(Generalized) zeta functions *over zeros of* zeta functions

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Book: *Zeta Functions over Zeros of Zeta Functions*, Springer (2010)

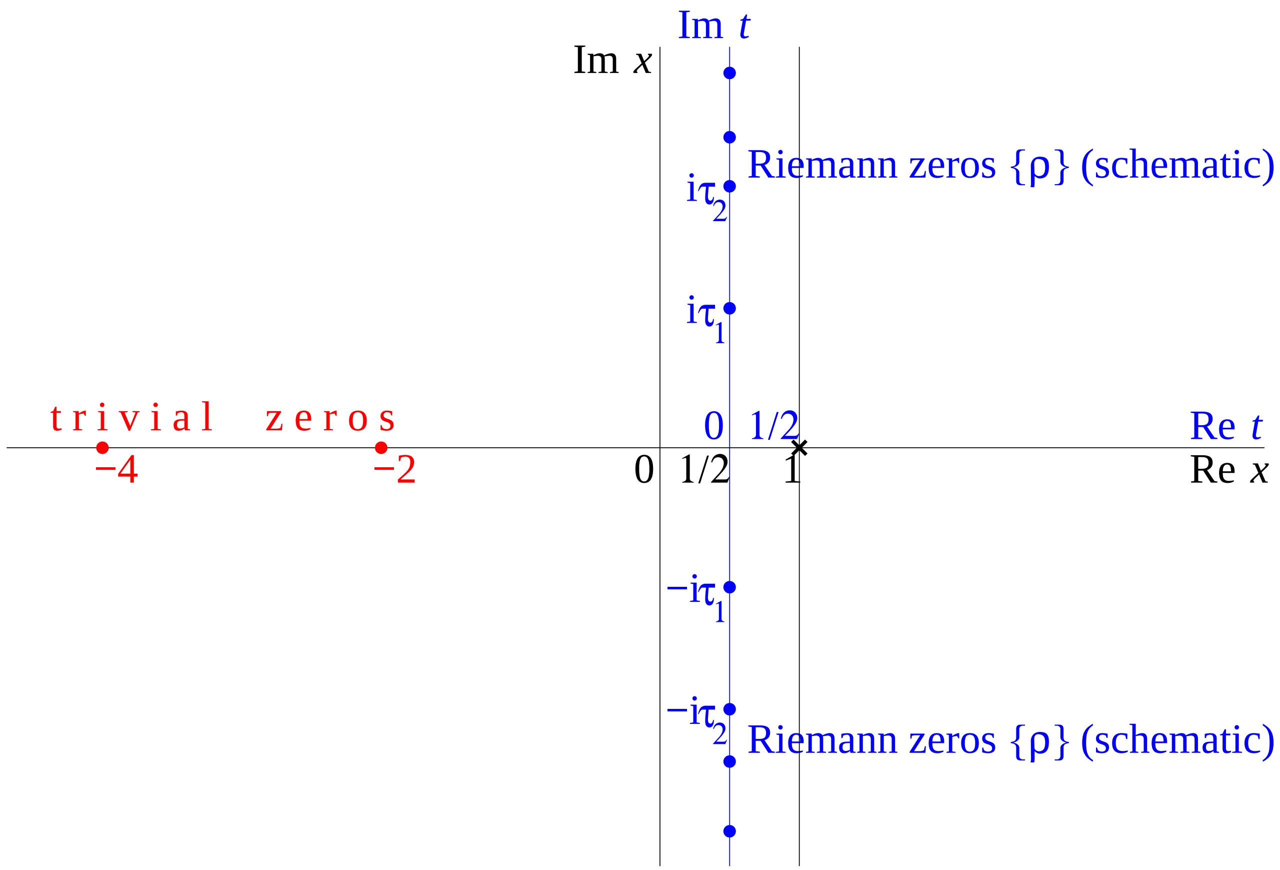

Figure 1: Plane of **Riemann zeta function** $\zeta(x) = \sum_{k=1}^{\infty} k^{-x}$. Handy variable is $t \equiv x - \frac{1}{2}$; remarkable values: $t = 0$ (center) and $\frac{1}{2}$ (the pole).

