Seminar series on motives and period integrals in Quantum field theory and String theory
IPhT CEA-Saclay/University of Oxford, March 6, 2024

Symbolic Computation and Feynman Integrals

Carsten Schneider

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University Linz

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DESY-cooperation: J. Bluemlein, P. Marquard

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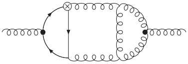
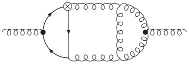
FWF Der Wissenschaftsfonds.

Evaluation of Feynman Integrals



behavior of particles

Evaluation of Feynman Integrals

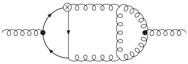


behavior of particles

$$\int \Phi(n, \epsilon, x) dx$$

Feynman integrals

Evaluation of Feynman Integrals



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$$\int \Phi(n, \epsilon, x)dx$$

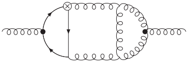
Feynman integrals

DESY

$$\sum f(n, \epsilon, k)$$

complicated multi-sums

Evaluation of Feynman Integrals



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$$\int \Phi(n,\epsilon,x)dx$$

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$$\sum f(n,\epsilon,k)$$

complicated
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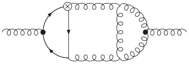
RISC
(Sigma-package)

expression in
special functions

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(n, \epsilon, x)dx$$

Feynman integrals

DESY

LHC at CERN

applicable

$$\sum f(n, \epsilon, k)$$

complicated multi-sums

RISC

(Sigma-package)

expression in special functions

$$F(\varepsilon, n) = \iiint \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^n}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

||?

$$F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + F_0(n)\varepsilon^0 + \dots$$

$$F(\varepsilon, n) = \iiint \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^n}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\sum_{k=1}^{n} (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\Big(-1-\frac{3\varepsilon}{2}\Big) \times$$
$$\times B\Big(2+k, \frac{\varepsilon}{2}\Big) B(-\varepsilon+k, -\varepsilon) B\Big(1-\frac{\varepsilon}{2}+k, 1+\frac{\varepsilon}{2}\Big) \binom{n}{k}$$

where

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

$$F(\varepsilon,n)=\iiint\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^n}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\underbrace{\sum_{k=1}^{n}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{n}{k}}_{=f_{-3}(n,k)\varepsilon^{-3}+f_{-2}(n,k)\varepsilon^{-2}+f_{-1}(n,k)\varepsilon^{-1}+\dots}$$

for general expansion methods see
J. Blümlein, CS, M. Saragnese, Ann. Math. Artif. Intell. 91(5), pp. 591-649. 2023.

$$F(\varepsilon,n)=\iiint\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^n}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\underbrace{\sum_{k=1}^{n}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{n}{k}}_{=f_{-3}(n,k)\varepsilon^{-3}+f_{-2}(n,k)\varepsilon^{-2}+f_{-1}(n,k)\varepsilon^{-1}+\dots}$$

$$||$$

$$\underbrace{\Big(\sum_{k=1}^{n}f_{-3}(n,k)\Big)}_{F_{-3}(n)}\varepsilon^{-3}+\underbrace{\Big(\sum_{k=1}^{n}f_{-2}(n,k)\Big)}_{F_{-2}(n)}\varepsilon^{-2}+\underbrace{\Big(\sum_{k=1}^{n}f_{-1}(n,k)\Big)}_{F_{-1}(n)}\varepsilon^{-1}+\dots$$

$$F(\varepsilon,n)=\iiint\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^n}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\sum_{k=1}^{n}\underbrace{(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{n}{k}}_{=f_{-3}(n,k)\varepsilon^{-3}+f_{-2}(n,k)\varepsilon^{-2}+f_{-1}(n,k)\varepsilon^{-1}+\dots}$$

$$||$$

$$\Big(\underbrace{\sum_{k=1}^{n}f_{-3}(n,k)}_{F_{-3}(n)}\Big)\varepsilon^{-3}+\Big(\underbrace{\sum_{k=1}^{n}f_{-2}(n,k)}_{F_{-2}(n)}\Big)\varepsilon^{-2}+\Big(\underbrace{\sum_{k=1}^{n}f_{-1}(n,k)}_{F_{-1}(n)}\Big)\varepsilon^{-1}+\dots$$

Simplify

$$F_{-1}(n) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \Big)$$

where

$$S_a(n) = \sum_{i=1}^{n} \frac{\mathsf{sign}(a)^i}{i^{|a|}} \text{ and } \zeta_a = \sum_{i=1}^{\infty} \frac{1}{i^a}$$

Simplify

$$F_{-1}(n) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \Big)$$

$\downarrow$ (summation package `Sigma.m`)

$$\begin{aligned}\big(16n^3+144n^2+413n+384\big)(n+1)^2 F_{-1}(n)& \\ -(n+2)(2n+5)\big(16n^3+112n^2+221n+113\big)F_{-1}(n+1)& \\ +(n+3)^2\big(16n^3+96n^2+173n+99\big)F_{-1}(n+2)& \\ =\frac{1}{2}\big(4n^2+21n+29\big)\zeta_2+\tfrac{-64n^5-500n^4-1133n^3+203n^2+3516n+3090}{3(n+2)(n+3)}&\end{aligned}$$

Simplify

$$F_{-1}(n)=\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

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$$\begin{aligned}\{c_1\frac{1-4n}{n+1}+c_2\frac{-14n-13}{(n+1)^2}\\ +\frac{(4n-1)S_1(n)}{n+1}+\frac{(1-4n)S_1(n)^2}{6(n+1)}+\frac{(14n+13)S_1(n)}{3(n+1)^2}\\ +\frac{175n^2+334n+155}{12(n+1)^3}+\frac{(1-4n)S_2(n)}{6(n+1)}+\frac{\zeta_2}{8(n+1)}\,|c_1,c_2\in\mathbb{R}\}\end{aligned}$$

Simplify

$$F_{-1}(n) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \Big)$$

$$\bigcap$$

$$\Big\{ c_1 \frac{1-4n}{n+1} + c_2 \frac{-14n-13}{(n+1)^2} + \frac{(4n-1)S_1(n)}{n+1} + \frac{(1-4n)S_1(n)^2}{6(n+1)} + \frac{(14n+13)S_1(n)}{3(n+1)^2} + \frac{175n^2+334n+155}{12(n+1)^3} + \frac{(1-4n)S_2(n)}{6(n+1)} + \frac{\zeta_2}{8(n+1)} \Big| c_1, c_2 \in \mathbb{R} \Big\}$$

Simplify

$$F_{-1}(n)=\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

|| (recurrence finding and solving)

$$\Big(\frac{1}{12}-\frac{1}{8}\zeta_2\Big)\frac{1-4n}{n+1}+1\,\frac{-14n-13}{(n+1)^2}$$
$$+\frac{(4n-1)S_1(n)}{n+1}+\frac{(1-4n)S_1(n)^2}{6(n+1)}+\frac{(14n+13)S_1(n)}{3(n+1)^2}$$
$$+\frac{175n^2+334n+155}{12(n+1)^3}+\frac{(1-4n)S_2(n)}{6(n+1)}+\frac{\zeta_2}{8(n+1)}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $F(n)$

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FIND a recurrence for $F(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \ldots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)F(n) + \cdots + a_d(n)F(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, JSC, 2021)

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Special cases:

$$S_{2,1}(n) = \sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j} \qquad \text{(harmonic sums)}$$

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];
J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

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Special cases:

$$\sum_{k=1}^{n}\frac{2^k}{k}\sum_{i=1}^{k}\frac{2^{-i}}{i}\sum_{j=1}^{i}\frac{S_1(j)}{j}$$

(generalized harmonic sums)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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Special cases:

$$\sum_{k=1}^{n} \frac{1}{(1+2k)^2} \sum_{j=1}^{k} \frac{1}{j^2} \sum_{i=1}^{j} \frac{1}{1+2i}$$

(cyclotomic harmonic sums)

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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$$\sum_{j=1}^{n}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}$$ (binomial sums)

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, JSC, 2021)

Special cases:

$$\sum_{h=1}^{n} 2^{-2h}(1-\eta)^h \binom{2h}{h} \sum_{k=1}^{h} \frac{2^{2k}}{k^2\binom{2k}{k}}$$

(generalized binomial sums)

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].
J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916].

