



Bethe Center for
Theoretical Physics

UNIVERSITÄT **BONN**

2D fishnet integrals and Calabi-Yau geometries

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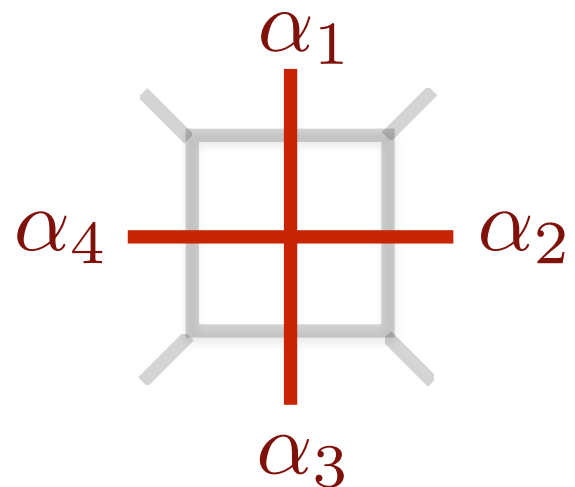
based on work in collaboration with
Albrecht Klemm, Florian Löbbert, Christoph Nega, Franziska Porkert
[2209.05291], + work in preparation.

Online seminar on motives and period integrals in
Quantum field theory and String theory
12 October 2022

- Feynman integrals are the building blocks for multi-loop scattering amplitudes.
 - ➔ Important both for collider and gravitational wave phenomenology.
 - ➔ A window into the mathematical structure of pQFT.
- They exhibit a rich mathematical structure, with connections to (algebraic) geometry.
- We would like to understand the underlying mathematics as well as we can!

- **Example:** All 1-loop integrals compute the volumes of some polytope in hyperbolic space.

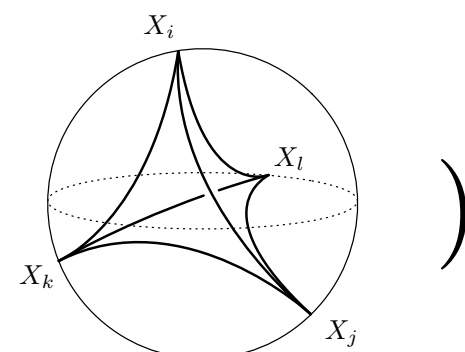
[Davydychev, Delbourgo; Schnetz; ...]



$$= \int \frac{d^4 \xi}{i\pi^2} \prod_{i=1}^4 \frac{1}{(\xi - \alpha_i)^2} = -\frac{4}{\alpha_{13}^2 \alpha_{24}^2} \frac{D(z)}{z - \bar{z}}$$

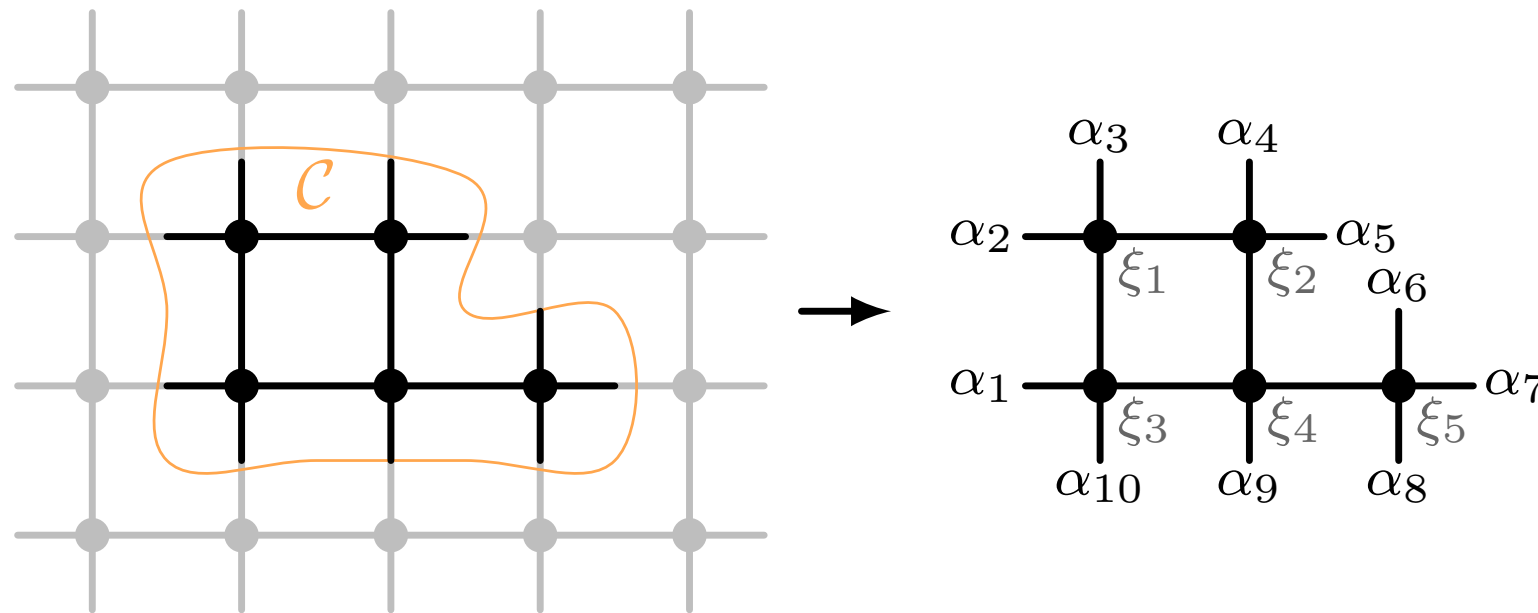
$$z\bar{z} = \frac{\alpha_{12}^2 \alpha_{34}^2}{\alpha_{13}^2 \alpha_{24}^2} \quad (1 - z)(1 - \bar{z}) = \frac{\alpha_{14}^2 \alpha_{23}^2}{\alpha_{13}^2 \alpha_{24}^2}$$

Bloch-Wigner Dilogarithm:

$$D(z) = \text{Im} [\text{Li}_2(z) + \log |z| \log(1 - z)] \sim \text{Vol} \left(\text{tetrahedron} \right)$$


[Picture taken from 1004.3498]

- Here: We focus on **fishnet integrals** (in D dimensions).



