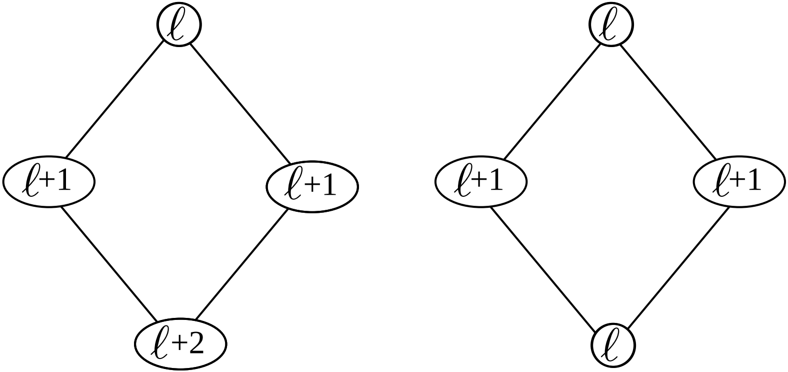
Each vertex v receives a label $\ell(v)$ equal to its graph distance from the origin.

The Schaeffer bijection



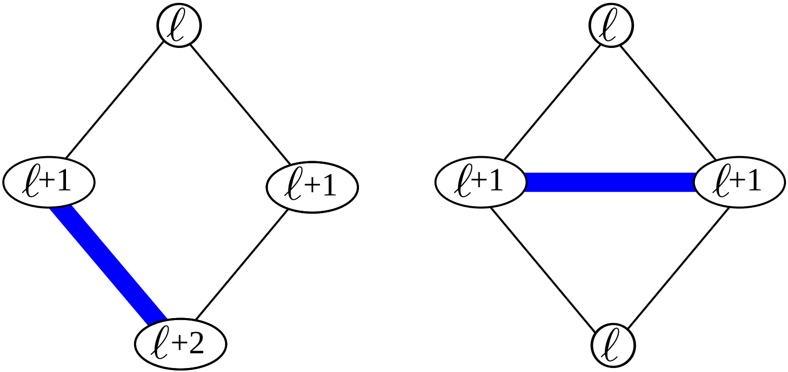
$|\ell(v) - \ell(v')| = 1$ if v and v' are neighbors on the quadrangulation.

The Schaeffer bijection



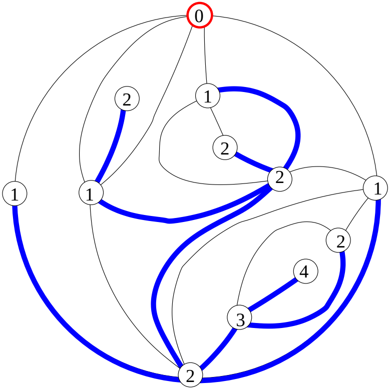
Two types of faces.

The Schaeffer bijection



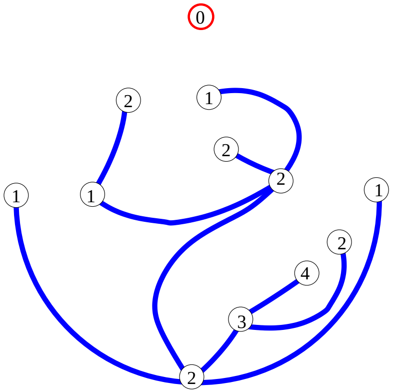
Create a new edge within each face depending on the type.

The Schaeffer bijection



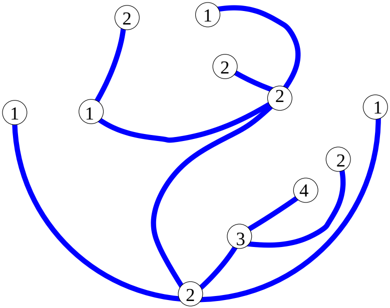
Apply the Schaeffer rules independently within each face.