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A more general example:

$$\sum_{k=1}^{n} \left(\prod_{i=1}^{k} \frac{1+i+i^2}{i+1} \right) \left(\sum_{j=1}^{k} \frac{1}{j\binom{4j}{3j}^2} \right) \left(\sum_{j=1}^{k} \begin{bmatrix} 2j \\ j \end{bmatrix}_q \right)$$

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3. Find a "closed form"

F(n)=combined solutions in terms of indefinite nested sums.

This summation machinery is based on

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In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

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In[4]:= $\mathbf{mySum} =$

$$\sum_{k=1}^{n}(-1)^{k}e^{-\frac{3\varepsilon\gamma}{2}}\left(-2-\frac{3\varepsilon}{2}\right)!\,B\left[2+k,\frac{\varepsilon}{2}\right]B[-\varepsilon+k,-\varepsilon]B\left[1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\right]\binom{n}{k};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum, \{\}, \{n\}, \{1\}, ExpandIn \to \{\varepsilon, -3, -3\}]}$

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In[5]:= $\mathbf{EvaluateMultiSum[mySum, \{\}, \{n\}, \{1\}, ExpandIn \to \{\varepsilon, -3, -3\}]}$

Out[5]= $\Big\{\frac{59n^2+120n+49}{9(n+1)^2}-\frac{2(n+3)S[1,n]}{3(n+1)}\Big\}$

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$$-\frac{2(20n^3+58n^2+57n+22)}{3(n+1)^3}+\frac{2(n+2)(2n-1)S[1,n]}{3(n+1)^2}-\frac{S[1,n]^2}{n+1}-\frac{S[2,n]}{n+1}\Big\}$$

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Out[5]=
$$\Big\{\frac{59n^2+120n+49}{9(n+1)^2}-\frac{2(n+3)S[1,n]}{3(n+1)},$$
$$-\frac{2(20n^3+58n^2+57n+22)}{3(n+1)^3}+\frac{2(n+2)(2n-1)S[1,n]}{3(n+1)^2}-\frac{S[1,n]^2}{n+1}-\frac{S[2,n]}{n+1},$$
$$\left(\frac{1}{12}-\frac{1}{8}z2\right)\frac{1-4n}{n+1}+\frac{-14n-13}{(n+1)^2}+\frac{(4n-1)S[1,n]}{n+1}+\frac{(1-4n)S[1,n]^2}{6(n+1)}+$$
$$\frac{(14n+13)S[1,n]}{3(n+1)^2}+\frac{175n^2+334n+155}{12(n+1)^3}+\frac{(1-4n)S[2,n]}{6(n+1)}+\frac{z2}{8(n+1)}\Big\}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\begin{array}{l}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\\ \frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)\end{array}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\boxed{\sum_{j=0}^{n-2} \sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+ \frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^{r}(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$||$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\left(\left(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\right)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\right)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

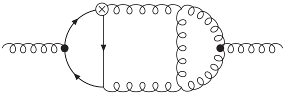
$$\sum_{j=0}^{n-2}\left(\left(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\right)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\right)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

note: $S_a(n)=\sum_{i=1}^{n}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.

Feynman integral



a 3-loop massive ladder diagram [arXiv:1509.08324]

||

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\binom{n-1}{j+2}\binom{j+1}{k+1}$$

$$\times \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

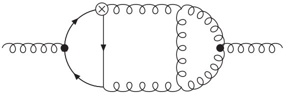
$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-\varepsilon/2}$$

$$\Big[\left[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k$$

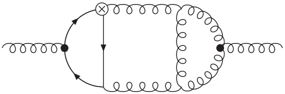
$$+\left[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k \Big]$$

$$\times (1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{n-3-j}$$

$$\times \left[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3\right]^{n-3-j}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

Simplify

$$\|$$

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+n-3}\sum_{s=1}^{-l+n-q-3}\sum_{r=0}^{-l+n-q-s-3}(-1)^{-j+k-l+n-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{n-1}{j+2}\binom{-j+n-3}{q}\binom{-l+n-q-3}{s}\binom{-l+n-q-s-3}{r}r!(-l+n-q-r-s-3)!(s-1)!}{(-l+n-q-2)!(-j+n-1)(n-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+n-1)-4S_1(-j+n-2)-2S_1(k)$$

$$-(S_1(-l+n-q-2)+S_1(-l+n-q-r-s-3)-2S_1(r+s))$$

$$+2S_1(s-1)-2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}$$

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$
$$+\Big(-\frac{4(13n+5)}{n^2(n+1)^2} + \Big(\frac{4(-1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)$$
$$+\big(2+2(-1)^n\big)S_{2,1}(n) - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n) + \Big(\frac{3}{4} + (-1)^n\Big)S_2(n)^2$$
$$-2(-1)^n S_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26+4(-1)^n\big)S_1(n) + \frac{4(-1)^n}{n+1}\Big)$$
$$+\Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$
$$+\frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22+6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$
$$+\Big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\Big)S_3(n) + \Big(\frac{19}{2} - 2(-1)^n\Big)S_4(n) + \big(-6+5(-1)^n\big)S_{-4}(n)$$
$$+\Big(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\Big)S_{2,1}(n) + \big(20+2(-1)^n\big)S_{2,-2}(n) + \big(-17+13(-1)^n\big)S_{3,1}(n)$$
$$-\frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24+4(-1)^n\big)S_{-3,1}(n) + \big(3-5(-1)^n\big)S_{2,1,1}(n)$$
$$+32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^n S_{-2}(n)\right)\zeta(2)$$

$$\boxed{F_0(n)} =$$

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$$+ \Big(\cdots\frac{\cdots)^n(2n+1)}{\cdots(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)$$

$$+ \big(2 + \cdots 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n) + \Big(\frac{3}{4} + (-1)^n\Big)S_2(n)^2$$

$$- 2(-1)^n S_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26 + 4(-1)^n\big)S_1(n) + \frac{4(-1)^n}{n+1}\Big)$$

$$+ \Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$

$$+ \frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22 + 6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$

$$+ \Big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\Big)S_3(n) + \Big(\frac{19}{2} - 2(-1)^n\Big)S_4(n) + \big(-6 + 5(-1)^n\big)S_{-4}(n)$$

$$+ \Big(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\Big)S_{2,1}(n) + \big(20 + 2(-1)^n\big)S_{2,-2}(n) + \big(-17 + 13(-1)^n\big)S_{3,1}(n)$$

$$- \frac{8(-1)^n(2n+1) + 4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24 + 4(-1)^n\big)S_{-3,1}(n) + \big(3 - 5(-1)^n\big)S_{2,1,1}(n)$$

$$+ 32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^n S_{-2}(n)\right)\zeta(2)$$

