Matrix models as conformal field theories

Ivan Kostov, IPhT

1. Large N matrix integrals have conformal symmetry

2. The collective theory is a bosonic CFT defined on a Riemann surface

3. The gaussian approximation reproduces the leading and the subleading order in 1/N (genus 0 and 1). The complete 1/N expansion is obtained by "dressing" the branch points of the Riemann surface by the modes of the bosonic current so that the conformal symmetry is restored.

4. Diagram technique for the 1/N expansion having only finite number of terms for given genus

1. Large N matrix integrals have conformal symmetry

$$\mathcal{Z} = \int d\mathbf{X}\, e^{-V[\mathbf{X}]} \quad = \langle 1 \rangle \qquad\qquad V[M] \sim N^2$$

$U(N)$ symmetry => $\quad V[\mathbf{X}] = V(W_1, W_2, \ldots), \quad W_n = \mathrm{tr}\, \mathbf{X}^n$

The Ward identity $\quad \int d\mathbf{X}\, \mathrm{tr} \left(\frac{\partial}{\partial \mathbf{X}} \mathbf{X}^n \right) e^{-V[\mathbf{X}]} = 0$

can be written as Virasoro constraint for the chiral bosonic collective field

$$\partial\phi(x) = \sum_{n\geq 0} x^{-n-1} W_n + \tfrac{1}{2} \sum_{n\geq 1} n\, x^{n-1} \left(-\frac{\partial V}{\partial W_n} + \frac{\partial}{\partial W_n} \right)$$

with stress-energy tensor $\quad T(z) = \tfrac{1}{2} : \partial\phi(z)\partial\phi(z) := \sum_n L_n z^{-n-2}$

$$\langle L_n \rangle = 0, \quad n \geq -1 \qquad \text{or} \qquad \langle T(x) \rangle_{\mathrm{sing}} = 0$$

Generalizes obviously
for multi-matrix models

2. The collective theory is a bosonic CFT defined on a Riemann surface

The 1/N expansion = quasiclassical expansion for the bosonic field

$$\mathcal{Z} = \exp\left(N^2 \mathcal{F}^{(0)} + N \mathcal{F}^{(1)} + \sum_{g \geq 2} N^{2-2g} \mathcal{F}^{(g)} \right)$$

classical action — gaussian fluctuations — quantum corrections

Up to g>1 terms the collective field is a free chiral boson defined on the Riemann surface $\sum$ of the classical solution

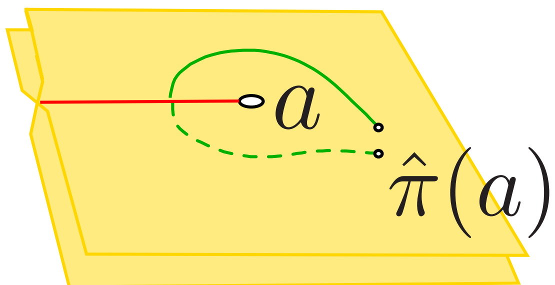
$$J_{\text{cl}}(x) = \partial \Phi_{\text{cl}}(x)$$

The two-point function of the collective field is given by the Laplace kernel on the Riemann surface $\sum$

$$D(x, x') \stackrel{\text{def}}{=} \langle\, \Phi(x) \Phi(x') \,\rangle = \tfrac{1}{2} \log(x - x') + \text{regular}$$

The gaussian field on the Riemann surface is conformal invariant only in the leading order because the conformal transformations displace the branch points. This can be cured by placing special operators at the branch points.

Let a be one of the branch points of the Riemann surface $\sum$. The bosonic field splits into a twisted and untwisted components, each satisfying its Virasoro condition.



the projector onto the twisted component: $\hat{P}(a) = \frac{1}{2}(1 - \hat{\pi}(a))$

$$J^{[a]}(x) \stackrel{\text{def}}{=} \hat{P}(a) J(x)$$

$$T^{[a]}(x) = \lim_{x' \to x} \left[J^{[a]}(x) J^{[a]}(x') - \frac{1}{2} \frac{1}{(x - x')^2} \right] = \sum_n L_n(a)\, (x - a)^{-n-2}.$$