The Schaeffer bijection



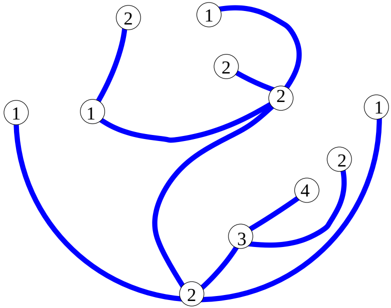
Remove the isolated origin.

The Schaeffer bijection



Obtain a *well-labeled tree* (with minimal label 1).

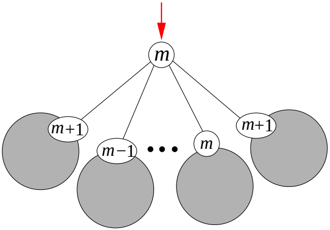
The Schaeffer bijection



Extension to more general classes of maps: [BDG04-07]

Integrability

We consider generating functions for quadrangulations with a weight g per face (per edge for trees).
For well-labeled trees we easily find a recursive equation:



$$R_\ell = \sum_{k \geq 0} \left(g\left(R_{\ell-1} + R_\ell + R_{\ell+1}\right)\right)^k = \frac{1}{1 - g(R_{\ell+1} + R_\ell + R_{\ell-1})}$$

This is valid for $\ell > 0$ with the boundary condition $R_0 = 0$.

Integrability

The solution is

$$R_\ell = R\frac{[\ell][\ell+3]}{[\ell+1][\ell+2]}$$

where

$$[\ell] \equiv \frac{1-x^\ell}{1-x}$$

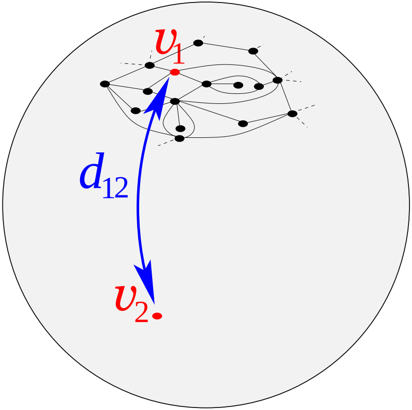
and

$$R(g) = \frac{1-\sqrt{1-12g}}{6g} \qquad x(g) + \frac{1}{x(g)} + 1 = \frac{1}{gR(g)^2}.$$

The property of "integrability" appears in a more general context [BDG03].

The two-point function

Find the law for the distance between two random vertices in a random quadrangulation.



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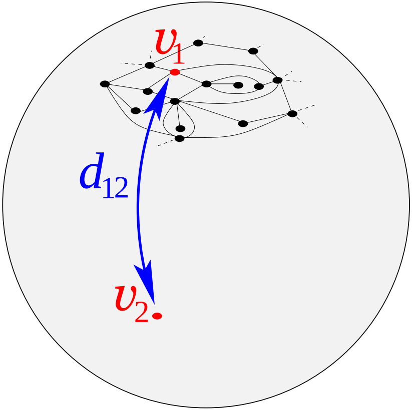
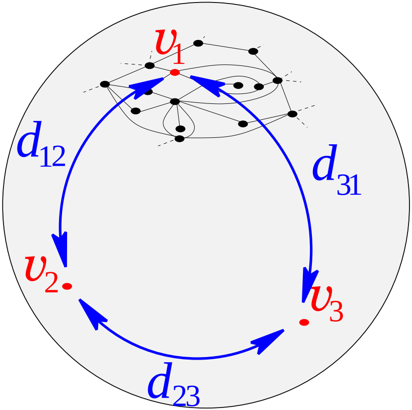


Solved above ! [BDG 2003] The generating function for quadrangulations with two marked vertices at distance $\leq d$ is $\log R_d$.

The three-point function

Find the probability distribution for the pairwise distances between three random vertices in a random quadrangulation.