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i}$$

$\boxed{F_0(n)} =$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$

$$+ \Big(\cdots)^n(2n+1)}{(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{\cdots}(-1)^n\Big)S_{\cdots}(n)\Big)$$

$$+ \Big(2 + 2\cdots 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)\cdots\Big)S_2(n)^2$$

$$- 2(-1)^n S_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26+4(-1)^n\big)S_1(n) + \frac{\cdots^n}{n+1}\Big)$$

$$+ \Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$

$$+ \frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22+6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$

$$+ \Big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\Big)S_3(n) + \Big(\frac{19}{2} - 2(-1)^n\Big)S_4(n) + \big(-6+5(-1)^n\big)S_{-4}(n)$$

$$+ \Big(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\Big)S_{2,1}(n) + \big(20+2(-1)^n\big)S_{2,-2}(n) + \big(-17+13(-1)^n\big)S_{3,1}(n)$$

$$- \frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24+4(-1)^n\big)S_{-3,1}(n) + \big(3-5(-1)^n\big)S_{2,1,1}(n)$$

$$+ 32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^n S_{-2}(n)\right)\zeta(2)$$

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$

$$+ \Big(\quad \frac{\ldots)^n(2n+1)}{(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{\ldots} \quad (-1)^n\Big)S_{\ldots}(n)$$

$$+ \Big(2 + \quad 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big) \quad \Big)S_2(n)^2$$

$$- 2(-1)^nS_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26 + 4(-1)^n\big)S_1(n) + \frac{\quad ^n}{n+1}\Big)$$

$$+ \Big(\frac{(-1)^n}{2n^2} \quad + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$

$$+ \frac{4(3n \ldots}{n(n \ldots} \quad -1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$

$$+ \Big(\frac{(-1)^n}{n(} \quad n) + \big(-6 + 5(-1)^n\big)S_{-4}(n)$$

$$+ \Big(- \frac{2(}{n(n+1)} \quad \frac{}{n} \quad \Big)S_{2,-2}(n) + \big(-17 + 13(-1)^n\big)S_{3,1}(n)$$

$$- \frac{8(-1)^n(2n+1) + 4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24 + 4(-1)^n\big)S_{-3,1}(n) + \big(3 - 5(-1)^n\big)S_{2,1,1}(n)$$

$$+ 32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^nS_{-2}(n)\right)\zeta(2)$$

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

$$S_{-2,1,1}(n) = \sum_{i=1}^{n}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{1}{k}}{j}}{i^2}$$

The general tactic

Feynman integrals

The general tactic

Feynman integrals

$\downarrow$ non-trivial transformations (DESY)

multiple sums

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

Tactic 1: Expand the summand and simplify

Ablinger, Blümlein, Klein, CS, LL2010, arXiv:1006.4797 [math-ph]
Blümlein, Hasselhuhn, CS, RADCOR'10, arXiv:1202.4303 [math-ph]
CS, ACAT 2013, arXiv:1310.0160 [cs.SC]

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

Tactic 2: Expand a recurrence in ε

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656 [cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(n)=\sum_{k=1}^{n}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\big(-1-\frac{3\varepsilon}{2}\big)B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{n}{k}$$

$\downarrow$ (summation package `Sigma.m`)

$$2(n+1)^2F(n)+\big(3\varepsilon^2+3\varepsilon n+9\varepsilon-4n^2-12n-8\big)F(n+1)$$
$$-(2\varepsilon-n-1)(\varepsilon+2n+6)F(n+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(n)=\sum_{k=1}^{n}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\big(-1-\frac{3\varepsilon}{2}\big)B\big(2+k,\frac{\varepsilon}{2}\big)B(-\varepsilon+k,-\varepsilon)B\big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\big)\binom{n}{k}$$

$\downarrow$ (summation package `Sigma.m`)

$$2(n+1)^2F(n)+\big(3\varepsilon^2+3\varepsilon n+9\varepsilon-4n^2-12n-8\big)F(n+1)$$
$$-(2\varepsilon-n-1)(\varepsilon+2n+6)F(n+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^{0}+\dots$$

$$F(1)=\frac{2}{3}\varepsilon^{-3}-\frac{11}{6}\varepsilon^{-2}+\big(\frac{\zeta_2}{4}+\frac{79}{24}\big)\varepsilon^{-1}+\dots$$
$$F(2)=\frac{8}{9}\varepsilon^{-3}-\frac{73}{27}\varepsilon^{-2}+\big(\frac{\zeta_2}{3}+\frac{1415}{324}\big)\varepsilon^{-1}+\dots$$

$\downarrow$

$$\boxed{F(n)=F_{-3}(n)\,\varepsilon^{-3}+F_{-2}(n)\,\varepsilon^{-2}+F_{-1}(n)\,\varepsilon^{-1}+\dots}$$

Ansatz (for power series)

$$
\begin{aligned}
&\ a_0(\varepsilon,n)\Big[F(n)\Big]\\
&+a_1(\varepsilon,n)\Big[F(n+1)\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[F(n+d)\Big]
\end{aligned}
$$

$$= h_0(n) + h_1(n)\varepsilon + h_2(n)\varepsilon^2 + \dots$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F(n+1)\Big]\\
&+\\
&\vdots\\
&+a_d(\varepsilon,n)\Big[F(n+d)\Big]\\
&\qquad\qquad= h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F(n+d)\Big]\\
&\qquad\qquad\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad=h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad=h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}$$

Ansatz (for power series)

$$\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad=h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}$$

REC solver: Given the initial values $F_0(1), F_0(2), \dots, F_0(d-1)$, **decide** if $F_0(n)$ can be written in terms of indefinite nested sums and products.

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad=h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n{+}1)+\dots+a_d(0,n)F_0(n{+}d)=h_0(n)}
$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n) + F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1) + F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d) + F_1(n+d)\varepsilon + F_2(n+d)\varepsilon^2 + \dots\Big]\\
&\qquad\qquad = h_0(n) + h_1(n)\varepsilon + h_2(n)\varepsilon^2 + \dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,n)F_0(n) + a_1(0,n)F_0(n+1) + \dots + a_d(0,n)F_0(n+d) = h_0(n)}
$$

$$
\begin{aligned}
&\quad a_0(\varepsilon, n)\Big[F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots\Big] \\
&+a_1(\varepsilon, n)\Big[F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots\Big] \\
&+ \\
&\;\vdots \\
&+a_d(\varepsilon, n)\Big[F_1(n+d)\varepsilon + F_2(n+d)\varepsilon^2 + \dots\Big] \\
&\qquad\qquad\qquad = h_0'(n) + h_1'(n)\varepsilon + h_2'(n)\varepsilon^2 + \dots
\end{aligned}
$$

$$
\begin{aligned}
&\ a_0(\varepsilon,n)\Big[F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots\Big] \\
&+a_1(\varepsilon,n)\Big[F_1(n{+}1)\varepsilon + F_2(n{+}1)\varepsilon^2 + \dots\Big] \\
&+ \\
&\ \vdots \\
&+a_d(\varepsilon,n)\Big[F_1(n{+}d)\varepsilon + F_2(n+d)\varepsilon^2 + \dots\Big] \\
&\qquad\qquad\qquad = \underbrace{h_0'(n)}_{\mathbf{=0}} + h_1'(n)\varepsilon + h_2'(n)\varepsilon^2 + \dots
\end{aligned}
$$

Devide by ε

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_1(n)+F_2(n)\varepsilon+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_1(n+1)+F_2(n+1)\varepsilon+\dots\Big]\\
+&\\
\vdots&\\
+&a_d(\varepsilon,n)\Big[F_1(n+d)+F_2(n+d)\varepsilon+\dots\Big]=h_1'(n)+h_2'(n)\varepsilon+\dots
\end{aligned}
$$

now repeat for $F_1(n), F_2(n), \dots$

Remark: Works the same for Laurent series.