(Generalized) zeta functions

For many "natural" sequences $\{x_k\}$, zeta functions of the types

$$Z(s) = \sum_k {x_k}^{-s} \quad \text{(ordinary)}, \qquad Z(s,a) = \sum_k (x_k + a)^{-s} \quad \text{(generalized)},$$

enjoy *numerous explicit properties.*

For the Riemann zeros $\rho = \frac{1}{2} \pm \mathrm{i}\tau_k$, they
- have been *very scantily* studied up to year 2K ($\lesssim$ a dozen papers);
- resist naive *Explicit-Formula* approaches;
- yield numerous explicit properties indeed.

$$\mathscr{Z}(s \,|\, t) \stackrel{\rm def}{=} \sum_\rho (\tfrac{1}{2} + t - \rho)^{-s}, \qquad \mathrm{Re}\ s > 1 \qquad (1^{\rm st}\ \text{kind}),$$

$$\mathcal{Z}(\sigma \,|\, t) \stackrel{\rm def}{=} \sum_{k=1}^{\infty} ({\tau_k}^2 + t^2)^{-\sigma}, \qquad \mathrm{Re}\ \sigma > \tfrac{1}{2} \qquad (2^{\rm nd}\ \text{kind}),$$

$$\mathfrak{Z}(s \,|\, \tau) \stackrel{\rm def}{=} \sum_{k=1}^{\infty} (\tau_k + \tau)^{-s}, \qquad \mathrm{Re}\ s > 1 \qquad (3^{\rm rd}\ \text{kind}).$$

The Hurwitz zeta function $\zeta(x, w)$

is the basic template for later properties.

It is meromorphic in x, its only pole is $x = 1$, simple of residue 1; and it has *special-value formulae* $\left(\frac{\text{rational}}{\text{transcendental}}\right)$ at all $x = \pm n$ integer:

x	$\zeta(x, w) = \sum_{k=0}^{\infty} (k + w)^{-x}$
$-n \leq 0$	$-\dfrac{B_{n+1}(w)}{n+1}$
0	$\frac{1}{2} - w$
0 *(x-derivative)*	$\boxed{\zeta'(0, w) = \log\left[\Gamma(w)/\sqrt{2\pi}\right]}$
$+1$ *(finite part)*	$\boxed{\mathrm{FP}_{x=1}\zeta(x, w) = -\psi(w)}$
$+n > 1$	$\dfrac{(-1)^n}{(n-1)!}\,\psi^{(n-1)}(w)$

$B_n(w)$: Bernoulli polynomial; $\psi(w) \stackrel{\text{def}}{=} [\Gamma'/\Gamma](w)$ (digamma function).

Basic analytical continuation formula

for the zeta function *of 1st kind* over the Riemann zeros,

$$\mathscr{Z}(s\,|\,t) \stackrel{\mathrm{def}}{=} \sum_{\rho} (\tfrac12 + t - \rho)^{-s}, \qquad \mathrm{Re}\ s > 1.$$

Continuation uses the partner zeta function over the *trivial zeros* of $\zeta(x)$,

$$\mathbf{Z}(s\,|\,t) \stackrel{\mathrm{def}}{=} \sum_{k=1}^{\infty} (\tfrac12 + t + 2k)^{-s} \equiv 2^{-s}\zeta(s, \tfrac54 + \tfrac12 t),$$

and a *Mellin transform* of ζ'/ζ :

$$\mathscr{Z}(s\,|\,t) = -\,\mathbf{Z}(s\,|\,t) + (t - \tfrac12)^{-s} + \frac{\sin \pi s}{\pi} \int_0^{\infty} \frac{\zeta'}{\zeta}(\tfrac12 + t + y)\, y^{-s}\, \mathrm{d}y, \qquad \mathrm{Re}\ s < 1,$$

valid in a cut t-plane: $\frac12 + t$ must avoid all the negatively oriented half-lines drawn from the Riemann zeros and pole (whose t-cut, $(-\infty, +\frac12]$, forces a special treatment for the nicest t-values: 0 and $\frac12$ as in Fig. 1).