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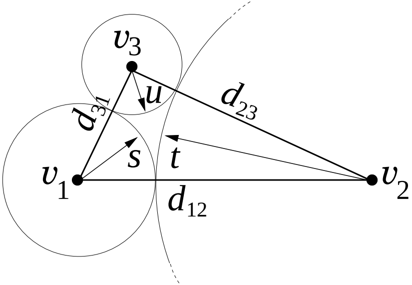
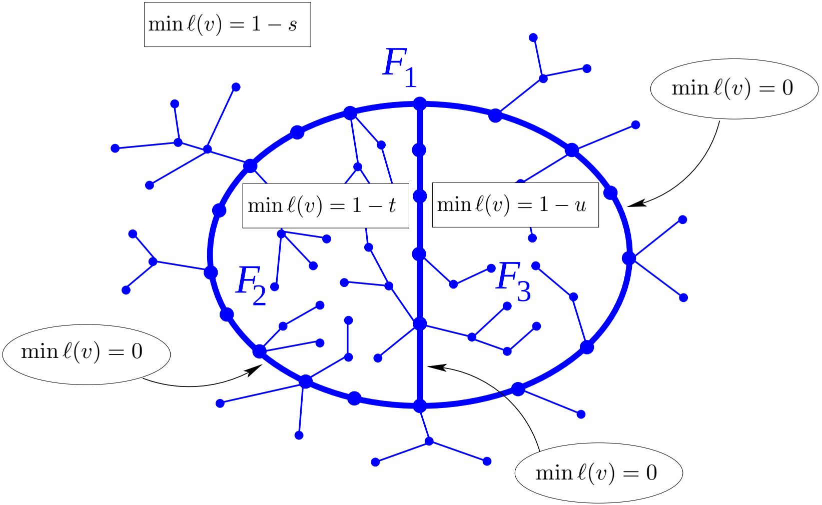


We introduce the useful parametrization:

$$d_{12} = s + t \qquad d_{23} = t + u \qquad d_{31} = u + s$$

The three-point function

Planar quadrangulations with three marked points at prescribed pairwise distances d_{12}, d_{23}, d_{31} are in one-to-one correspondence with well-labeled maps of this generic type [BG08]:



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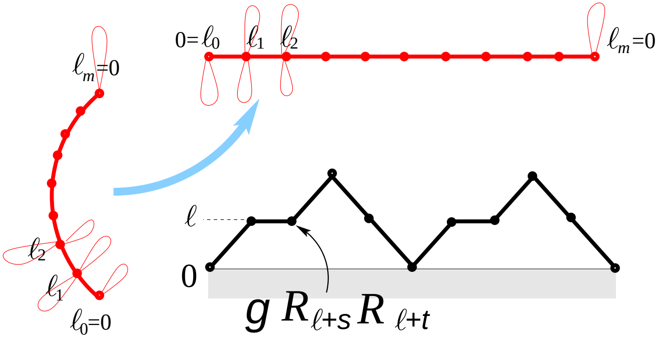

$$X_{s,t} = 1 + gR_sR_t\,X_{s,t}\,(1 + gR_{s+1}R_{t+1}\,X_{s+1,t+1})$$

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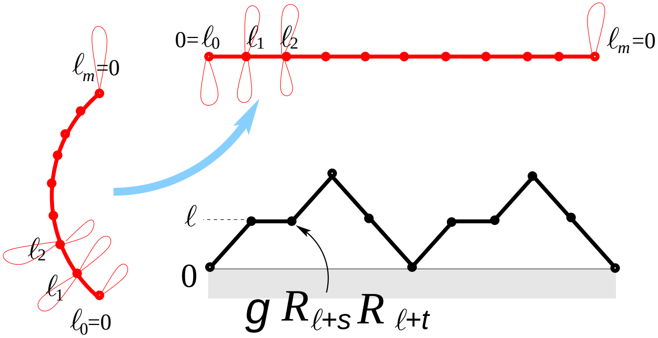

$$X_{s,t} = 1 + gR_sR_t\,X_{s,t}\,(1 + gR_{s+1}R_{t+1}\,X_{s+1,t+1})$$

$$X_{s,t} = \frac{[3][s+1][t+1][s+t+3]}{[1][s+3][t+3][s+t+1]}$$

The three-point function

In the end, the generating function for triply-pointed quadrangulations is

$$G(d_{12}, d_{23}, d_{31}) = \Delta_s \Delta_t \Delta_u F(s, t, u)$$

where

$$F(s,t,u) = \frac{[3]([s+1][t+1][u+1][s+t+u+3])^2}{[1]^3[s+t+1][s+t+3][t+u+1][t+u+3][u+s+1][u+s+3]}$$