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↓ (summation package Sigma.m)

$$2(n+1)^2F(n)+\big(3\varepsilon^2+3\varepsilon n+9\varepsilon-4n^2-12n-8\big)F(n+1)$$
$$-(2\varepsilon-n-1)(\varepsilon+2n+6)F(n+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\big(2\zeta_2-\frac{68}{3}\big)\varepsilon^0+\dots$$

$$F(1)=\frac{2}{3}\varepsilon^{-3}-\frac{11}{6}\varepsilon^{-2}+\big(\frac{\zeta_2}{4}+\frac{79}{24}\big)\varepsilon^{-1}+\dots$$
$$F(2)=\frac{8}{9}\varepsilon^{-3}-\frac{73}{27}\varepsilon^{-2}+\big(\frac{\zeta_2}{3}+\frac{1415}{324}\big)\varepsilon^{-1}+\dots$$

↓ (summation package Sigma.m)

$$F(n)=\tfrac{4n}{3(n+1)}\varepsilon^{-3}-\Big(\tfrac{2(2n+1)}{3(n+1)}S_1(n)+\tfrac{2n(2n+3)}{3(n+1)^2}\Big)\varepsilon^{-2}$$
$$\Big(\tfrac{(1-4n)}{6(n+1)}S_1(n)^2-\tfrac{n\left(n^2-2\right)}{3(n+1)^3}+\tfrac{(3n+2)(4n+5)}{3(n+1)^2}S_1(n)+\tfrac{(1-4n)}{6(n+1)}S_2(n)+\tfrac{n\zeta_2}{2(n+1)}\Big)\varepsilon^{-1}+\dots$$

Find a recurrence for the integral

$$F(n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$

$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \ldots$$

ε-**recurrence solver**

multivariate
Almquist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon, n)F(n) + \ldots + a_d(\varepsilon, n)F(n+d) = h(\varepsilon, n)$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

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The integrand is

- hyperexponential in x,y,z:

$$\frac{D_x f(\varepsilon,n,x,y,z)}{f(\varepsilon,n,x,y,z)}\in\mathbb{Q}(\varepsilon,n,x,y,z)$$

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The integrand is

- hyperexponential in x,y,z:

$$\frac{D_y f(\varepsilon,n,x,y,z)}{f(\varepsilon,n,x,y,z)}\in\mathbb{Q}(\varepsilon,n,x,y,z)$$

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The integrand is

- hyperexponential in x,y,z:

$$\frac{D_z f(\varepsilon,n,x,y,z)}{f(\varepsilon,n,x,y,z)}\in\mathbb{Q}(\varepsilon,n,x,y,z)$$

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$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

The integrand is

- hyperexponential in x,y,z:
- hypergeometric in n:

$$\frac{f(\varepsilon,n+1,x,y,z)}{f(\varepsilon,n,x,y,z)}\in\mathbb{Q}(\varepsilon,n,x,y,z)$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

$$\Big\downarrow \quad \text{Ablinger's MultiIntegrate.m (9 hours)}$$

$$a_0(\varepsilon,n)F(\varepsilon,n)+a_1(\varepsilon,n)F(\varepsilon,n+1)+\cdots+a_5(\varepsilon,n)F(\varepsilon,n+5)=0$$