Mode expansion near the branch point:

$$J^{[a]}(x) = J^{[a]}_{\text{cl}}(x) + \sum_{r \in \mathbb{Z} + \frac{1}{2}} J^{[a]}_r\, (x - a)^{-r-1},$$

$$J^{[a]}_{\text{cl}}(x) = \sum_{r \geq 3/2} \mu^{[a]}_r\, (x - a)^{r-1}. \qquad [J^{[a]}_r, J^{[a]}_s] = \frac{1}{2} r \delta_{r+s,0}.$$

The quantum field associated with the branch point at $x=a$ is the twist operator $\sigma(a)$: $J^{[a]}_{1/2}\sigma(a) = J^{[a]}_{3/2}\sigma(a) = J^{[a]}_{5/2}\sigma(a) = \cdots = 0.$

Operator formalism:

$$|0_{\rm tw}\rangle = \prod_k \sigma(a_k)|0\rangle$$

$$\mathcal{Z}_{\rm classical+gauss} = \langle\, \Sigma|0_{\rm tw}\,\rangle$$

The twist operator is not conformal invariant. To make it conformal invariant, we dress it with the modes of the twisted current.
The complete solution is obtained by inserting the dressing operator

$$\hat\Omega = \prod_k \hat\Omega(a_k)$$

The conformal invariance $\quad L_n^{[a_k]}\,\hat\Omega\,|0_{\rm tw}\rangle = 0 \qquad (n \geq -1)$,
completely determines the coefficients in the formal expansion

$$\mathcal{Z} = \langle\Sigma\big|\hat\Omega|0_{\rm tw}\rangle = \langle\Sigma|0_{\rm tw}\rangle\ \langle 0_{\rm tw}|\ e^{2J\hat G J}\ \hat\Omega\ |0_{\rm tw}\rangle$$

$J_{\rm cl}(x)$

$$\langle\, J(x)J(x')\,\rangle = \partial_x\partial_{x'}D(x,x')$$

Explicit expression for the *1/N* expansion of the free energy:

$$J^{[a_k]}_{-n-1/2} = -\tfrac{1}{2} t^{[a_k]}_n\,, \quad J^{[a_k]}_{n+1/2} = -(n+\tfrac{1}{2})\,\partial^{[a_k]}_n \qquad (n \geq 0) \qquad \partial^{[a_k]}_n \equiv \partial/\partial t^{[a_k]}_n.$$

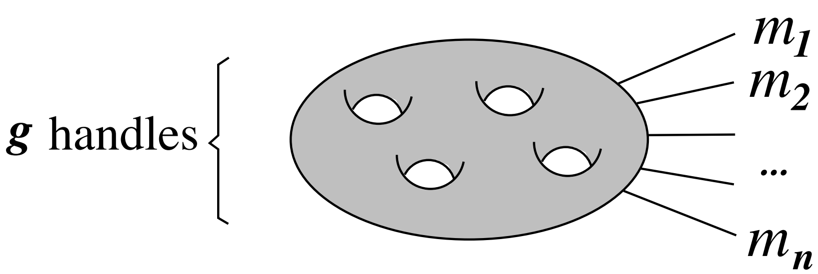
$$e^{\sum_{g\geq 2} N^{2-2g}\mathcal{F}^{(g)}} = \exp\left(\frac{1}{2}\sum_{i,j=1}^{2p}\sum_{m,n\geq 0} G^{[a_i a_k]}_{m,n}\partial^{[a_i]}_m \partial^{[a_k]}_n\right)\exp\left(\sum_{j=1}^{2p}\sum_{n\geq 0}\mu^{[a_k]}_n \partial^{[a_k]}_n\right)$$
$$\times \exp\left(\sum_{g\geq 0}\sum_{n\geq 0}\sum_{k_1,\ldots,k_n\geq 0}\left(\mu^{[a_k]}_1\right)^{2-2g-n} w^{(g)}_{k_1,\ldots,k_n}\frac{t^{[a_k]}_{k_1}\ldots t^{[a_k]}_{k_n}}{n!}\right)_{t^{(\cdot)}_{\cdot}=0}.$$