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We can now deduce from our expression the probability distribution for distances in random planar quadrangulations of large size n. This is obtained through a contour integral and a saddle point expansion around the critical point $g_c = 1/12$:

$$g = \frac{1}{12}(1 - \Lambda\epsilon) \qquad d = D\epsilon^{-1/4} \qquad \epsilon \ll 1$$

The two-point function yields

$$G(d;g) \equiv \log R_d/R_{d-1} \sim \epsilon^{3/4}\mathcal{G}(D;\alpha)$$

with

$$\mathcal{G}(D;\alpha) \equiv 4\alpha^3 \frac{\cosh(\alpha D)}{\sinh^3(\alpha D)} \qquad \alpha \equiv \sqrt{\frac{3}{2}}\Lambda^{1/4}$$

in agreement with [Ambjørn-Watabiki 1996].

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Going back to the canonical ensemble (fixed size) we find the probability density for D:

$$\rho(D) = \frac{2}{\mathrm{i}\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi\, \xi\, e^{-\xi^2} \mathcal{G}(D; \sqrt{-3\mathrm{i}\xi/2})$$

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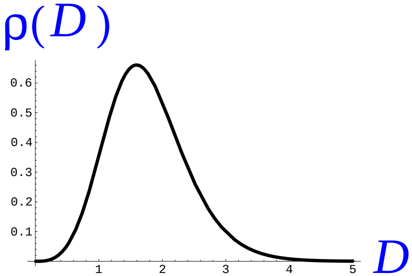

Applying the same method to the three-point function we compute $\rho(D_{12}, D_{23}, D_{31})$. Plots are easier with the conditional density:

$$\rho(D_{23}, D_{31}|D_{12}) \equiv \frac{\rho(D_{12}, D_{23}, D_{31})}{\rho(D_{12})}$$

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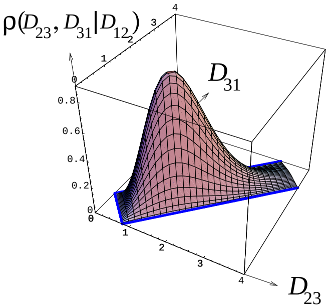
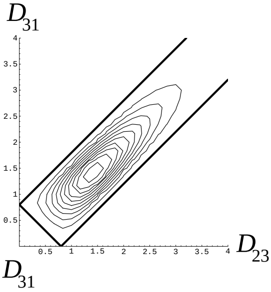
$D_{12} = 0.8$





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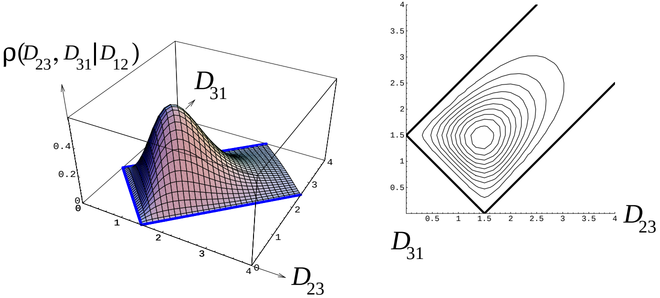
$$D_{12} = 1.5$$