$$
\begin{aligned}
a_0(n,\varepsilon) = {} & (n+1)(n+2)\big(8\varepsilon^{10} + 104\varepsilon^9(n+3) + 4\varepsilon^8\big(96n^2 + 601n + 887\big) \\
& + 4\varepsilon^7\big(12n^3 + 414n^2 + 1583n + 1393\big) \\
& - 8\varepsilon^6\big(264n^4 + 2436n^3 + 8643n^2 + 14518n + 9947\big) \\
& - 16\varepsilon^5\big(156n^5 + 1690n^4 + 6847n^3 + 12661n^2 + 9537n + 717\big) \\
& + 32\varepsilon^4\big(68n^6 + 1158n^5 + 8155n^4 + 30114n^3 + 61712n^2 + 67616n + 31693\big) \\
& + 64\varepsilon^3\big(40n^7 + 560n^6 + 2755n^5 + 3729n^4 - 14194n^3 - 61920n^2 - 89140n - 46600\big) \\
& - 128\varepsilon^2(n+2)\big(12n^7 + 254n^6 + 2249n^5 + 10758n^4 + 30173n^3 + 50610n^2 \\
& + 49122n + 22706\big) \\
& + 256\varepsilon(n+2)^2(n+3)(n+4)\big(44n^4 + 501n^3 + 2044n^2 + 3455n + 1976\big) \\
& - 512(n+1)(n+2)^3(n+3)^2(n+4)\big(6n^2 + 47n + 95\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_1(n,\varepsilon) = (n+2)\big(&-22\varepsilon^{11} - 2\varepsilon^{10}(157n+435) - \varepsilon^{9}\big(1500n^2+8611n+11745\big)\\
&- \varepsilon^{8}\big(2548n^3+22936n^2+63597n+54229\big)\\
&+ 4\varepsilon^{7}\big(266n^4+1857n^3+6065n^2+14351n+15987\big)\\
&+ 8\varepsilon^{6}\big(994n^5+12961n^4+67246n^3+174692n^2+226821n+116092\big)\\
&+ 16\varepsilon^{5}\big(336n^6+5348n^5+33569n^4+104918n^3+165290n^2+108259n+6100\big)\\
&- 16\varepsilon^{4}\big(404n^7+7578n^6+61778n^5+284762n^4+802660n^3+1382074n^2\\
&+ 1340455n+560287\big)\\
&- 64\varepsilon^{3}\big(94n^8+1823n^7+14305n^6+55870n^5+96299n^4-37256n^3\\
&- 447044n^2-704959n-379338\big)\\
&+ 128\varepsilon^{2}(n+3)\big(30n^8+715n^7+7667n^6+48253n^5+194086n^4+507439n^3\\
&+ 835393n^2+785327n+320382\big)\\
&- 256\varepsilon(n+2)(n+3)^2\big(107n^6+2070n^5+16342n^4+67226n^3+151557n^2\\
&+ 176932n+83196\big)\\
&+ 256(n+2)^3(n+3)^3(n+4)\big(30n^3+331n^2+1193n+1386\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_2(n,\varepsilon) = \big(&12\varepsilon^{12} + 12\varepsilon^{11}(17n+45) + 2\varepsilon^{10}\big(620n^2 + 3553n + 4795\big)\\
&+ 2\varepsilon^9\big(1504n^3 + 14190n^2 + 41901n + 38907\big)\\
&+ 4\varepsilon^8\big(172n^4 + 4983n^3 + 30942n^2 + 69119n + 50850\big)\\
&- 4\varepsilon^7\big(1996n^5 + 24056n^4 + 113313n^3 + 269119n^2 + 337198n + 185290\big)\\
&- 16\varepsilon^6\big(450n^6 + 8210n^5 + 59749n^4 + 227386n^3 + 486841n^2 + 563176n + 275664\big)\\
&+ 16\varepsilon^5\big(340n^7 + 4314n^6 + 19137n^5 + 25532n^4 - 55105n^3 - 206516n^2 - 191528n\\
&- 23458\big)\\
&+ 32\varepsilon^4\big(140n^8 + 2940n^7 + 26550n^6 + 139926n^5 + 493839n^4 + 1240186n^3\\
&+ 2161699n^2 + 2304248n + 1100084\big)\\
&+ 64\varepsilon^3\big(4n^9 + 506n^8 + 8651n^7 + 63510n^6 + 236215n^5 + 395334n^4 - 105413n^3\\
&- 1551017n^2 - 2362944n - 1217770\big)\\
&- 128\varepsilon^2(n+3)\big(12n^9 + 314n^8 + 3782n^7 + 29105n^6 + 160727n^5 + 640273n^4\\
&+ 1750874n^3 + 3052505n^2 + 3017094n + 1276604\big)\\
&+ 256\varepsilon(n+2)(n+3)^2(n+4)\big(26n^6 + 825n^5 + 8967n^4 + 46529n^3 + 125411n^2\\
&+ 168628n + 88652\big)\\
&- 512(n+1)(n+2)^2(n+3)^3(n+4)^2\big(6n^3 + 98n^2 + 459n + 655\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_3(n,\varepsilon) = \big(& -64\varepsilon^{12} - 8\varepsilon^{11}(113n+298) - 8\varepsilon^{10}\big(519n^2+2948n+3896\big) \\
& - 4\varepsilon^9\big(1444n^3+13839n^2+39746n+34305\big) \\
& + 4\varepsilon^8\big(1948n^4+17868n^3+63837n^2+112966n+84655\big) \\
& + 16\varepsilon^7\big(1456n^5+20460n^4+112365n^3+304963n^2+412258n+221769\big) \\
& - 8\varepsilon^6\big(320n^6+2050n^5+4192n^4+27408n^3+174901n^2+411759n+324872\big) \\
& - 16\varepsilon^5\big(1756n^7+33154n^6+265889n^5+1186719n^4+3218059n^3+5349388n^2 \\
& + 5071913n+2113696\big) \\
& + 32\varepsilon^4\big(188n^8+4802n^7+59527n^6+439922n^5+2025336n^4+5813984n^3 \\
& + 10076450n^2+9621283n+3878602\big) \\
& + 64\varepsilon^3\big(140n^9+2768n^8+22500n^7+99545n^6+287700n^5+723136n^4 \\
& + 1854572n^3+3714620n^2+4272517n+2031600\big) \\
& - 128\varepsilon^2\big(24n^{10}+830n^9+14362n^8+152630n^7+1053620n^6+4834279n^5 \\
& + 14824351n^4+29964399n^3+38244797n^2+27875896n+8824032\big) \\
& + 256\varepsilon(n+2)(n+3)(n+4)\big(118n^7+2639n^6+24247n^5+118311n^4+329565n^3 \\
& + 520306n^2+426076n+136854\big) \\
& - 512(n+1)(n+2)^2(n+3)^2(n+4)^2(n+5)\big(12n^3+97n^2+230n+144\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_4(n,\varepsilon) = \big(&64\varepsilon^{12} + 192\varepsilon^{11}(5n+14) + 16\varepsilon^{10}\big(297n^2 + 1769n + 2451\big)\\
&+ 16\varepsilon^9\big(453n^3 + 4462n^2 + 13094n + 11244\big)\\
&- 8\varepsilon^8\big(1084n^4 + 11117n^3 + 47258n^2 + 103981n + 94650\big)\\
&- 8\varepsilon^7\big(3304n^5 + 51138n^4 + 311957n^3 + 948722n^2 + 1440105n + 858544\big)\\
&+ 16\varepsilon^6\big(420n^6 + 5507n^5 + 36275n^4 + 169650n^3 + 536911n^2 + 952507n + 694370\big)\\
&+ 16\varepsilon^5\big(1828n^7 + 38868n^6 + 353301n^5 + 1801014n^4 + 5604391n^3 + 10664390n^2\\
&+ 11433064n + 5260048\big)\\
&- 32\varepsilon^4\big(316n^8 + 8356n^7 + 105800n^6 + 802421n^5 + 3836854n^4 + 11588223n^3\\
&+ 21401558n^2 + 22066744n + 9745752\big)\\
&- 64\varepsilon^3\big(116n^9 + 2424n^8 + 19923n^7 + 82966n^6 + 208191n^5 + 530980n^4 + 1847484n^3\\
&+ 4687014n^2 + 6120858n + 3111104\big)\\
&+ 128\varepsilon^2\big(24n^{10} + 826n^9 + 14897n^8 + 172000n^7 + 1314686n^6 + 6710299n^5\\
&+ 22873183n^4 + 51298261n^3 + 72551278n^2 + 58573022n + 20544948\big)\\
&- 256\varepsilon(n+2)(n+3)\big(106n^8 + 3278n^7 + 42903n^6 + 310942n^5 + 1366350n^4\\
&+ 3729418n^3 + 6173159n^2 + 5657732n + 2191212\big)\\
&+ 512(n+1)(n+2)^2(n+3)^2(n+4)(n+5)(n+6)\big(12n^3 + 121n^2 + 396n + 431\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_5(n,\varepsilon) = (n+5)\big(&-128\varepsilon^{11} - 128\varepsilon^{10}(11n+26) - 32\varepsilon^9\big(115n^2+592n+647\big)\\
&+ 32\varepsilon^8\big(63n^3+430n^2+1665n+2384\big)\\
&+ 16\varepsilon^7\big(714n^4+7881n^3+33802n^2+66225n+47654\big)\\
&- 16\varepsilon^6\big(234n^5+2444n^4+13989n^3+50862n^2+104083n+87848\big)\\
&- 16\varepsilon^5\big(580n^6+10181n^5+76586n^4+319207n^3+772120n^2+1012046n+547832\big)\\
&+ 16\varepsilon^4\big(244n^7+5456n^6+61605n^5+401216n^4+1536277n^3+3408574n^2\\
&+ 4066436n+2026928\big)\\
&+ 64\varepsilon^3\big(26n^8+357n^7+583n^6-11139n^5-65193n^4-120264n^3+11864n^2\\
&+ 272830n+222624\big)\\
&- 64\varepsilon^2(n+3)\big(12n^8+298n^7+4684n^6+49024n^5+306907n^4+1122441n^3\\
&+ 2350650n^2+2607576n+1185072\big)\\
&+ 256\varepsilon(n+2)(n+3)\big(25n^7+743n^6+8856n^5+55358n^4+197497n^3+404131n^2\\
&+ 439902n+196128\big)\\
&- 256(n+1)(n+2)^2(n+3)^2(n+4)(n+6)(n+7)\big(6n^2+35n+54\big)\big).
\end{aligned}
$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

Ablinger's MultiIntegrate.m $\Big\downarrow$ (9 hours)

$$a_0(\varepsilon,n)F(\varepsilon,n)+a_1(\varepsilon,n)F(\varepsilon,n+1)+\cdots+a_5(\varepsilon,n)F(\varepsilon,n+5)=0$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

Ablinger's MultiIntegrate.m $\downarrow$ (9 hours)

$$a_0(\varepsilon,n)F(\varepsilon,n)+a_1(\varepsilon,n)F(\varepsilon,n+1)+\cdots+a_5(\varepsilon,n)F(\varepsilon,n+5)=0$$

`Sigma.m` $\downarrow$ (2 hours)

$$F(\varepsilon,n)=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+\cdots+F_4(n)\varepsilon^4+O(\varepsilon^5)$$