[Gürdoğan, Kazakov; Chicherin, Kazakov, Löbbert, Müller, Zhong]

- Feynman rules:

$$\text{---} \bullet \begin{array}{c} | \\ | \\ | \\ | \\ | \\ | \\ | \\ | \\ | \\ | \\ | \\ | \end{array} \text{---} \underset{\xi}{=} \int d^D \xi$$

$$\text{---} \bullet \underset{\xi_1}{\text{---}} \text{---} \bullet \underset{\xi_2}{\text{---}} = \frac{1}{[(\xi_1 - \xi_2)^2]^{D/4}}$$

$$\alpha \text{---} \bullet \underset{\xi_1}{\text{---}} = \frac{1}{[(\xi - \alpha)^2]^{D/4}}$$

- They are invariant under the Yangian $Y(\mathfrak{so}(1, D + 1))$ of the conformal group in D (Euclidean) dimensions (with conformal weight 1 at each external point).

[Chicherin, Kazakov, Löbbert, Müller, Zhong;
Löbbert, Miczajka, Müller, Munkler]

➔ Conformal (level 0) generators: $J^A = \sum_{j=1}^n J_j^A$

$$\begin{aligned} P_j^\mu &= -i\partial_j^\mu, & L_j^{\mu\nu} &= ix_j^\mu \partial_j^\nu - ix_j^\nu \partial_j^\mu, \\ D_j &= -ix_{j\mu} \partial_j^\mu - i\Delta_j, & K_j^\mu &= -i(2x_j^\mu x_j^\nu - \eta^{\mu\nu} x_j^2) \partial_{j,\nu} - 2i\Delta_j x_j^\mu, \end{aligned}$$

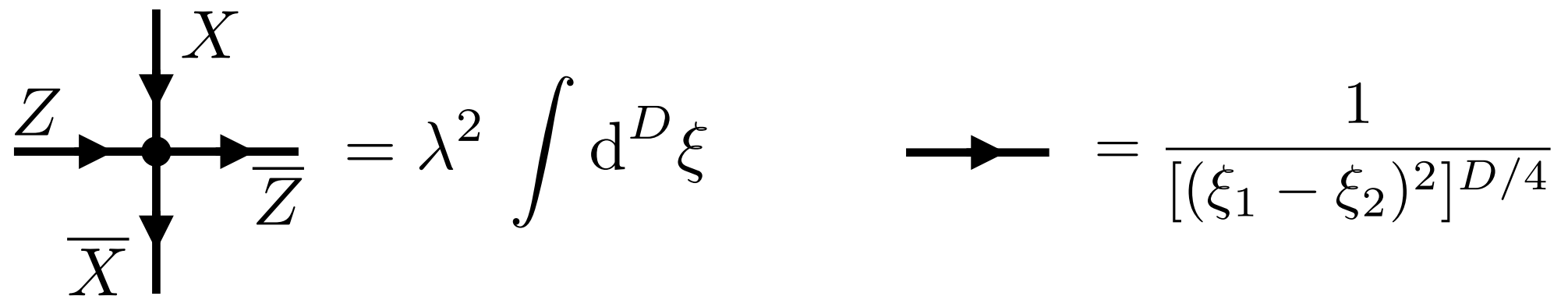
➔ Level 1 generators: $\hat{J}^A = \frac{1}{2} f^A{}_{BC} \sum_{k=1}^n \sum_{j=1}^{k-1} J_j^C J_k^B + \sum_{j=1}^n s_j J_j^A$

➔ Invariance of the integral associated to the fishnet graph G :

$$J^A I_G = \hat{J}^A I_G = 0$$

- They compute correlators in the D -dimensional fishnet theory:

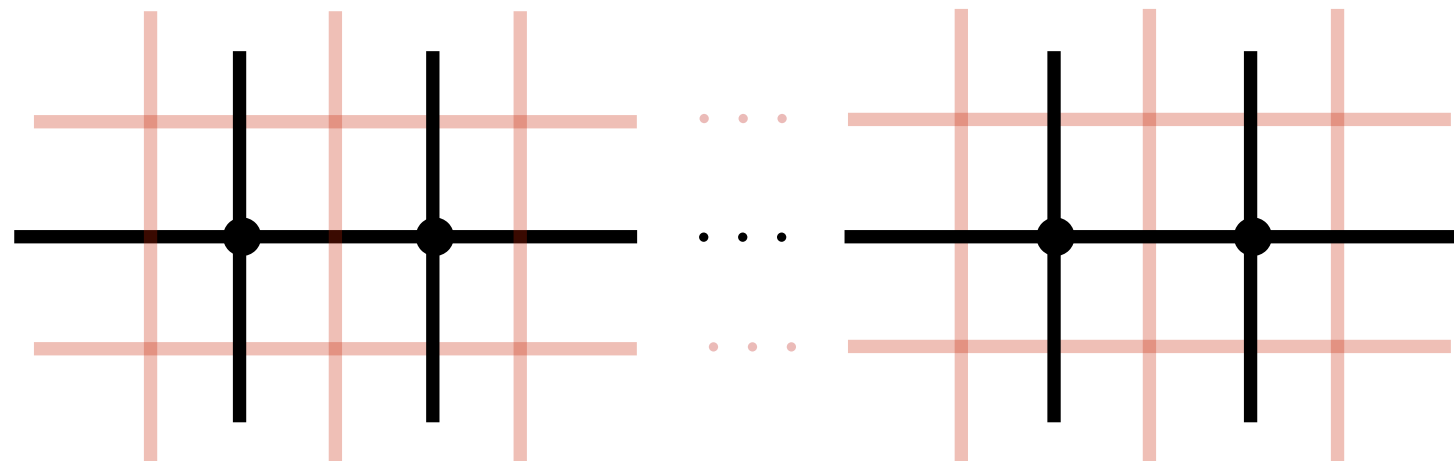
$$\mathcal{L}_{\text{FN},D} = N_c \text{tr} \left[\bar{X}(-\square)^{D/4} X + \bar{Z}(-\square)^{D/4} Z + \lambda^2 \bar{X} \bar{Z} X Z \right] \quad [\text{Gürdogan, Kazakov; Kazakov, Olivucci}]$$



