Scattering Amplitudes in Gauge Theories

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Simplicity of scattering amplitudes

Example: 2 gluons in, \( n - 2 \) gluons out.

\[
A(p_1^+, p_2^+; p_3^+, p_4^+, \ldots, p_n^+) = \frac{\langle \lambda_1 \lambda_2 \rangle^4}{\langle \lambda_1 \lambda_2 \rangle \langle \lambda_2 \lambda_3 \rangle \ldots \langle \lambda_{n-1} \lambda_n \rangle \langle \lambda_n \lambda_1 \rangle}
\]
Feynman rules in QCD

\[
\begin{align*}
\text{a}_i^{\mu} \rightarrow \text{b}_v^{\nu} & \quad -i \delta^{ab} \left[ \left( q_{\mu \nu} - \frac{k_{\mu} k_{\nu}}{k^2} \right) / k^2 + a k_{\mu} k_{\nu} / k^4 \right] \\
\text{i} \rightarrow \text{j} & \quad -i \delta^{ij} / k^2 \\
\text{b}_i^{\mu} \rightarrow \text{c}_v^{\nu} & \quad i \delta^{ij} k / k^2 \\
\text{a}_i^{\lambda} \rightarrow \text{c}_v^{\sigma} \quad \text{p} & \quad -g f^{abc} \left[ (p-q)_{\nu} g_{\lambda \mu} + (q-r)_{\lambda} g_{\mu \nu} + (r-p)_{\mu} g_{\nu \lambda} \right] \\
\text{b}_i^{\lambda} \rightarrow \text{d}_v^{\sigma} \quad \text{c} & \quad -i g^2 f^{abc} f^{cde} \left( g_{\lambda \nu} g_{\mu \sigma} - g_{\lambda \sigma} g_{\mu \nu} \right) \\
\text{b}_i^{\mu} \rightarrow \text{d}^{\nu} \quad \text{c} & \quad -i g^2 f^{abc} f^{cde} \left( g_{\lambda \mu} g_{\nu \sigma} - g_{\lambda \sigma} g_{\mu \nu} \right) \\
\text{i} \rightarrow \text{j} & \quad -i g f^{\alpha \beta \gamma} \left( g_{\lambda \nu} g_{\mu \sigma} - g_{\lambda \sigma} g_{\mu \nu} \right) \\
\text{p} & \quad g f^{abc} p^\mu \\
\text{i} \rightarrow \text{e}^{\alpha} & \quad -i g a^{\alpha} f_{ij} 
\end{align*}
\]
Too many Feynman diagrams

Already at tree level in pure gauge theory:

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Each diagram is a long mathematical expression.
We are discovering new perspectives.

Construct amplitude from global properties.
From momentum vectors to spinors

Change from Lorentz 4-vector to spinor indices with Pauli matrices:

$$ p_{a\dot{a}} = \sigma^{\mu}_{a\dot{a}} p_{\mu} \quad a, \dot{a} = 1, 2 $$

For a null vector (massless particle):

$$ 0 = p^2 = \det(p_{a\dot{a}}) \implies p_{a\dot{a}} = \lambda_a \bar{\lambda}_{\dot{a}}. $$

Lorentz-invariant spinor products:

$$ \langle \lambda \lambda' \rangle \equiv \epsilon_{ab} \lambda^a \lambda'^b $$

$$ [\bar{\lambda} \bar{\lambda}'] \equiv \epsilon_{\dot{a}\dot{b}} \bar{\lambda}^\dot{a} \bar{\lambda}'^\dot{b} $$
Spinors and Twistors

\((p_1, p_2, p_3, p_4) \rightarrow (\lambda^1, \lambda^2, \tilde{\lambda}^1, \tilde{\lambda}^2)\)

\(\rightarrow (\lambda^1, \lambda^2, \mu^1, \mu^2),\)

with

\[\mu^\dot{a} \equiv -i \frac{\partial}{\partial \tilde{\lambda}^\dot{a}}.\]

Breaks symmetry between \(\lambda\) and \(\tilde{\lambda}\).
Simplicity of scattering amplitudes

\[ A(p_1^+, p_2^+; p_3^+, p_4^+, \ldots, p_n^+) = \frac{\langle \lambda_1 \lambda_2 \rangle^4}{\langle \lambda_1 \lambda_2 \rangle \langle \lambda_2 \lambda_3 \rangle \ldots \langle \lambda_{n-1} \lambda_n \rangle \langle \lambda_n \lambda_1 \rangle} \]

- \( n \) is arbitrary
- Only half of the spinors are involved \((\lambda \text{ but not } \tilde{\lambda})\)
- A line in twistor-space geometry
Aims

1. Exploit the simplicity. LHC applications.

2. Seek deeper structure.
One-loop amplitudes

2006: 6 gluons. Complexity of $2 \rightarrow 4$ scattering in QCD at one-loop order

[Ellis, Giele, Zanderighi; Bedford, Berger, Bern, Bidder, Bjerrum-Bohr, Brandhuber, RB, Buchbinder, Cachazo, Dixon, Dunbar, Feng, Forde, Ita, Kosower, Mastrolia, Perkins, Spence, Travaglini, Xiao, Yang, Zhu]

Both numerical and analytic results.

Analytic techniques extend readily to larger numbers of particles.

Completed 2009: analytic results for $pp \rightarrow \text{Higgs} + 2$ jets.