The correlation functions of the Kontsevich model

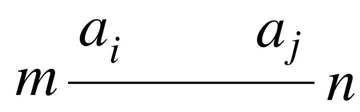
$$G^{[a_i a_k]}_{m,n} = 4\int\frac{dx}{2\pi i}\int\frac{dx'}{2\pi i}\frac{\langle J(x)J(x')\rangle}{(x-a_i)^{m+1/2}(x'-a_k)^{n+1/2}}$$

$$\mu^{[a_k]}_n = -2\oint_{a_k}\frac{dx}{2\pi i}\frac{J_{\rm cl}(x)}{(x-a_k)^{n+1/2}} \qquad (n\geq 1)$$

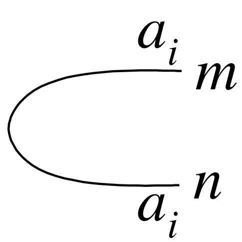
Diagram technique for the 1/N expansion:



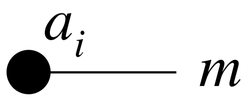
$W^{(g)}_{m_1 m_2 \ldots m_n}$



$G^{[a_i,a_j]}_{m,n}$



$G^{[a_i,a_i]}_{m,n}$



$\mu^{[a_i]}_{m}$

$$\sum_{j=1}^{n} m_j = 3(g-1)+n.$$

$$G^{[a_i a_k]}_{m,n} = 4\int \frac{dx}{2\pi i}\int \frac{dx'}{2\pi i}\,\frac{\langle J(x)J(x')\rangle}{(x-a_i)^{m+1/2}(x'-a_k)^{n+1/2}}$$

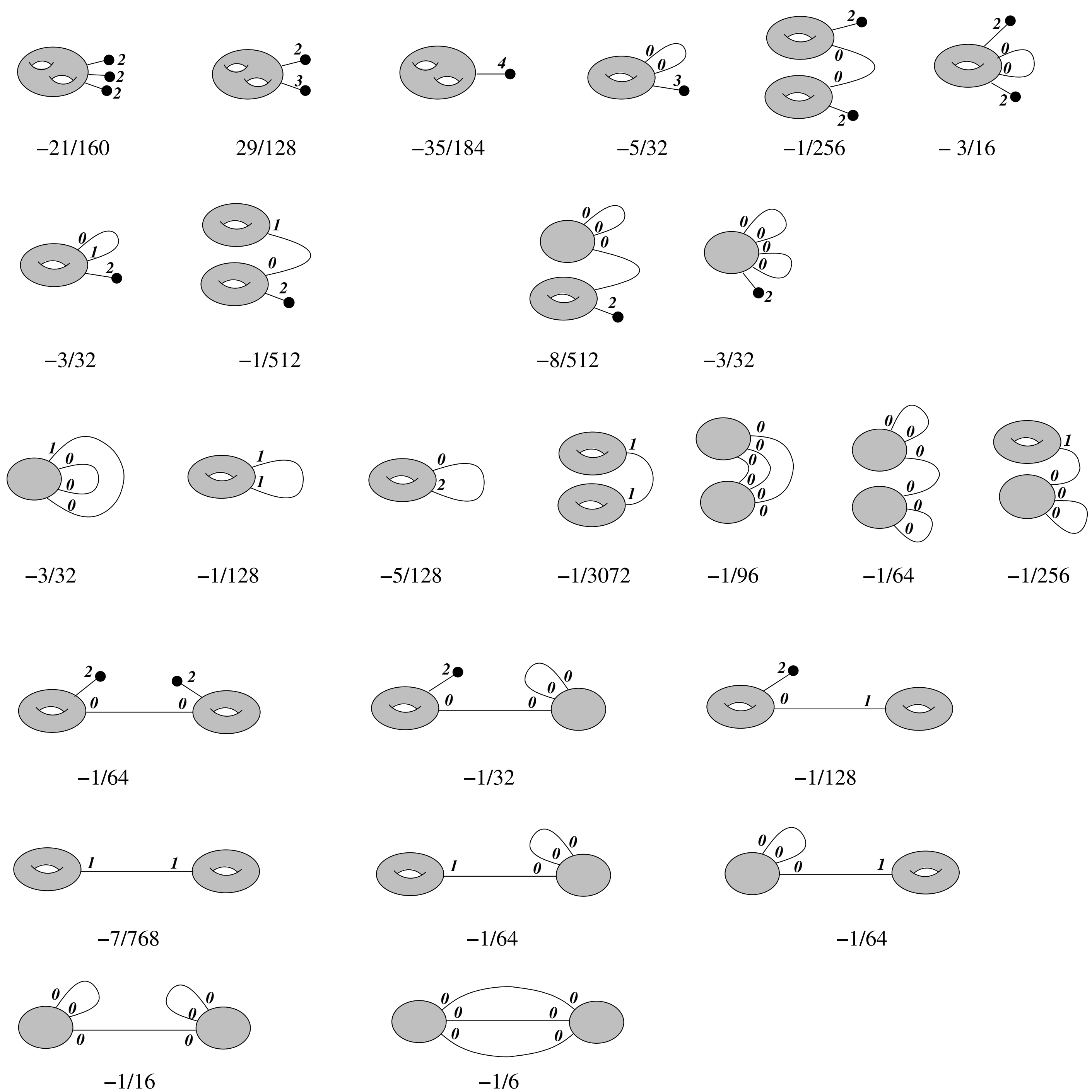
$$\mu^{[a_k]}_{n} = -2\oint_{a_k}\frac{dx}{2\pi i}\,\frac{J_{\rm cl}(x)}{(x-a_k)^{n+1/2}} \qquad (n\geq 1)$$