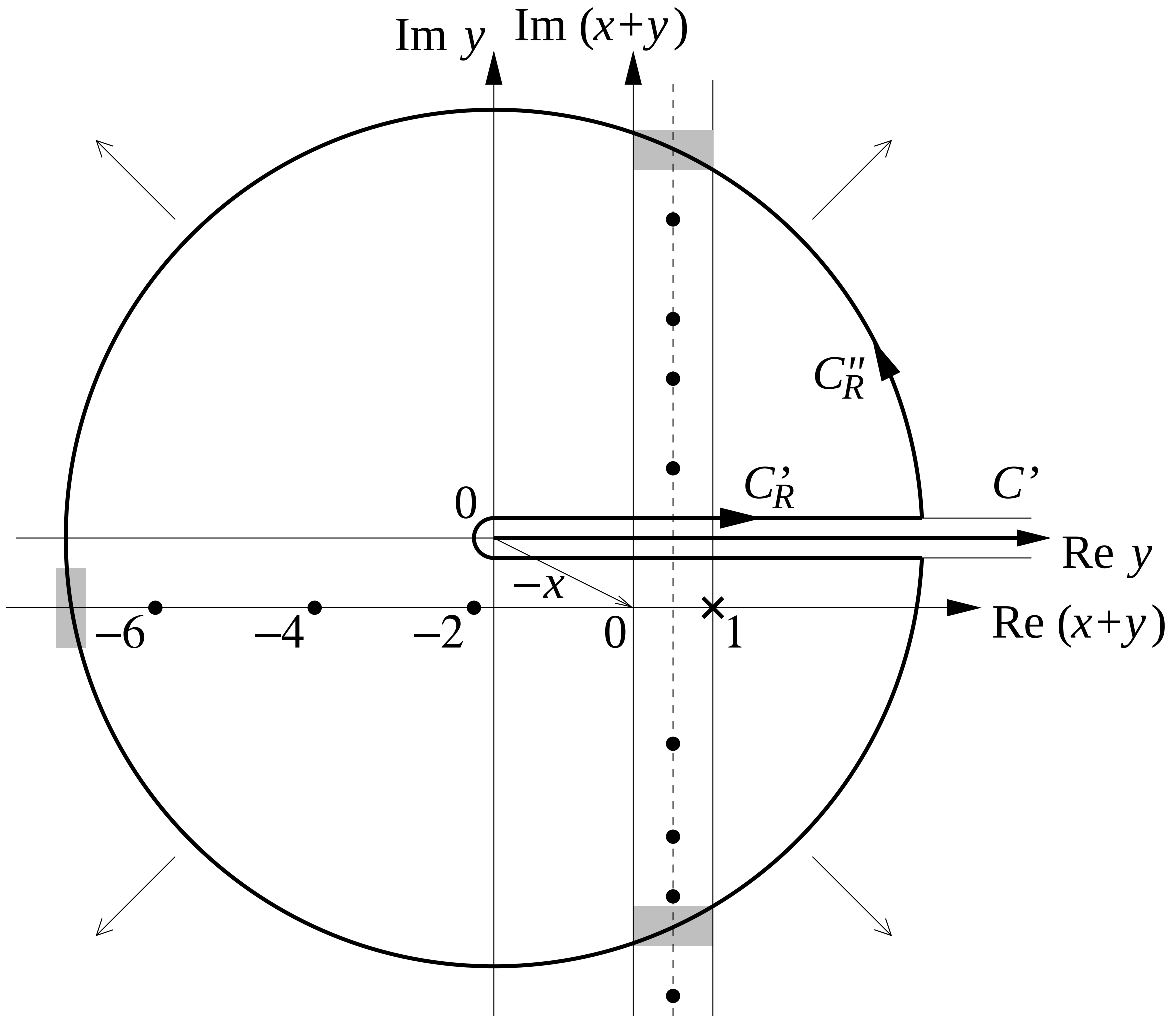

Figure 2: Initial integration contour for the Mellin transform of ζ'/ζ, leading to the above analytical continuation formula for $\mathscr{Z}(s\,|\,t)$.

Explicit special values for the functions of 1st kind

($\mathscr{Z}(s\,|\,t)$ is meromorphic in s; its only pole is $s=1$, simple of residue $-\frac{1}{2}$.)

General-t special values $\left(\frac{\text{rational}}{\text{transcendental}}\right)$; (n : integer)

s	$\mathscr{Z}(s\,|\,t) = \sum_\rho (\frac{1}{2}+t-\rho)^{-s}$
$-n \le 0$	$\dfrac{2^n}{n+1}\, B_{n+1}(\frac{1}{4}+\frac{1}{2}t) + (t+\frac{1}{2})^n + (t-\frac{1}{2})^n$
0	$\frac{1}{2}(t+\frac{7}{2})$
0 (*s-derivative*)	$\boxed{\mathscr{Z}'(0\,|\,t) = -\frac{1}{2}(\log 2\pi)\,t + \frac{1}{4}(\log 8\pi) - \log \Xi(\frac{1}{2}+t)}$
$+1$ (*finite part*)	$\boxed{\mathrm{FP}_{s=1}\, \mathscr{Z}(s\,|\,t) = \frac{1}{2}\log 2\pi + (\log \Xi)'(\frac{1}{2}+t)}$
$+n \ge 1$	$\dfrac{(-1)^{n-1}}{(n-1)!}\,(\log \Xi)^{(n)}(\frac{1}{2}+t)$

$\Xi(x) \stackrel{\text{def}}{=} x(x-1)\pi^{-x/2}\Gamma(x/2)\zeta(x)$ (completed Riemann zeta function).