[Badger, Berger, Campbell, Del Duca, Dixon, Ellis, Giele, Glover, Mastrolia, Risager, Sofianatos, Williams, Zanderighi]
Seeking deeper structure

Two recent discoveries:

- Residue theorem → Recursion Relations
- Branch cuts → Loop Constructions
Define the following function of a complex variable $z$:

$$A(z) = A(p_1, \ldots, p_{k-1}, p_k(z), p_{k+1}, \ldots, p_{n-1}, p_n(z))$$

where

$$p_k(z) = p_k - zq, \quad p_n(z) = p_n + zq,$$

and $q$ satisfies

$$p_k(z)^2 = 0, \quad p_n(z)^2 = 0.$$

The shift preserves momentum conservation while staying on shell.
On-shell recursion relation for amplitudes.

Residue theorem gives formula in terms of smaller, simpler amplitudes.

$$ A(0) = \sum_{k=1}^{n-3} A(p_n(z_k), p_1, \ldots, p_k, -P_{n,k}(z_k)) $$

$$ \times \frac{1}{P_{n,k}^2} A(P_{n,k}(z_k), p_{k+1}, \ldots, p_{n-2}, p_{n-1}(z_k)) $$

where $z_k$ is the solution to $P_{n,k}(z_k)^2 = 0$. New results obtained quickly and compactly.
Reduction to "master integrals":

\[ A^{\text{1-loop}} = c_1 + c_2 + c_3 + \cdots \]

For amplitudes involving many particles, this is not yet enough simplification.
Unitarity Cuts: Loops from Trees

\[ \Delta A^{1\text{-loop}} = \int d^4 \ell \ A^{\text{tree}}_{\text{Left}} \times A^{\text{tree}}_{\text{Right}} \]

Cut conditions: \[ \ell_1^2 = 0, \quad \ell_2^2 = 0. \]

By unitarity, this is the discontinuity of the amplitude across a branch cut.
Amplitudes from unitarity cuts

4-dimensional cuts suffice to determine certain one-loop amplitudes! [Bern, Dixon, Dunbar, Kosower 1994]
Match cuts of amplitudes with cuts of master integrals from Passarino-Veltman reduction: essence of loop momentum integral is done once and for all.

\[ \Delta A^{1-\text{loop}} = \sum c_i \Delta I_i \]

\[ \begin{align*}
\bullet & \quad \bullet \\
\quad & = c_1 \quad + c_2 \quad + c_3 \quad + \cdots
\end{align*} \]
Spinor integration

[Anastasiou, RB, Buchbinder, Cachazo, Feng, Kunszt, Mastrolia]

- Change loop momentum to spinor variables in unitarity cut integral.

\[ \ell \rightarrow \lambda, \bar{\lambda} \]

- Each term of integrand takes the form:

\[ \frac{(K^2)^{n+1}}{\langle \lambda | K | \bar{\lambda} \rangle^{n+2}} \prod_{i=1}^{n+k} \frac{\langle \lambda | R_i | \bar{\lambda} \rangle}{\langle \lambda | Q_j | \bar{\lambda} \rangle} \]

- Evaluate with residue theorem.
- Identify expressions with cuts of basis integrals and read off coefficients.
- We have given formulas for the resulting coefficients.
Close to four dimensions

Orthogonal decomposition, keeping external momenta in 4 dimensions. [Bern, Chalmers, Mahlon, Morgan]

\[ \int d^{4-2\epsilon} \ell_{4-2\epsilon} = \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)} \int_0^1 du \ u^{-1-\epsilon} \int d^4 \tilde{\ell}. \]

where \( \ell_{-2\epsilon}^2 = \frac{K^2}{4} u. \)

The integral over \( u \) will remain. The \( u \)-dependence is controlled:

\[ \Delta A = \int_0^1 du \ u^{-1-\epsilon} \int d^4 \ell \ \delta(\ell^2) \ \delta(\sqrt{1-u} \ K^2 - 2K \cdot \ell) \]

Recognize and perform the 4-d integral as before.

(Cf. methods by Ossola, Papadopoulos, Pittau; Forde; Ellis, Giele, Kunszt; Kilgore; Giele, Kunszt, Melnikov)
Incorporating Masses

Cut amplitude:

\[
\int_{0}^{1} du \, u^{-1-\epsilon} \int \langle \ell \, d\ell \rangle \langle \ell \, d\ell \rangle [\ell \, d\ell] \left( (1 - 2z) + \frac{M_1^2 - M_2^2}{K^2} \right) \frac{(K^2)^{n+1}}{\langle \ell | K | \ell \rangle^{n+2}} \prod_{j=1}^{n+k} \langle \ell | R_j | \ell \rangle \prod_{i=1}^{k} \langle \ell | Q_i | \ell \rangle
\]

- For scalar particles, the formalism/formulas for integral coefficients will look the same. [See also: Kilgore]
- Integral coefficients are polynomials in \( u \).
- New element: tadpole and massless bubble integrals. Coefficients of tadpoles might be found formally by cutting an artificial extra propagator.
- Self-energy and mass renormalization contributions may require gauge fixing. [Ellis, Giele, Kunszt, Melnikov].
Currently...

Implementing analytic algorithm in Mathematica. Seeking optimal order of operations.

$ZZggg$ with massless fermion loop, comparison with Feynman-diagram derivation. Starting with 4d cuts.

Investigating issues of masses.
Closing comments

- Conceptual breakthroughs in quantum field theory have come from tackling specific, difficult calculations.
- Searches for new high-energy physics at LHC demand such calculations.
- Current ideas lead to both formal and LHC-driven advances.