$$w^{(0)}_{m_1,\ldots,m_n} = (-1)^n \prod_{i=1}^{n}(2m_j-1)!!\,\frac{(m_1+\ldots+m_n)!}{m_1!\ldots m_n!},$$

$$w^{(0)}_{0,0,0}=-1,\ \ w^{(1)}_{1}=-\frac{1}{24},\ \ w^{(1)}_{1,1}=\frac{1}{24},\ \ w^{(1)}_{0,2}=-\frac{1}{8},\ \ w^{(1)}_{0,1,2}=-\frac{1}{4},\ \ w^{(1)}_{0,0,3}=-\frac{5}{8},$$

etc.

Example: The genus two free energy for the one-cut solution (two branch points) of the one-matrix model



$d = |a_1 - a_2|.$

$$\begin{aligned}\mathcal{F}^{(2)} \;=\;& \frac{1}{\mu_1^2}\left(-\frac{21}{160}\frac{\mu_2^3}{\mu_1^3}+\frac{29}{128}\frac{\mu_2\mu_3}{\mu_1^2}-\frac{35}{384}\frac{\mu_4}{\mu_1}+\frac{5}{32}\frac{\mu_3}{\mu_1 d}-\frac{49}{256}\frac{\mu_2^2}{\mu_1^2 d}-\frac{105}{512}\frac{\mu_2}{d^2\mu_1}-\frac{175}{1024}\frac{1}{d^3}\right)\\ &+\;\{\mu\leftrightarrow\mu'\}\;+\frac{1}{\mu_1\mu_1'}\left(-\frac{1}{64}\frac{\mu_2\mu_2'}{\mu_1\mu_1' d}-\frac{5}{128}\frac{\mu_2}{\mu_1 d^2}-\frac{5}{128}\frac{\mu_2'}{\mu_1' d^2}-\frac{69}{256}\frac{1}{d^3}\right)\end{aligned}$$

Conclusion

1. Any (multi) matrix integral is dercribed at large N by a CFT of a bosonic field on a Riemann surface.

2. The $1/N$ expansion is completely determined by the classical solution and the conformal invariance.

3. Analogy with the quasiclassical one-dimensional motion:
-- Planck constant <==> 1/N
-- Classical trajectory <==> Riemann surface
-- Turning points <==> branch points (more strictly, ramification points)
-- Airy function <==> Kontsevich integral

4. The diagram technique for the 1/N expansion obtained from CFT is a partial resummation of the diagram technique obtained by the Eynard-Orantin topological recursion procedure.
[I.K. - N. Orantin, 2010]