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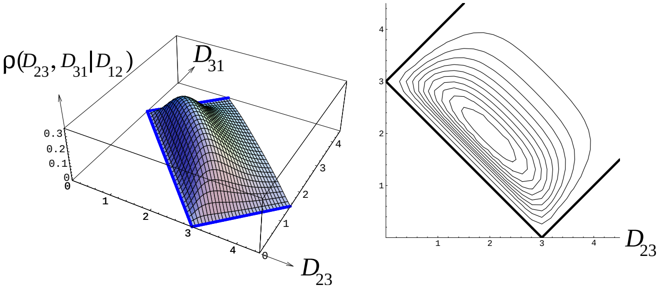
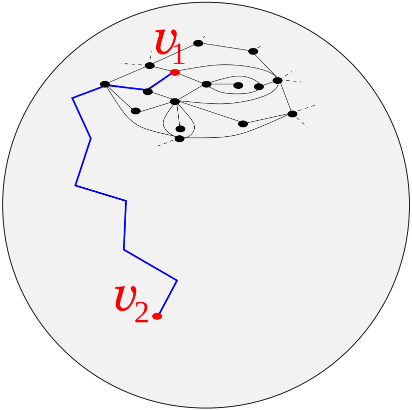
$D_{12} = 3.0$



Properties of geodesics

In another work [BG07-08] we have studied the properties of geodesic paths themselves.



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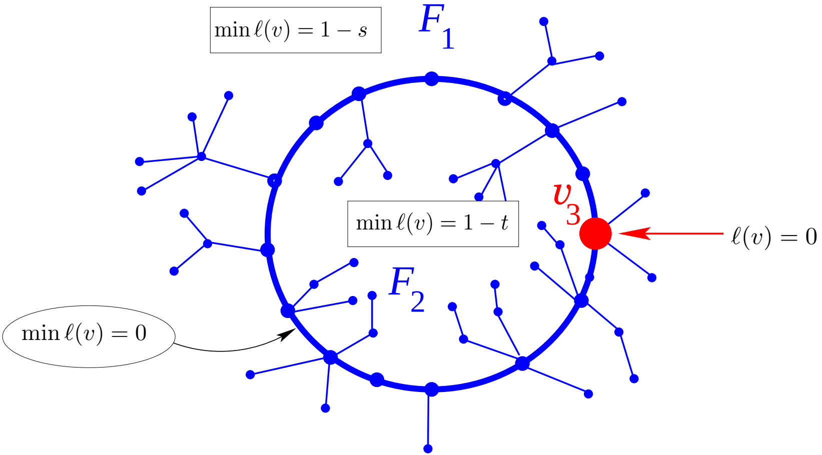
- However for two generic vertices, any two geodesics connecting them are indistinguishable on a macroscopic ($\propto n^{1/4}$) scale.

Properties of geodesics

This can be seen by studying a specific case of the previous construction: $u = 0$, $d_{12} = s + t$, $d_{23} = t$, $d_{31} = s$.



[Miermont 2007]

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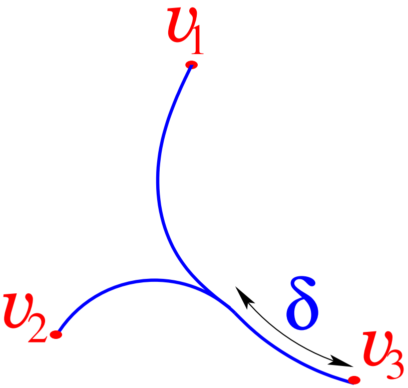
- There are some pairs of exceptional vertices connected by several macroscopically disjoint geodesics. We find that for k geodesics the number of such pairs is of order $n^{(11-3k)/4}$. See also [Le Gall 2008].

Confluence of geodesics

Le Gall has shown the surprising phenomenon of *confluence* of geodesics.