We get
$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$

We get

$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

$$F_{-2}(n) = -\frac{4(-1)^n\left(3n^3+18n^2+31n+18\right)}{3(n+1)^3(n+2)^2} - \frac{4\left(6n^3+32n^2+51n+26\right)}{3(n+1)^3(n+2)^2}$$

We get

$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

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$$\begin{aligned} F_{-1}(n) &= (-1)^n\left(\frac{2\left(9n^5+81n^4+295n^3+533n^2+500n+204\right)}{3(n+1)^4(n+2)^3} + \frac{\zeta_2}{(n+1)(n+2)}\right) \\ &+ \frac{2\left(18n^5+150n^4+490n^3+755n^2+536n+132\right)}{3(n+1)^4(n+2)^3} + \frac{(2n+3)\zeta_2}{(n+1)^2(n+2)} \\ &+ \left(-\frac{4}{(n+1)^2(n+2)} + \frac{4(-1)^n}{(n+1)(n+2)}\right)S_2(n) \\ &+ \left(\frac{4(-1)^n}{3(n+1)(n+2)} - \frac{4(n+9)}{3(n+1)^2(n+2)}\right)S_{-2}(n) \end{aligned}$$

Find a recurrence for the integral

$$F(n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$

$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

ε-recurrence solver

multivariate
Almquist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon, n)F(n) + \dots + a_d(\varepsilon, n)F(n + d) = h(\varepsilon, n)$$

Find a recurrence for the integral

$$F(n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
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ε-recurrence solver

multivariate
Almquist/Zeilberger
(J. Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \ldots, i_7)$$

Wegschaider's MultiSum
Package (F. Stan)

$$a_0(\varepsilon, n)F(n) + \ldots + a_d(\varepsilon, n)F(n+d) = h(\varepsilon, n)$$

Find a recurrence for the integral

$$F(n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \ldots$$

ε-recurrence solver

multivariate
Almquist/Zeilberger
(J. Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \ldots, i_7)$$

Wegschaider's MultiSum
Package (F. Stan)

Holonomic/difference ring
approach (M. Round)

$$a_0(\varepsilon, n)F(n) + \ldots + a_d(\varepsilon, n)F(n+d) = h(\varepsilon, n)$$

The general tactic

Feynman integrals

↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

Tactic 2: Expand a recurrence in ε

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656 [cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

Calculations based on Tactic 1 and 2:

- I. Bierenbaum, J. Blümlein, S. Klein, and CS. Two–Loop Massive Operator Matrix Elements for Unpolarized Heavy Flavor Production to $O(\epsilon)$. *Nucl.Phys. B* 803(1-2):1–41, 2008.
- J. Ablinger, J. Blümlein, S. Klein, C. Schneider. Modern Summation Methods and the Computation of 2- and 3-loop Feynman Diagrams. Nucl. Phys. B (Proc. Suppl.) 205-206, pp. 110-115, 2010.
- J. Ablinger, I. Bierenbaum, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider, F. Wissbrock. Heavy Flavor DIS Wilson coefficients in the asymptotic regime. Nucl. Phys. B (Proc. Suppl.) 205-206, pp. 242-249, 2010.
- J. Ablinger, J. Blümlein, S. Klein, CS, F. Wissbrock. The $O(\alpha_s^3)$ Massive Operator Matrix Elements of $O(n_f)$ for the Structure Function $F_2(x, Q^2)$ and Transversity. Nucl. Phys. B, 844: 26-54, 2011.
- J. Ablinger, J. Blümlein, A. Hasselhuhn, S. Klein, CS, F. Wissbrock Massive 3-loop Ladder Diagrams for Quarkonic Local Operator Matrix Elements. Nuclear Physics B. 864: 52-84, 2012.
- J. Blümlein, A. Hasselhuhn, S. Klein, CS. The $O(\alpha_s^3 n_f T_F^2 C_{A,F})$ Contributions to the Gluonic Massive Operator Matrix Elements. Nuclear Physics B: 866: 196-211, 2013.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, CS, F. Wißbrock. Three Loop Massive Operator Matrix Elements and Asymptotic Wilson Coefficients with Two Different Masses. Nucl. Phys. B. 921, pp. 585-688. 2017.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. The Two-mass Contribution to the Three-Loop Gluonic Operator Matrix Element $A_{gg,Q}^{(3)}$. Nucl. Phys. B 932, pp. 129-240. 2018.
- J. Ablinger, J. Blümlein, A. De Freitas, M. Saragnese, CS, K. Schönwald. The three-loop polarized pure singlet operator matrix element with two different masses. Nuclear Physics B 952(114916), pp. 1-18. 2020.

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- I. Bierenbaum, J. Blümlein, S. Klein, and CS. Two–Loop Massive Operator Matrix Elements for Unpolarized Heavy Flavor Production to $O(\epsilon)$. *Nucl.Phys. B* 803(1-2):1–41, 2008.
- J. Ablinger, J. Blümlein, S. Klein, C. Schneider. Modern Summation Methods and the Computation of 2- and 3-loop Feynman Diagrams. Nucl. Phys. B (Proc. Suppl.) 205-206, pp. 110-115, 2010.
- J. Ablinger, I. Bierenbaum, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider, F. Wissbrock. Heavy Flavor DIS Wilson coefficients in the asymptotic regime. Nucl. Phys. B (Proc. Suppl.) 205-206, pp. 242-249, 2010.
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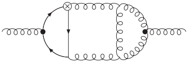
An alternative approach is the method of hyperlogarithms (see, e.g., F. Brown/E. Panzer with his HyperInt). We used an extension of it in

- J. Ablinger, J. Blümlein, C. Raab, CS, F. Wissbrock. Calculating Massive 3-loop Graphs for Operator Matrix Elements by the Method of Hyperlogarithms. Nuclear Physics B 885, pp. 409-447. 2014.

Evaluation of Feynman Integrals

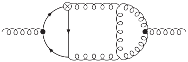


Behavior of particles

$$\int \Phi(n, \epsilon, x)dx$$

Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(n, \epsilon, x) dx$$

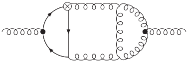
Feynman integrals

DESY

$$Dy = A\,y$$

coupled systems of linear DEs

Evaluation of Feynman Integrals



Behavior of particles → $\int \Phi(n,\epsilon,x)dx$

Feynman integrals

↓ **DESY**

$$Dy = A\,y$$

coupled systems of linear DEs

← **RISC** (coupled system solver)

expression in special functions

Tactic 3: Solve coupled systems of differential equations

[coming, e.g., from IBP methods]

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x),\dots,\hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x),\dots,\hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix}\hat{I}_1(x)\\ \dots \\ \hat{I}_\lambda(x)\end{pmatrix} = A(x) \begin{pmatrix}\hat{I}_1(x)\\ \dots \\ \hat{I}_\lambda(x)\end{pmatrix} + \begin{pmatrix}\hat{R}_1(x)\\ \dots \\ \hat{R}_\lambda(x)\end{pmatrix}$$

given

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x),\dots,\hat{R}_\lambda(x)$ (in terms of special functions)
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given

A whole industry (for solutions of ε-expansions) started with

[J. Henn. Multiloop integrals in dimen- sional regularization made simple. Phys. Rev. Lett., 110:251601, 2013.]

Here we follow another successful tactic.

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x),\dots,\hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x),\dots,\hat{I}_\lambda(x)$ (for given initial values) s.t.

