Liouville Quantum Gravity & KPZ

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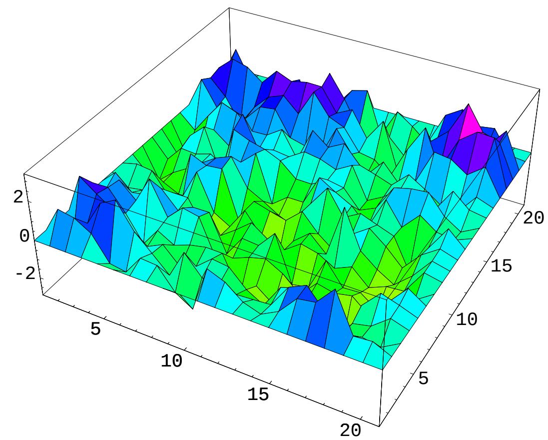
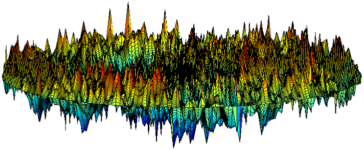
Gaussian Free Field (GFF)



Distribution h with *Gaussian weight* $\exp\left[-\frac{1}{2}(h,h)_\nabla\right]$, and **Dirichlet inner product** in domain D

$$(f_1, f_2)_\nabla := (2\pi)^{-1}\int_D \nabla f_1(z)\cdot\nabla f_2(z)\,d^2z$$
$$= \mathrm{Cov}\big((h, f_1)_\nabla, (h, f_2)_\nabla\big)$$

♢ STARRING THE GFF! *(Courtesy of N.-G. Kang)* ♢



Liouville QG

Random Measure

$$d\mu = \text{“}e^{\gamma h} d^2 z\text{”}$$

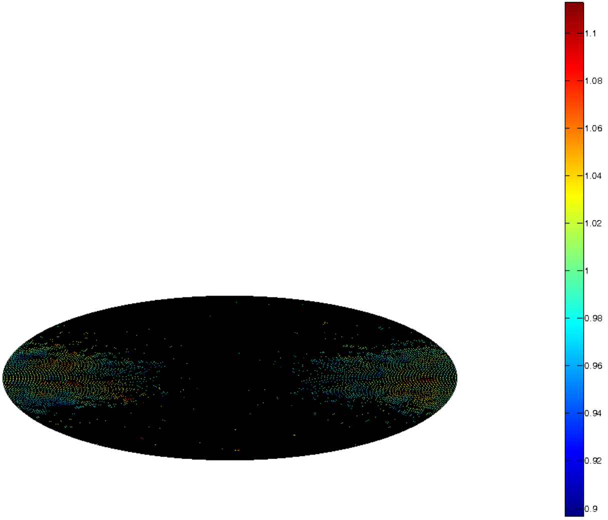
◊

The Emergence of Quantum Gravity

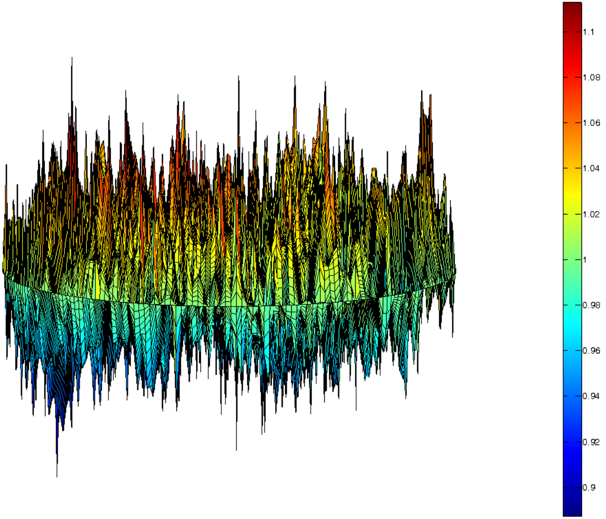
(Courtesy of N.-G. Kang)