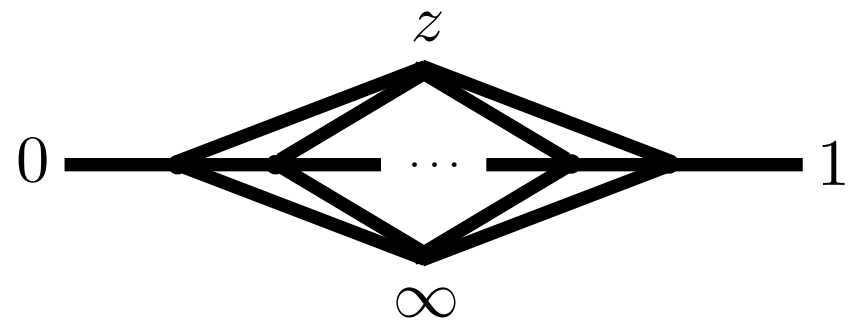
$$\begin{array}{c} X \\ \downarrow \\ Z \rightarrow \bullet \rightarrow \bar{Z} \\ \uparrow \\ \bar{X} \end{array} = \lambda^2 \int d^D \xi \quad \longrightarrow \quad = \frac{1}{[(\xi_1 - \xi_2)^2]^{D/4}}$$

- Traintrack integrals compute specific supercomponents of $\mathcal{N} = 4$ amplitudes in $D = 4$.

[Caron-Huot, Larsen; Bourjaily, He, McLeod, von Hippel, Wilhelm]



- **Ladder integrals:** (single-valued) classical polylogs (for all loops).



[Davydychev, Ussyukina]

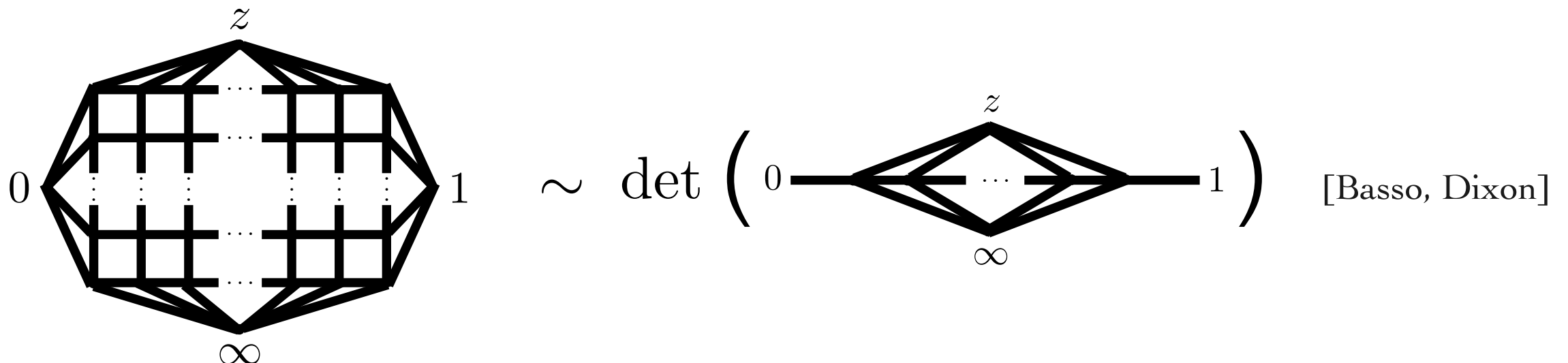
- **Two-loop train track:** elliptic polylogs.

➔ Explicit result for two lightlike external legs. [Kristensson, Wilhelm, Zhang]

- ℓ -loop traintrack: geometry is Calabi-Yau $(\ell - 1)$ -fold.

[Bourjaily, He, McLeod, von Hippel, Wilhelm]

- **Basso-Dixon Formula:**

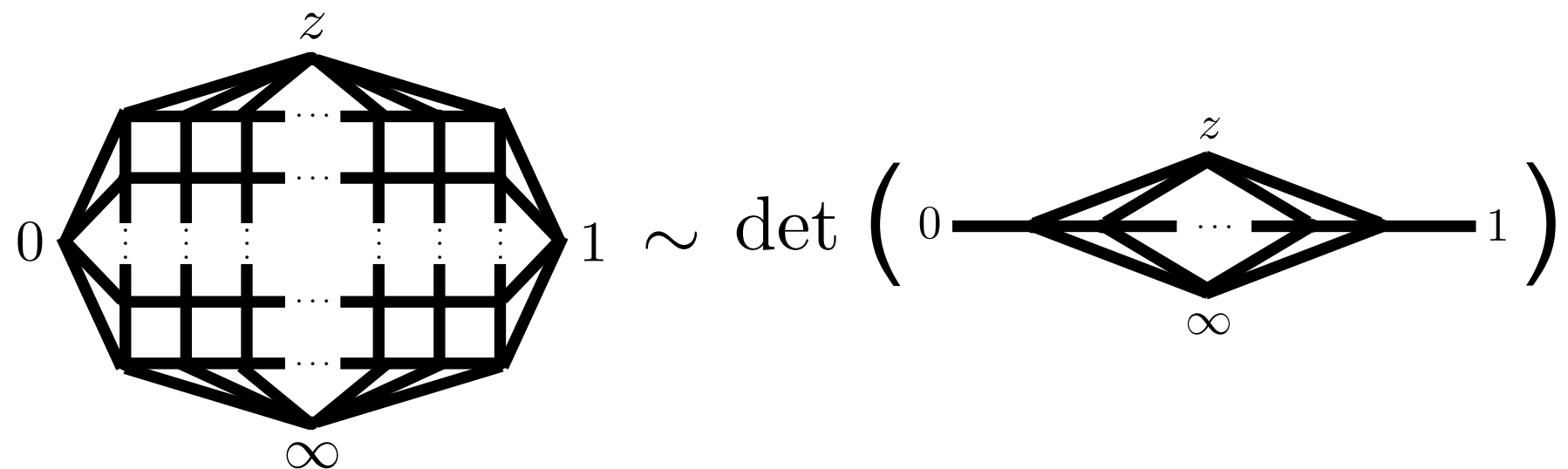


[Basso, Dixon]

- ℓ -loop ladder integral: bilinear in $\ell+1 F_\ell$ -functions.

[Derkachov, Kazakov, Olivucci]

- Basso-Dixon formula in 2D:



- No known results for multi-variable traintracks.