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$\Big\downarrow$ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

$\downarrow$ uncoupling algorithms
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2. For $i = 2, \dots, r$ we get

$$\hat{I}_i(x) = \mathsf{LinComb}(\hat{I}_1(x), \dots, D_x^{\lambda-1}\hat{I}_1(x)) + \mathsf{LinComb}(\dots, D^i\hat{R}_i(x), \dots)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

↓ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

DE-solver

A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x)f(x) + \cdots + b_\lambda(x)D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x),\dots,b_\lambda(x)\in\mathbb{K}[x]$

$$b_0(x)f(x)+\cdots+b_\lambda(x)D^\lambda f(x)=0;$$

together with initial values $f(0),\dots,D^{\lambda-1}f(x)|_{x=0}\in\mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

A differential equation solver (`HarmonicSums.m`)

GIVEN a linear differential equation $b_0(x),\dots,b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x)f(x)+\dots+b_\lambda(x)D^\lambda f(x)=0;$$

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DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

Special cases of iterated integrals over hyperexponential functions:

$$H_{1,-1}(x)=\int_0^x \frac{1}{1-\tau_1}\int_0^{\tau_1}\frac{1}{1+\tau_2}d\tau_2 d\tau_1$$ (harmonic polylogarithms)

E. Remiddi, E. and J.A.M. Vermaseren, Int. J. Mod. Phys. **A15** (2000) [arXiv:hep-ph/9905237]

A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x)f(x) + \cdots + b_\lambda(x)D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

Special cases of iterated integrals over hyperexponential functions:

$$H_{2,-2}(x) = \int_0^x \frac{1}{2-\tau_1} \int_0^{\tau_1} \frac{1}{2+\tau_2} d\tau_2 d\tau_1$$ (generalized polylogarithms)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x) f(x) + \cdots + b_\lambda(x) D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1} f(x)|_{x=0} \in \mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

Special cases of iterated integrals over hyperexponential functions:

$$\int_0^x \frac{1}{1+\tau_1+\tau_1^2} \int_0^{\tau_1} \frac{1}{1+\tau_2^2} d\tau_2 d\tau_1$$ (cyclotomic polylogarithms)

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

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together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

Special cases of iterated integrals over hyperexponential functions:

$$\int_0^x \frac{1}{\sqrt{1+\tau_1}} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1 \qquad \text{(radical integrals)}$$

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

A differential equation solver (HarmonicSums.m)

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DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

Special cases of iterated integrals over hyperexponential functions:

$$\int_0^x \frac{1}{1-\tau_1+\eta\tau_1} \int_0^{\tau_1} \sqrt{1-\tau_2}\sqrt{1-\tau_2+\eta\tau_2}\, d\tau_2 d\tau_1$$

(generalized radical integrals)

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].
J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

A differential equation solver (HarmonicSums.m)

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$$b_0(x)f(x) + \dots + b_\lambda(x)D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

A more general example:

$$\int_0^x e^{\int_1^{\tau_1} \frac{1}{1+y+y^2}dy} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1$$

`HarmonicSums` can also deal with Liouvillian solutions (i.e., it contains Kovacic's algorithm):

$$(11+20x)f'(x)+(1+x)(35+134x)f''(x) + 3(1+x)^2(4+37x)f^{(3)}(x)+18x(1+x)^3f^{(4)}(x)=0$$

$$\downarrow$$

$$\Big\{c_1+c_2\int_0^x\frac{1}{1+\tau_1}\,d\tau_1+c_3\int_0^x\frac{1}{1+\tau_1}\int_0^{\tau_1}\frac{\sqrt[3]{1+\sqrt{1+\tau_2}}}{1+\tau_2}d\tau_2 d\tau_1 + c_4\int_0^x\frac{1}{1+\tau_1}\int_0^{\tau_1}\frac{\sqrt[3]{1-\sqrt{1+\tau_2}}}{1+\tau_2}d\tau_2 d\tau_1 \mid c_1,c_2,c_3,c_4\in\mathbb{K}\Big\}$$

Connection: DE $\longleftrightarrow$ REC

Let

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

be a (formal) power series. Then:

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$$b_0(x) f(x) + \cdots + b_\lambda(x) D^\lambda f(x) = 0.$$

$$\Updownarrow$$

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$\Big\Updownarrow$ algorithmic

There exist $a_0(x), \dots, a_\delta(x) \in \mathbb{K}[x]$ with

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Example 1: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

for

$$-\left(x^4 - 64x^3\right) f^{(4)}(x) - 2\left(5x^3 - 144x^2\right) f^{(3)}(x)$$
$$- \left(25x^2 - 208x\right) f''(x) - (15x - 8) f'(x) - f(x) = 0$$

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$$F(n) = \frac{1}{\binom{2n}{n}^3} = \frac{(1)_n (1)_n (1)_n (1)_n}{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n n!} \frac{1}{64^n}$$

Example 1: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{\binom{2n}{n}^3} = {}_4F_3\left[\begin{matrix}1,1,1,1\\ \frac{1}{2},\frac{1}{2},\frac{1}{2}\end{matrix};\frac{x}{64}\right]$$

for

$$-\left(x^4-64x^3\right)f^{(4)}(x) - 2\left(5x^3-144x^2\right)f^{(3)}(x) - \left(25x^2-208x\right)f''(x) - (15x-8)f'(x) - f(x) = 0$$

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Example 2: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

for

$$\begin{aligned}\left(x^6 - 32x^5 + 256x^4\right) f^{(6)}(x) + \left(23x^5 - 528x^4 + 2560x^3\right) f^{(5)}(x) \\ + \left(171x^4 - 2552x^3 + 6272x^2\right) f^{(4)}(x) + 2\left(245x^3 - 2002x^2 + 1728x\right) f^{(3)}(x) \\ + 2\left(253x^2 - 786x + 72\right) f''(x) + 4(35x - 12)f'(x) + 4f(x) = 0\end{aligned}$$

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$$(n+2)(n+1)^3 F(n) - 4(n+2)(2n+1)^2(2n+3)F(n+1) + 16(2n+1)^2(2n+3)^2 F(n+2) = 0$$

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$$\Big\downarrow \texttt{Sigma.m}$$

$$F(n)=\frac{1}{\binom{2n}{n}^2}\Big(c_1+c_2S_1(n)\Big)=\frac{(1)_n(1)_n(1)_n}{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n n!}\frac{1}{16^n}\Big(c_1+c_2S_1(n)\Big)$$

Example 2: Find a power series solution

$$f(x) = c_1 \cdot {}_3F_2\left[\begin{matrix}1,1,1\\ \frac{1}{2},\frac{1}{2}\end{matrix};\frac{x}{16}\right] + c_2 \sum_{n=0}^{\infty} \frac{S_1(n)}{\binom{2n}{n}^2} x^n$$

for

$$\begin{aligned}&\left(x^6-32x^5+256x^4\right)f^{(6)}(x)+\left(23x^5-528x^4+2560x^3\right)f^{(5)}(x)\\&+\left(171x^4-2552x^3+6272x^2\right)f^{(4)}(x)+2\left(245x^3-2002x^2+1728x\right)f^{(3)}(x)\\&+2\left(253x^2-786x+72\right)f''(x)+4(35x-12)f'(x)+4f(x)=0\end{aligned}$$

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Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

$\downarrow$ uncoupling algorithms (Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