◊

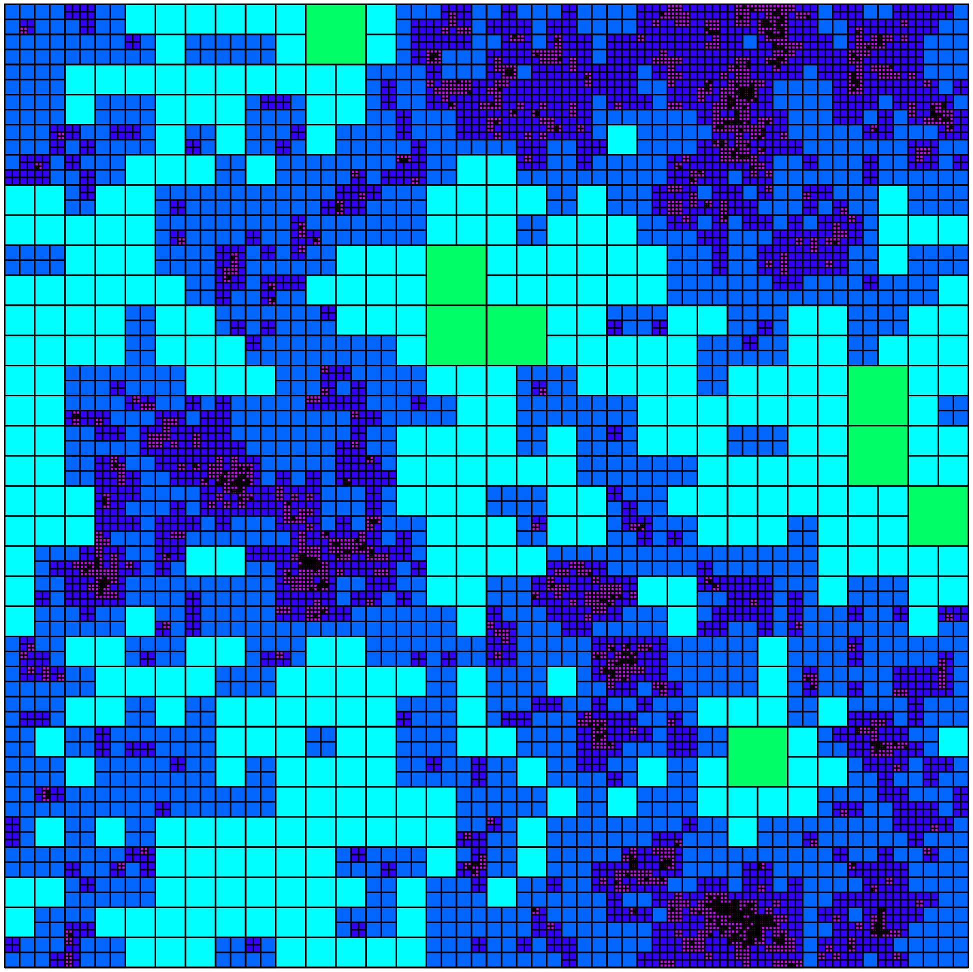
gamma = 0.02



gamma = 0.02

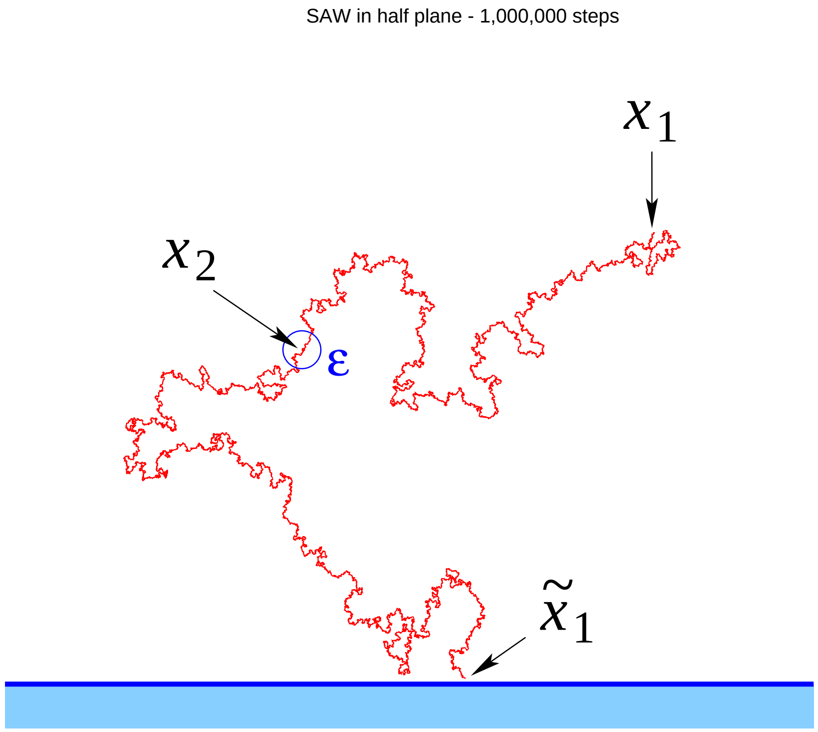


Discrete Quantum Gravity Measure ($\gamma = 3/2$)



Euclidean squares of similar quantum area δ

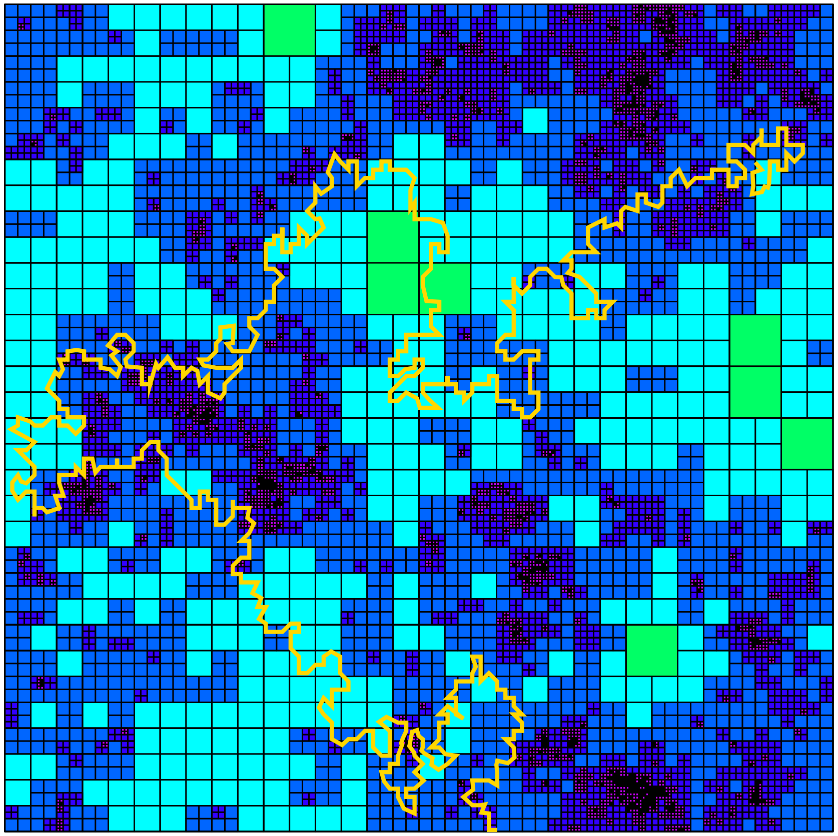
Scaling Exponents of (Random) Fractals in $\mathbb{H}$



Probabilities & Hausdorff Dimensions (e.g., SLE_κ)

$$\mathbb{P} \asymp \varepsilon^{2x}, \quad \tilde{\mathbb{P}} \asymp \varepsilon^{\tilde{x}}, \qquad D = 2 - 2x_2 \qquad (= 1 + \kappa/8)$$

Quantum Gravity Scaling Exponents Δ & $\tilde{\Delta}$



$$\mathbb{P} \asymp \delta^{\Delta}, \qquad \tilde{\mathbb{P}} \asymp \tilde{\delta}^{\tilde{\Delta}}$$