- **Observation:** 1-loop can be expressed as elliptic integrals:

$$\begin{array}{l}
 \begin{array}{c} \alpha_2 \\ | \\ \alpha_1 - \bullet - \alpha_3 \\ | \\ \alpha_4 \end{array} = \frac{4}{|a_{12}a_{34}|} [\mathbf{K}(z) \mathbf{K}(1 - \bar{z}) + \mathbf{K}(1 - z) \mathbf{K}(\bar{z})] \quad \text{2 periods of an elliptic curve} \\
 \mathbf{K}(z) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-zx^2)}} \quad z = \frac{a_{23}a_{14}}{a_{21}a_{34}} \quad a_k = \alpha_k^1 + i \alpha_k^2 \quad \text{[Corcoran, Löbbert, Miczajka]}
 \end{array}$$

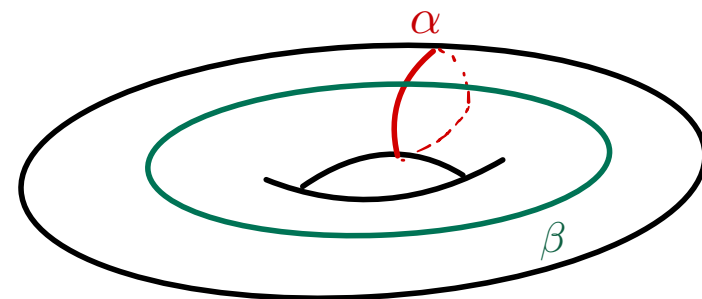
1. Brief overview of Calabi-Yau geometry.
2. 2d fishnets and Calabi-Yau geometry.
3. Fishnet integrals as volumes.

Brief overview of Calabi- Yau geometry

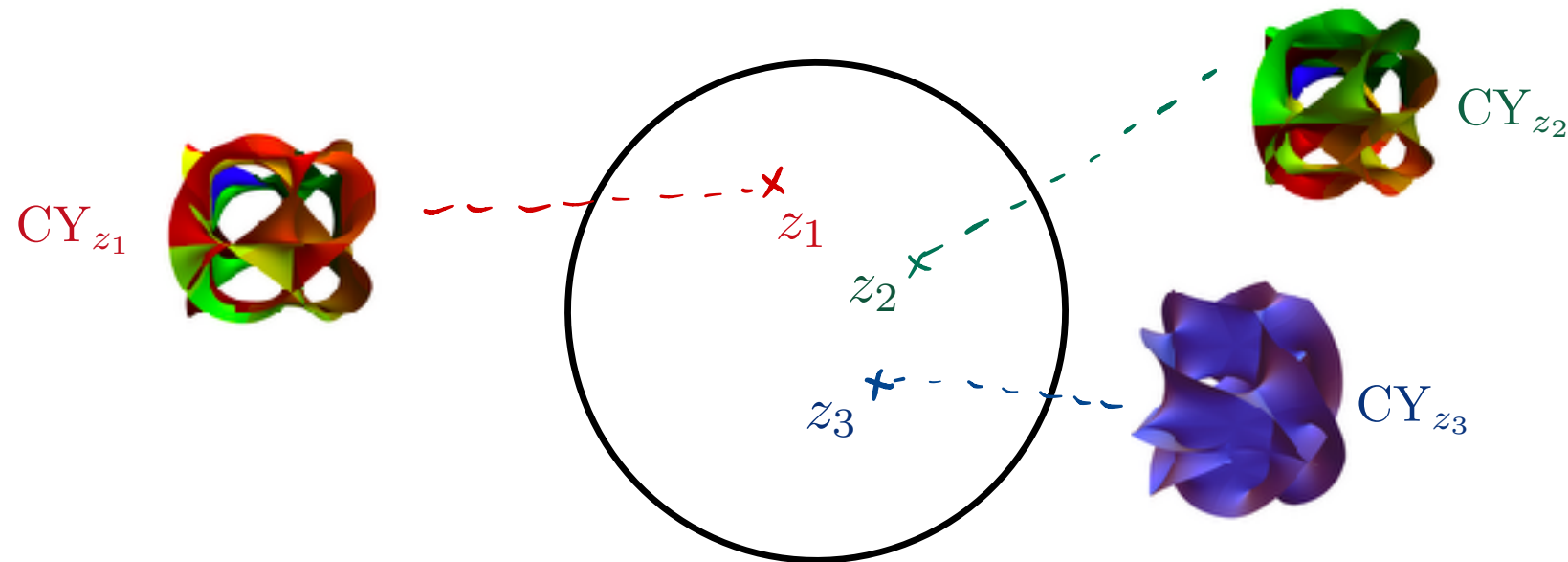
- A Calabi-Yau ℓ -fold is an ℓ -dimensional complex Kähler manifold with a unique nowhere vanishing $(\ell, 0)$ -form.
 - ➔ (p, q) -form: p holomorphic and q anti-holomorphic differentials.
 - ➔ Uniqueness of $(\ell, 0)$ -form is equivalent to vanishing of 1st Chern class.
- A Calabi-Yau ℓ -fold is uniquely defined by a triple

Complex manifold of dimension ℓ (M, Ω, ω) Kähler form; $(1, 1)$ -form (\sim metric)
 $(\ell, 0)$ -form (defines complex structure)
- Example: Calabi-Yau 1-fold = elliptic curve

$$\left(\mathcal{E}, dz = \frac{dx}{y}, A dz \wedge d\bar{z} \right)$$



- We are typically interested in families of CY varieties:



- ➔ For each z there is a CY variety M_z with a given top-form Ω_z .
- **Example:** Family of elliptic curves: $y^2 = x(1-x)(1-zx)$
 - ➔ For each z there is \mathcal{E}_z with $\Omega_z = \frac{dx}{y} = \frac{dx}{\sqrt{x(1-x)(1-zx)}}$.
 - ➔ This does not fix the Kähler form! (We can still scale the torus, i.e., its area is not fixed!)

- We can integrate Ω_z over a basis of cycles of $H_\ell(M_z, \mathbb{Z})$.