$t = 0$ (the confluent case)

s		$\mathscr{Z}_0(s) \equiv \sum_\rho (\rho - \frac{1}{2})^{-s}$
$-n \le 0$	even	$2^{-n+1}(1 - \frac{1}{8}E_n)$
	odd	$-\frac{1}{2}(1-2^{-n})\frac{B_{n+1}}{n+1}$
0		$7/4$
0 *(derivative)*		$\boxed{\mathscr{Z}_0'(0) = \log\left[2^{11/4}\pi^{1/2}\Gamma(\frac{1}{4})^{-1}\lvert\zeta(\frac{1}{2})\rvert^{-1}\right]}$
+1 *(finite part)*		$\boxed{\mathrm{FP}_{s=1}\,\mathscr{Z}_0(s) = \frac{1}{2}\log 2\pi}$
$+n \ge 1$	odd	0
	even	$2^{n+1} - \frac{1}{2}\left[(2^n-1)\,\zeta(n) + 2^n\beta(n)\right] - \dfrac{(\log\lvert\zeta\rvert)^{(n)}(\frac{1}{2})}{(n-1)!}$

E_n : Euler, B_n : Bernoulli number; $\beta(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-x}$ (Dirichlet β-function).

$t = \frac{1}{2}$

s	$\mathscr{Z}_*(s) \equiv \sum_\rho \rho^{-s}$
$-n < 0$	$1 - (2^n - 1)\dfrac{B_{n+1}}{n+1}$
0	2
0 *(derivative)*	$\boxed{\mathscr{Z}_*'(0) = \frac{1}{2}\log 2}$
$+1$ *(finite part)*	$\boxed{\mathrm{FP}_{s=1}\,\mathscr{Z}_*(s) = 1 - \frac{1}{2}\log 2 + \frac{1}{2}\gamma}$
$+1$	$1 - \frac{1}{2}\log 4\pi + \frac{1}{2}\gamma$
$+n > 1$	$1 - (1-2^{-n})\,\zeta(n) + \dfrac{g_n^{\mathrm{c}}}{(n-1)!} \;\equiv\; 1 - (-1)^n 2^{-n}\zeta(n) - \dfrac{(\log\lvert\zeta\rvert)^{(n)}(0)}{(n-1)!}$

γ : Euler constant; $\{g_n^{\mathrm{c}}\}$: cumulant sequence of the *Stieltjes constants*.

Explicit special values for the functions of 2nd kind

$\mathcal{Z}(\sigma\,|\,t)$, functionally independent from $\mathscr{Z}(s\,|\,t)$ for $t \neq 0$, is also meromorphic in σ, now with *double* poles $\sigma \in \{\frac{1}{2} - \mathbb{N}\}$ and all principal parts explicitly computable. E.g., $\mathcal{Z}(\frac{1}{2}+\varepsilon\,|\,t) = \frac{1}{8\pi}\,\varepsilon^{-2} - \frac{\log 2\pi}{4\pi}\,\varepsilon^{-1} + \mathrm{O}(1)_{\varepsilon\to 0}$.

General-t special values $\left(\frac{\text{rational}}{\text{transcendental}}\right)$; $(m$: integer$)$

σ	$\mathcal{Z}(\sigma\,\|\,t) = \sum_{k=1}^{\infty} ({\tau_k}^2 + t^2)^{-\sigma}$	
$-m \leq 0$	$(t^2 - \frac{1}{4})^m - 2^{-2m-3} \sum_{j=0}^{m} \binom{m}{j} (-1)^j E_{2j}\,(2t)^{2(m-j)}$	
0	7/8	
0 (*σ-/s-derivatives*)	$\boxed{\mathcal{Z}'_{(\sigma)}(0\,\|\,t) = \mathscr{Z}'_{(s)}(0\,\|\,t) + (\frac{1}{2}\log 2\pi)\,t}$	
$+m \geq 1$	$\sum_{n=1}^{m} \binom{2m-n-1}{m-1} (2t)^{-2m+n}\,\mathscr{Z}(n\,\|\,t)$	$(t \neq 0)$