KPZ *(Knizhnik, Polyakov, Zamolodchikov '88)*

x and Δ ($\tilde{x}$ and $\tilde{\Delta}$) are related by the **KPZ formula**

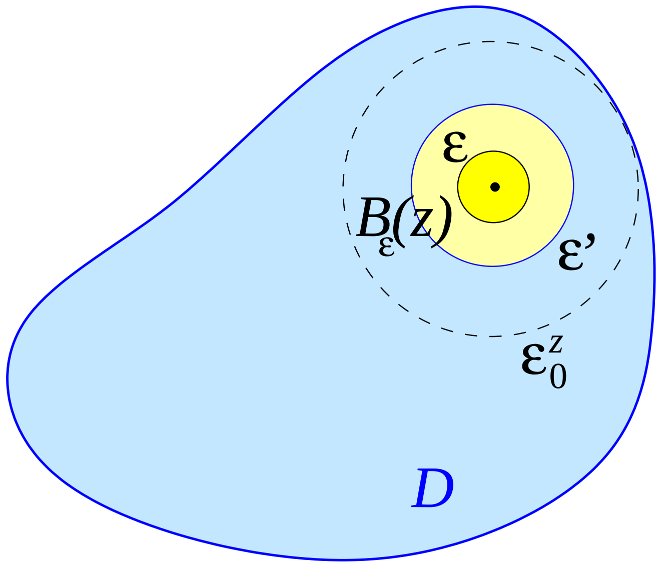
$$x = \left(1 - \frac{\gamma^2}{4}\right)\Delta + \frac{\gamma^2}{4}\Delta^2$$

KPZ is a Theorem *[D. & Sheffield, '08]*

arXiv:0808.1560 [math.PR] & PRL **102**, *150603 (2009)*

Kazakov '86; D. & Kostov '88 [Random matrices]
David; Distler & Kawai '88 [Liouville field theory]
Benjamini & Schramm '08; Rhodes & Vargas '08 [Math]
David & Bauer '08

Regularization: Circular Average of the GFF

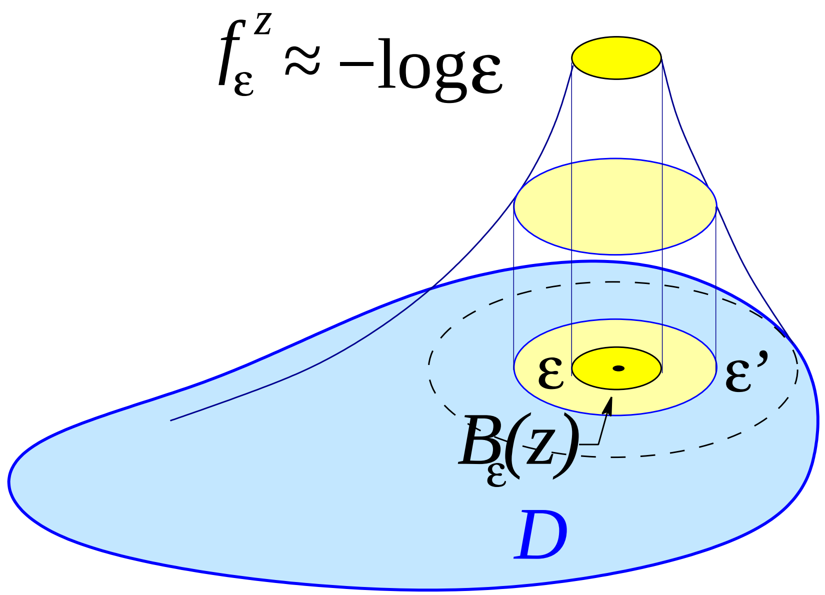


$$h_\varepsilon(z) := (h, \rho_\varepsilon^z) = (h, f_\varepsilon^z)_\nabla$$

$$(h, \rho) := \int_D h(y)\rho(y)d^2y$$

$\rho_\varepsilon^z(\cdot)$ *uniform distrib. of mass* 1 *on circle* $\partial B_\varepsilon(z)$