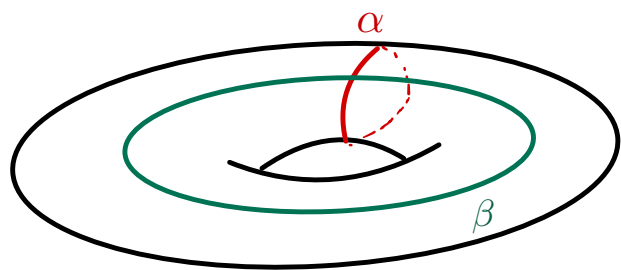
→ **Periods:** $\Pi(z) = (\Pi_0(z), \dots, \Pi_{b_\ell-1}(z))$, $\Pi_i(z) = \int_{\Gamma_i} \Omega_z$

- The periods are multivalued functions of z .

→ There is a monodromy-invariant bilinear pairing on periods:

$$\int_{M_z} \Omega_z \wedge \bar{\Omega}_z \sim \Pi(z)^\dagger \Sigma \Pi(z) \quad \Sigma^T = (-1)^\ell \Sigma$$

- **Example:** Family of elliptic curves: $y^2 = x(1-x)(1-zx)$



$$\Pi_\alpha(z) = \int_\alpha \frac{dx}{y} = 2K(z) \quad \Pi_\beta(z) = \int_\beta \frac{dx}{y} = 2iK(1-z)$$

$$\Pi(z)^\dagger \Sigma \Pi(z) = 4i [K(z)K(1-\bar{z}) + K(\bar{z})K(1-z)] \quad \Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

1-loop 2D fishnet integral!

- For every family of CYs, there is a set of differential operators whose solutions are precisely spanned by the periods!
 - ➔ They generate an ideal of differential operators, the Picard-Fuchs ideal (PFI) of the family of CYs.
 - ➔ For 1-parameter families, we only need a single operator, the Picard-Fuchs operator.

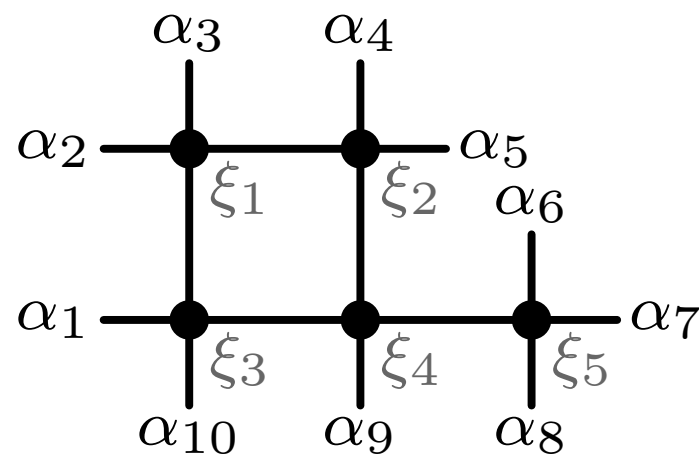
● **Example:** Family of elliptic curves: $y^2 = x(1-x)(1-zx)$

➔ Picard-Fuchs operator: $L = z^2(1-z)\partial_z^2 + z(1-2z)\partial_z - \frac{z}{4}$

$$Lf(z) = 0 \Leftrightarrow f(z) = A\Pi_\alpha(z) + B\Pi_\beta(z) \quad A, B \in \mathbb{C}$$

2D fishnets and
Calabi-Yau geometry

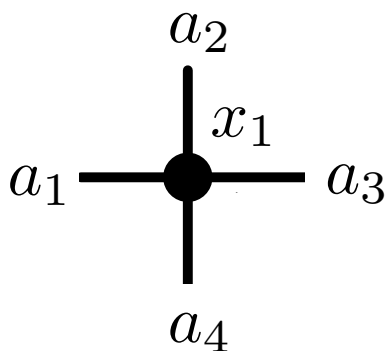
- Holomorphic factorisation: $x_k = \xi_k^1 + i\xi_k^2$ $a_k = \alpha_k^1 + i\alpha_k^2$



$$\frac{1}{\sqrt{(\xi_i - \xi_j)^2}} = \frac{1}{|x_i - x_j|}$$

$$d^2\xi_k = \frac{i}{2} dx_k \wedge d\bar{x}_k$$

$$I_G(a) = \int \left(\prod_{j=1}^{\ell} \frac{dx_j \wedge d\bar{x}_j}{-2i} \right) \frac{1}{\sqrt{|P_G(x, a)|^2}}$$

- Example: $G =$  $P_G(x, a) = (x_1 - a_1) \cdots (x_1 - a_4)$

- The Yangian also splits holomorphically:

$$Y(\mathfrak{so}(1, 3)) = Y(\mathfrak{sl}_2(\mathbb{R})) \oplus \overline{Y(\mathfrak{sl}_2(\mathbb{R}))}$$

- When does $y^2 = P_G(x, a)$ define a CY ℓ -fold?
 - ➔ The 1st Chern class must vanish.
 - ➔ This happens if $P_G(x, a)$ has degree 4 in each x_k .
 - ➔ This conditions is always satisfied for fishnet graphs, because all vertices are 4-valent!

Conclusion:

To every ℓ -loop fishnet graph G we can associate a family M_G of CY ℓ -folds parametrised by $a = (a_1, \dots, a_n)$ and holomorphic $(\ell, 0)$ -form

$$\Omega_G = \frac{dx_1 \wedge \dots \wedge dx_\ell}{\sqrt{P_G(x, a)}}$$

[CD, Klemm, Löbbert, Nega, Porkert]

- The Feynman integral is related to the periods:

$$I_G(a) = \int \left(\prod_{j=1}^{\ell} \frac{dx_j \wedge d\bar{x}_j}{-2i} \right) \frac{1}{\sqrt{|P_G(x, a)|^2}}$$

$$\sim \int_{M_G} \Omega_G \wedge \bar{\Omega}_G \sim \Pi_G(a)^\dagger \Sigma \Pi_G(a)$$

- ➔ Generalises the bilinear expression at 1-loop:

$$\Pi(z)^\dagger \Sigma \Pi(z) = 4i [K(z)K(1 - \bar{z}) + K(\bar{z})K(1 - z)] \quad \begin{array}{l} \text{1-loop 2D} \\ \text{fishnet integral!} \end{array}$$

- The periods are obtained by solving the PF differential equations.

- ➔ How can we find them?