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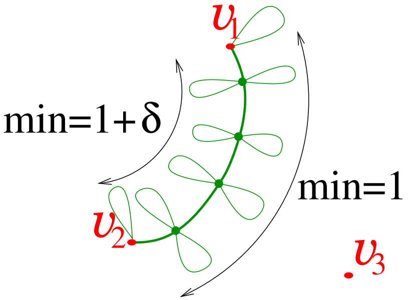

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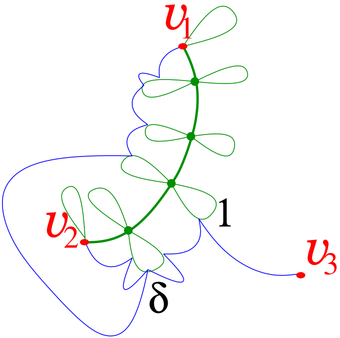

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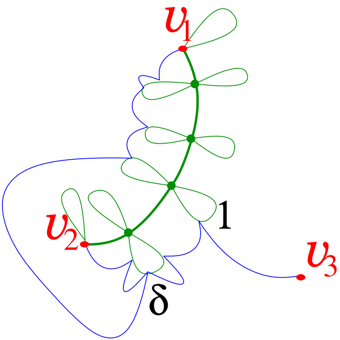


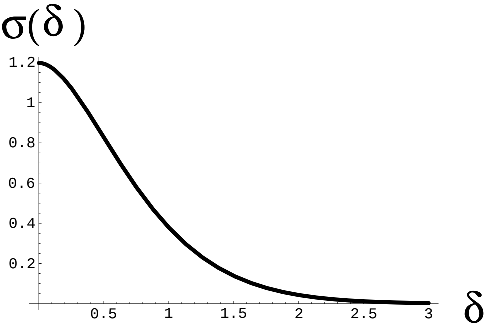
In the discrete setting these correspond to particular geodesics, nevertheless in the scaling limit this makes no difference. We have $\delta \propto n^{1/4}$.

Confluence of geodesics

We were able to compute the continuous law for δ ($\delta \to \delta \cdot n^{-1/4}$):

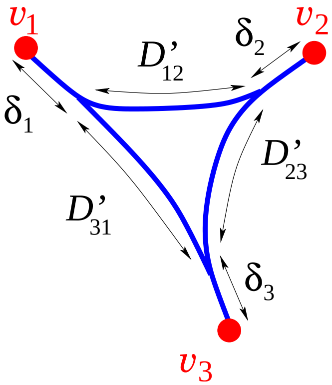
$$\sigma(\delta) = \frac{3}{\mathrm{i}\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi\, e^{-\xi^2} \sqrt{-3\mathrm{i}\xi/2}\, e^{-2\delta\sqrt{-3\mathrm{i}\xi/2}}$$

$$= \sqrt{\tfrac{3}{\pi}} \Big\{ \Gamma\!\left(\tfrac{3}{4}\right) {}_0F_2\!\left(\left\{\tfrac{1}{4},\tfrac{1}{2}\right\}, -\tfrac{9\delta^4}{64}\right) - 3\delta^2 \Gamma\!\left(\tfrac{5}{4}\right) {}_0F_2\!\left(\left\{\tfrac{3}{4},\tfrac{3}{2}\right\}, -\tfrac{9\delta^4}{64}\right) + \sqrt{3\pi}\,\delta^3 {}_0F_2\!\left(\left\{\tfrac{5}{4},\tfrac{7}{4}\right\}, -\tfrac{9\delta^4}{64}\right) \Big\}$$



Confluence of geodesics

The shape of a triangle will actually look like:



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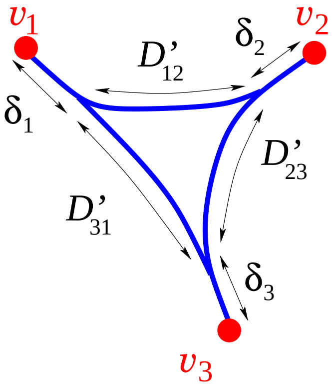

Our computation of the three-point function can be refined in order to obtain the joint law for all six parameters: $D'_{12}, D'_{23}, D'_{23}, \delta_1, \delta_2, \delta_3$. All these quantities have the same mean value.

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- Current work: a better understanding of the “integrability” property through a connection with continued fractions.