2D Logarithmic Potential



$$-\Delta f_\varepsilon^z = 2\pi\rho_\varepsilon^z$$

$$f_\varepsilon^z(\cdot) = -\log(|\cdot - z| \vee \varepsilon) + G_z(\cdot)$$

$G_z(\cdot)$ *harmonic extension of* $\log|\cdot - z|$ *in* D

STOCHASTIC AREA

$$d\mu_\varepsilon := \exp\left[\gamma h_\varepsilon(z)\right] \varepsilon^{\gamma^2/2} d^2 z$$

converges to a random measure as $\varepsilon \to 0$ *for*

$$\gamma < 2$$

(Høegh-Krohn '71)

Quantum Ball & Brownian Motion

Quantum area

- $\delta := \exp[\gamma h_\varepsilon(z)]\, \pi \varepsilon^{2+\gamma^2/2}$

Given z, $h_\varepsilon(z)$ *is standard Brownian motion* $\mathcal{B}_t$, $t = -\log \varepsilon$,
plus the deterministic term: $\gamma f_\varepsilon^z(z) = -\gamma \log \varepsilon = \gamma t$

$$\begin{aligned} \delta &= \exp(\gamma \mathcal{B}_t - a\,t), \qquad a := 2 - \gamma^2/2 \ (> 0) \\ -\log \delta &= a\,t - \gamma \mathcal{B}_t \qquad \square \qquad \text{(B. M. \& drift)} \end{aligned}$$

KPZ Theorem

Stochastic probability & stopping time

$$-\log \varepsilon_A = T_A = \inf\{t : a t - \gamma \mathcal{B}_t = -\log \delta =: A\}$$

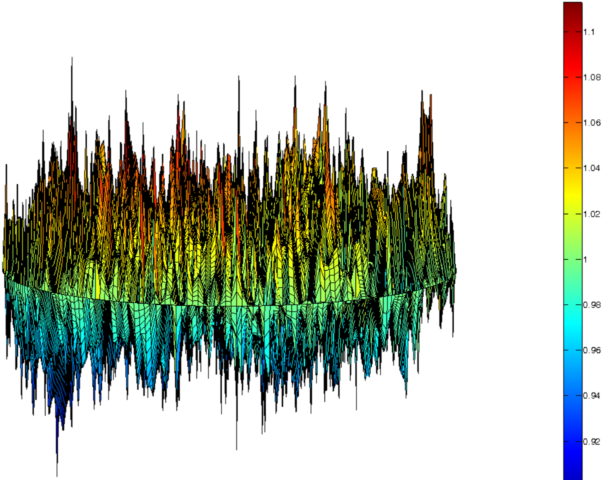
BROWNIAN MARTINGALE & LARGE DEVIATIONS

$$\begin{aligned} \mathbb{E}\left[{\varepsilon_A}^{2x}\right] &= \mathbb{E}\left[e^{-2xT_A}\right] = \exp(-\Delta A) = \delta^{\Delta} \\ 2x &= a\Delta + \frac{\gamma^2}{2}\Delta^2, \qquad a = 2 - \frac{\gamma^2}{2}\,(> 0) \qquad \text{(KPZ)} \quad \square \end{aligned}$$

Liouville Quantum Duality

$$\gamma > 2, \gamma' = 4/\gamma < 2$$

gamma = 0.02



LIOUVILLE QUANTUM DUALITY

Baby-Universes *Das, Dhar, Sengupta, Wadia '90; Jain & Mathur '92; Korchemsky '92; Alvarez-Gaumé, Barbón, Crnković '93; Durhuus '94; Ambjørn, Durhuus, Jonsson '94*

The Other Branch of Gravity *(Klebanov '95)*

Dual Dimensions

$$\gamma > 2, \gamma' = 4/\gamma < 2$$

$$\Delta_\gamma - 1 = \frac{4}{\gamma^2}(\Delta_{\gamma'} - 1)$$

D. & Sheffield, PRL **102**, *150603 (2009)*