- Let $J \in Y(\mathfrak{sl}_2(\mathbb{R}))$. Yangian-invariance implies:

$$0 = J [I_G(a)] \sim J [\Pi_G(a)^\dagger \Sigma \Pi_G(a)] \sim \Pi_G(a)^\dagger \Sigma J [\Pi_G(a)]$$

$$\rightarrow J [\Pi_G(a)] = 0$$

$$\rightarrow Y(\mathfrak{sl}_2(\mathbb{R})) \subset \text{PFI}$$

- The PFI contains more operators...

- Issue:

→ The Yangian is built on a cyclic ordering of the external points.

→ Fishnet graphs have more symmetries.

→ Example:

$$G = \begin{array}{c} a_2 \\ | \\ x_1 \\ | \\ a_1 \text{---} \bullet \text{---} a_3 \\ | \\ a_4 \end{array}$$

$$\text{Aut}(G) = S_4$$

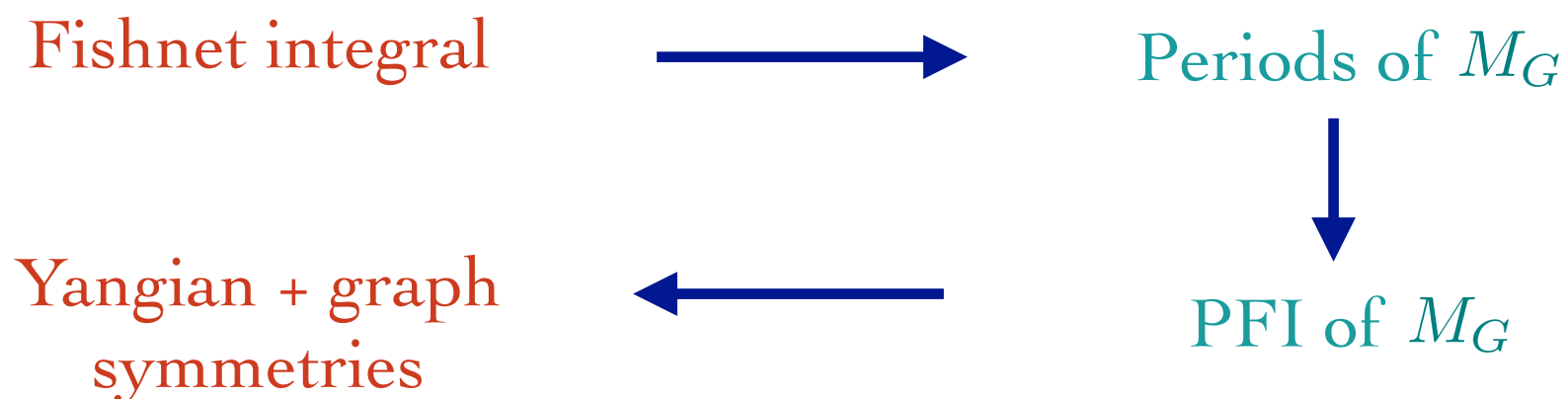
- For all examples we studied: we obtain the complete PFI if we add to the Yangian all its $\text{Aut}(G)$ permutations.

Conjecture:

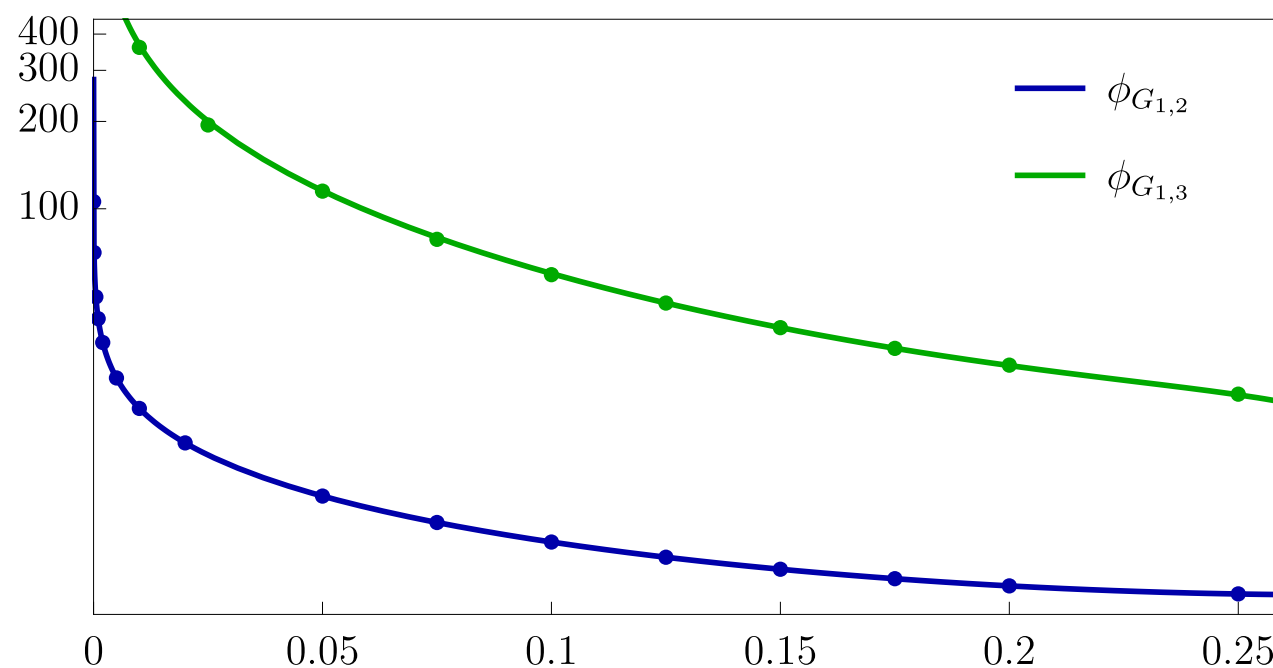
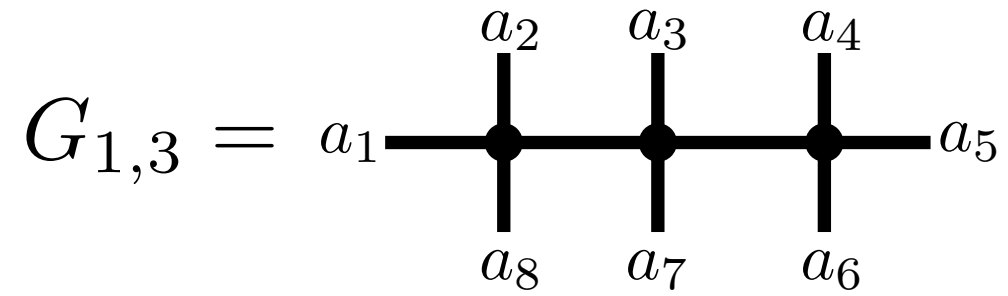
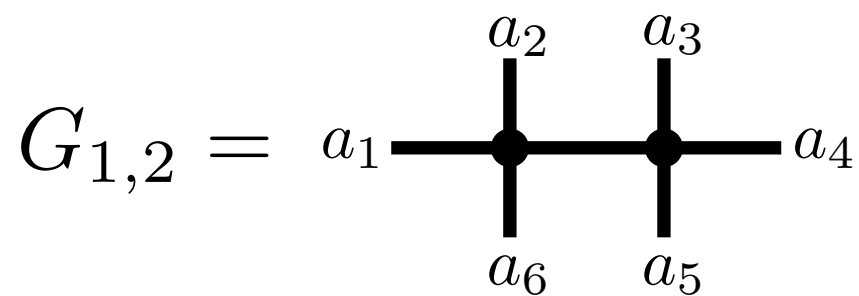
The PFI of M_G is generated by the Yangian $Y(\mathfrak{sl}_2(\mathbb{R}))$ and all its $\text{Aut}(G)$ permutations.