$t = 0$ (the confluent case)

σ	$\mathcal{Z}_0(\sigma) = \sum_{k=1}^{\infty} {\tau_k}^{-2\sigma} \equiv (2\cos\pi\sigma)^{-1}\, \mathscr{Z}_0(2\sigma)$
$-m \le 0$	$\frac{1}{2}(-1)^m\, \mathscr{Z}_0(-2m)$
0	$7/8$
0 (*σ-/s-derivatives*)	$\boxed{\mathcal{Z}_0'(0) = \mathscr{Z}_0'(0)}$
$+m \ge 1$	$\frac{1}{2}(-1)^m\, \mathscr{Z}_0(2m)$

$\boldsymbol{t = \frac{1}{2}}$

σ	$\mathcal{Z}_*(\sigma) = \sum_{k=1}^{\infty} ({\tau_k}^2 + \frac{1}{4})^{-\sigma}$
$-m < 0$	$-2^{-2m-3} \sum_{j=0}^{m} \binom{m}{j} (-1)^j E_{2j}$
0	$7/8$
0 *(derivative)*	$\boxed{\mathcal{Z}_*'(0) = \frac{1}{4} \log 8\pi}$
$+m \geq 1$	$\sum_{n=1}^{m} \binom{2m-n-1}{m-1} \mathscr{Z}_*(n)$

The functions of 3rd kind

$$\mathfrak{Z}(s\,|\,\tau) = \sum_{k=1}^{\infty} (\tau_k + \tau)^{-s}, \qquad \mathrm{Re}\ s > 1,$$

is still meromorphic but less regular than the others for general τ ($\neq 0$):

$$\mathfrak{Z}(s\,|\,\tau) \equiv \frac{1}{2\mathrm{i}\sin \pi s}\left[\mathrm{e}^{\mathrm{i}\pi s/2}\,\mathscr{Z}(s\,|\,-\mathrm{i}\tau) - \mathrm{e}^{-\mathrm{i}\pi s/2}\,\mathscr{Z}(s\,|\,+\mathrm{i}\tau)\right].$$

Its singular structure is computable just like before; on the other hand, only one special value remains explicit: the finite part of $\mathfrak{Z}(s\,|\,\tau)$ at $s = 0$, as

$$\left(\frac{7}{8} + \frac{\log 2\pi}{2\pi}\,\tau\right).$$

Extensions

currently include, but should not limited to,

(zeta functions over the) zeros of:

Dedekind zeta functions

Dirichlet L-functions (for real primitive Dirichlet characters)

Selberg zeta functions (for cocompact subgroups of $SL(2,\mathbb{R})$)

Asymptotic criterion for Riemann Hypothesis

The sequence of *Keiper–Li coefficients* is

$$\begin{aligned}\lambda_n &\overset{\text{def}}{=} \sum_\rho \left[1-(1-1/\rho)^n\right] \qquad\qquad (n=1,2,\ldots)\\ &\equiv -n\sum_{j=1}^{n}\frac{(-1)^j}{j}\binom{n+j-1}{2j-1}\,\mathcal{Z}_*(j)\\ &= \frac{(-1)^n n\,\mathrm{i}}{\pi}\oint_C \frac{\Gamma(\sigma+n)\Gamma(\sigma-n)}{\Gamma(2\sigma+1)}\,\mathcal{Z}_*(\sigma)\,\mathrm{d}\sigma,\end{aligned}$$

which for large n evaluates by the *saddle-point method* to give

- **if RH true,** a *tempered* growth to $+\infty$:

$$\lambda_n \sim \tfrac{1}{2}n\,(\log n - 1 + \gamma - \log 2\pi) \qquad (\mathrm{mod}\ \mathrm{o}(n));$$

- **if RH false,** an *exponentially growing* oscillatory behavior :

$$\lambda_n \sim \sum_{\{\arg\tau_k>0\}}\left(\frac{\tau_k+\mathrm{i}/2}{\tau_k-\mathrm{i}/2}\right)^n + \mathrm{c.c.} \qquad (\mathrm{mod}\ \mathrm{o}(\mathrm{e}^{\varepsilon n})\ \forall\varepsilon>0).$$