[CD, Klemm, Löbbert, Nega, Porkert]

- Geometry informs physics, physics informs geometry!



- We reproduce in this way results for 1-parameter ladder and fishnet graphs to high loop order!
- **New results:** 2- and 3-loop traintrack in general kinematics:



$$I_G(a) = |F_G(a)|^2 \phi_G(z)$$

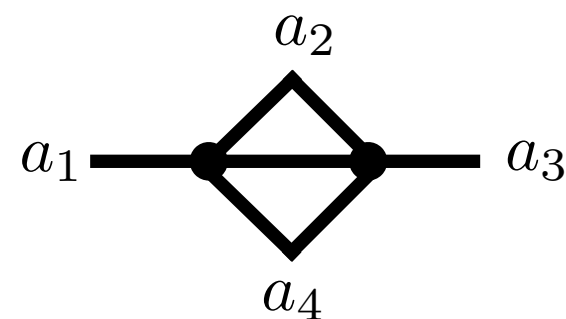
cross ratios

[CD, Klemm, Löbbert, Nega, Porkert]

- Periods of a 1-parameter family of CY 2-folds (K3 surfaces) can always be expressed in terms of products of elliptic integrals!

[Doran; Bogner]

- ➔ The two-loop ladder integral can be expressed in terms of elliptic integrals!



$$= \frac{32}{|a_{12}a_{34}a_{24}|} (K_+ \bar{K}_- + K_- \bar{K}_+)^2$$

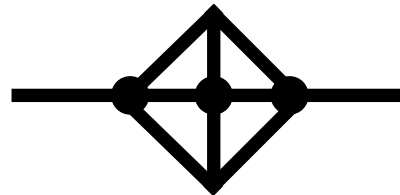
$$K_{\pm} = K\left(\frac{1}{2}(1 \pm \sqrt{1-z})\right) \quad z = \frac{a_{23}a_{14}}{a_{21}a_{34}}$$

[CD, Klemm, Löbbert, Nega, Porkert]

- For higher loops, no representation in terms of elliptic integrals is expected to exist.

- $Y(\mathfrak{sl}_2(\mathbb{R})) \subset \text{PFI}$ implies that the periods are Yangian invariants!

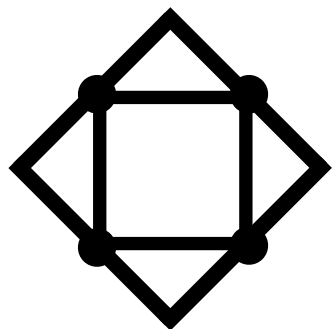
→ **Example:**



4 Yangian-invariant periods:

$$\left(\Pi_0(z), \Pi_1(z), \Pi_2(z), \Pi_3(z) \right)$$

- What about non-ladder graphs?



→ **Feynman integral:** Basso-Dixon formula.

→ **Periods:** $\left(D_{01}(z), D_{02}(z), D_{03}(z), D_{12}(z), D_{23}(z) \right)$

$$D_{ij}(z) = \det \begin{pmatrix} \Pi_i(z) & \Pi_j(z) \\ \partial_z \Pi_i(z) & \partial_z \Pi_j(z) \end{pmatrix}$$

$$D_{13}(z) = D_{02}(z)$$

- **In general:** the periods associated to $M \times N$ fishnets ($M \leq N$) are $M \times M$ determinants of the (derivatives) periods of $(M + N - 1)$ -loop ladders graphs.

→ Basso-Dixon formula for Yangian-invariant CY periods!

Physics

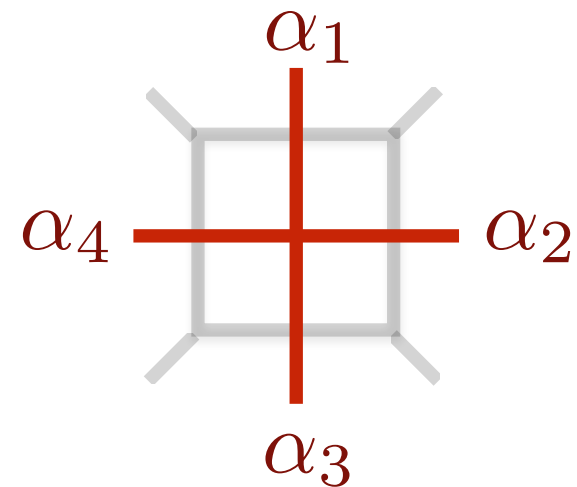
- ℓ -loop fishnet graph G
- Feynman integrand $\Omega_G \wedge \bar{\Omega}_G$
- Cross ratios of external points
- Yangian and graph symmetries
- Yangian invariants
- Basso-Dixon formula
- Feynman integral $I_G(a)$

CY geometry

- Family of CY ℓ -folds M_G
- $(\ell, 0)$ -form Ω_G
- Independent moduli
- Picard Fuchs ideal
- Periods
- Alternating products of PF operators.
- ?????

Fishnet integrals as volumes

- In 4D:



$$= -\frac{4}{\alpha_{13}^2 \alpha_{24}^2} \frac{D(z)}{z - \bar{z}} \sim \text{Vol} \left(\begin{array}{c} \text{A sphere with four points } x_i, x_j, x_k, x_l \text{ on its surface.} \\ \text{A tetrahedron is inscribed within the sphere, connecting these points.} \end{array} \right)$$

➔ No known extension beyond 1-loop.

- In 2D: Which metric shall we use?

➔ Ω_G defines a complex structure on M_G , but no Kähler structure!

➔ We have no canonical choice of metric to compute a volume...

- **Mirror symmetry:** CY ℓ -folds come in pairs (M, W) s.t.

$$\dim H^{p,q}(M) = \dim H^{\ell-p,q}(W)$$

$H^{p,q}(M)$ = cohomology classes of (p, q) -forms

- **Mirror symmetry exchanges complex and Kähler structures:**

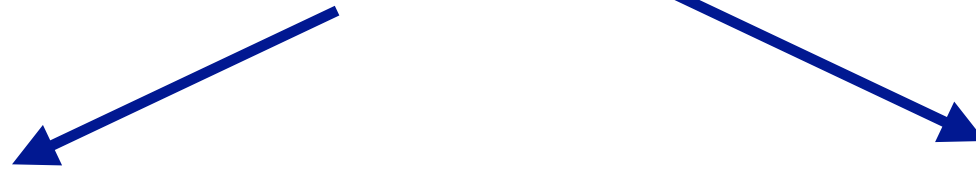
$$H^{\ell-1,1}(M) \quad \overset{\text{MS}}{\longleftrightarrow} \quad H^{1,1}(W)$$

Parametrises complex
structures on M

Parametrises Kähler
structures on W

$$\Omega_z \quad \overset{\text{MS}}{\longleftrightarrow} \quad \omega = \sum_i (\text{Im } t_i(z)) e_i \quad \text{Basis of } H^{1,1}(W)$$

Mirror map: $t_i(z) = \frac{\Pi_i(z)}{\Pi_0(z)} \sim \log z_i + \mathcal{O}(z^2)$ log-divergent solutions
holomorphic solution

Fishnet graph G 

Complex structure
 Ω_G on M_G

Kähler structure
 ω_G on W_G

- Classical volume:

$$\begin{aligned} \text{Vol}_{\text{cl.}}(W_G) &= \int_{W_G} \frac{\omega_G^l}{l!} & t_i^{\mathbb{R}}(z) &= \text{Im } t_i(z) \\ &= \frac{1}{l!} \sum_{i_1, \dots, i_\ell} C_{i_1, \dots, i_\ell}^{\text{cl.}} t_{i_1}^{\mathbb{R}}(z) \cdots t_{i_\ell}^{\mathbb{R}}(z) \end{aligned}$$

➔ $C_{i_1, \dots, i_\ell}^{\text{cl.}}$ are (explicit computable) intersection numbers.

- 1 loop:

$$I_{1\text{-loop}}(a) \sim K(z)K(1 - \bar{z}) + K(\bar{z})K(1 - z) = 2 |K(z)|^2 \text{Im } \tau(z)$$

$$\tau(z) = iK(1 - z)/K(z) \quad |\Pi_{1\text{-loop},0}(z)|^2; \text{ overall scale} \quad \text{Vol}_{\text{cl.}}(W_{1\text{-loop}})$$

- 2 loops:

$$I_{2\text{-loop}}(a) \sim (K_+ \bar{K}_- + K_- \bar{K}_+)^2 = 4 |K_-|^4 (\text{Im } \tau(z))^2$$

$$\tau(z) = iK_+/K_- \quad |\Pi_{2\text{-loop},0}(z)|^2 \quad \text{Vol}_{\text{cl.}}(W_{2\text{-loop}})$$

- 3 loops:

$$I_{3\text{-loop}}(a) \approx |\Pi_{3\text{-loop},0}(z)|^2 \text{Vol}_{\text{cl.}}(W_{3\text{-loop}})$$

- For $\ell \geq 3$, the volume receives instanton corrections:

➔ Quantum volume:

$$\begin{aligned}\Pi_G(z)^\dagger \Sigma \Pi_G(z) &\sim |\Pi_{G,0}(z)|^2 \text{Vol}_q(W_G) \\ &\sim |\Pi_{G,0}(z)|^2 \text{Vol}_{\text{cl.}}(W_G) + \mathcal{O}(e^{-t_i(z)})\end{aligned}$$

➔ The same notion of quantum volume appears in string theory and geometry. [cf. e.g. Lee, Lerche, Weigand]

- Instanton corrections are absent for elliptic curves and K3 surfaces.

➔ The classical and quantum volumes agree for $\ell = 1, 2$.

Physics

- ℓ -loop fishnet graph G
- Feynman integrand $\Omega_G \wedge \bar{\Omega}_G$
- Cross ratios of external points
- Yangian and graph symmetries
- Yangian invariants
- Basso-Dixon formula
- Feynman integral $I_G(a)$

CY geometry

- Family of CY ℓ -folds M_G
- $(\ell, 0)$ -form Ω_G
- Independent moduli
- Picard Fuchs ideal
- Periods
- Alternating products of PF operators.
- Quantum volume of W_G

- 2D Yangian-invariant fishnet integrals are closely related to Calabi-Yau geometries!
- This gives a new way to compute these integrals:
 - ➔ Computation of these fishnet integrals is reduced to the computation of the periods.
 - ➔ Periods are obtained from PF differential equations.
 - ➔ PF differential equations are generated by Yangian and permutation symmetries.
- **Bonus:** first interpretation of a multi-loop integral as a volume.
 - ➔ Receives instanton corrections starting from 3 loops.

- Implications of Yangian symmetry for geometry?
 - ➔ Basso-Dixon formula for periods?
- Implications for integrability of fishnet theories?
 - ➔ Explanation of instanton corrections at 3 loops?
 - ➔ Are there other Yangian invariants besides the periods?
- Is there a similar story for 4D fishnet integrals?
 - ➔ Volume interpretation in 4D?
 - ➔ Role of mirror symmetry in this context?