DE-solver

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DE-solver REC-solver

Tactic 3: the DE-REC approach

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

Tactic 3: the DE-REC approach

DE system	OreSys package (S. Gerhold) → uncoupling algorithm	uncoupled DE system
$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$		$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$ $\hat{I}_k(x) = \mathsf{expr}_k(\hat{I}_1(x)), k > 1$

Tactic 3: the DE-REC approach

DE system	OreSys package (S. Gerhold) $\longrightarrow$ uncoupling algorithm	uncoupled DE system
$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$		$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$ $\hat{I}_k(x) = \mathsf{expr}_k(\hat{I}_1(x)), k > 1$

$$\hat{I}_1(x) = \sum_{n=0}^{\infty} I_1(n) x^n$$

Tactic 3: the DE-REC approach

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)

→ uncoupling algorithm →

uncoupled DE system

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↓ holonomic closure prop. ↓

linear recurrence

$$\sum_i a'_i(n) I_1(n+i) = r'(n)$$

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linear recurrence

$$\sum_i a'_i(n) I_1(n+i) = r'(n)$$

← `Sigma`'s REC-solver ←

closed form solutions of $I_1(n)$ in the class of nested sums and products – if this is possible

Tactic 3: the DE-REC approach (SolveCoupledSystem package)

DE system
$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

→ OreSys package (S. Gerhold), uncoupling algorithm →

uncoupled DE system
$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$
$$\mathbf{\hat{I}_k(x)} = \text{expr}_{\mathbf{k}}(\mathbf{\hat{I}_1(x)}), \mathbf{k > 1}$$

↓ holonomic closure prop.

linear recurrence
$$\sum_i a'_i(n) I_1(n+i) = r'(n)$$

← `Sigma's` REC-solver ←

closed form solutions of
$I_1(n), \mathbf{I_2(n)}, \dots, \mathbf{I_n(n)}$
in the class of nested sums and products
– if this is possible

(from uncoupled DE system to closed form solutions:) **extract coefficient**
SumProduction package

J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
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General strategy: physical problem $\hat{P}(x)$

$\downarrow$ IBP methods

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$\downarrow$ plug into $\hat{P}(x) = \sum_{n=0}^{\infty} P(n) x^n$

$$P(n) = \varepsilon^{-3} P_{-3}(n) + \varepsilon^{-2} P_{-2}(n) + \varepsilon^{-1} P_{-1}(n) + \varepsilon^{0} P_0(n) + \ldots$$

Calculations based on Tactic 3:

- J. Ablinger, J. Blümlein, A. De Freitas A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The Transition Matrix Element $A_{gq}(n)$ of the Variable Flavor number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, pp. 263-288. 2014.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. Nuclear Physics B 885, pp. 280-317. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The 3-Loop non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity. Nuclear Physics B 886, pp. 733-823. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x, Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, pp. 48-151. 2015.
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- A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, CS. The $O(\alpha_s^3)$ Heavy Flavor Contributions to the Charged Current Structure Function $xF_3(x, Q^2)$ at Large Momentum Transfer. Physical Review D 92(114005), pp. 1-19. 2015.
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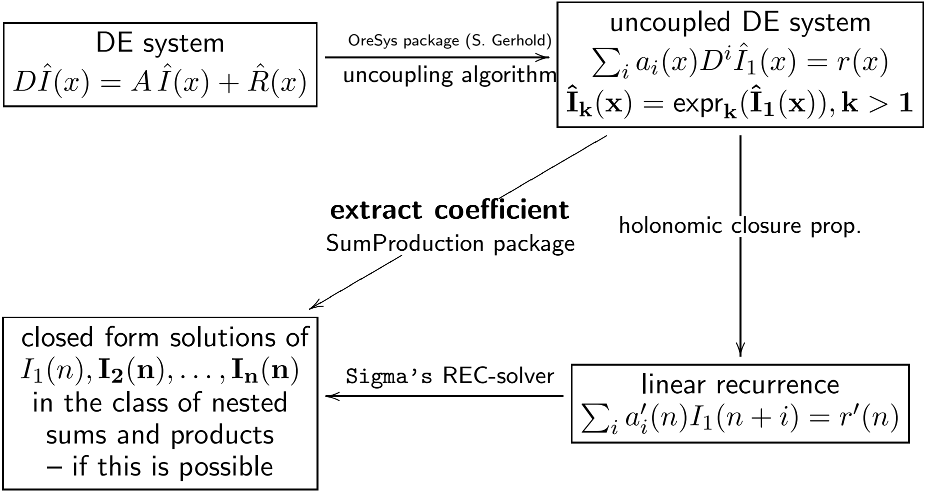
- J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, n. Rana, CS. The Heavy Quark Form Factors at Two Loops. Physical Review D 97(094022), pp. 1-44. 2018.
- J. Ablinger, J. Blümlein, A. De Freitas, CS, K. Schönwald. The two-mass contribution to the three-loop pure singlet operator matrix element. Nucl. Phys. B(927), pp. 339-367. 2018. ISSn 0550-3213.
- J. Blümlein, A. De Freitas, CS, K. Schönwald. The Variable Flavor number Scheme at next-to-Leading Order. Physics Letters B 782, pp. 362-366. 2018.
- J. Ablinger, J. Blümlein, P. Marquard, n. Rana, CS. Heavy Quark Form Factors at Three Loops in the Planar Limit. Physics Letters B 782, pp. 528-532. 2018.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Goedicke, A. von Manteuffel, CS, K. Schönwald. The Unpolarized and Polarized Single-Mass Three-Loop Heavy Flavor Operator Matrix Elements $A_{gg,Q}$ and $\Delta A_{gg,Q}$. Journal of High Energy Physics 2022(12), pp. 1-55. 2022.

Tactic 4: Compute large moments and guessing recurrences

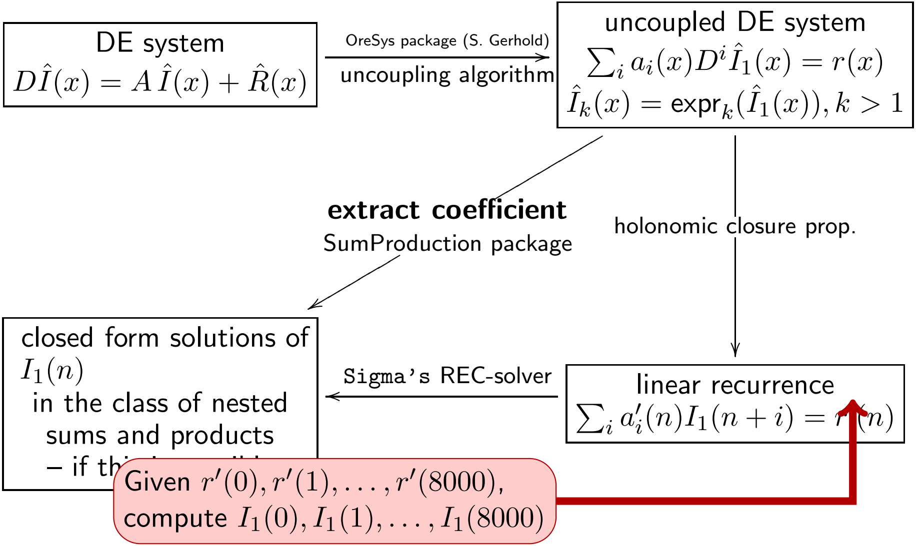
[coming, e.g., from IBP methods]

Tactic 3: the DE-REC approach (SolveCoupledSystem package)



J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

Tactic 4: compute large moments (SolveCoupledSystem package)



J. Blümlein, CS, 2017, arXiv:1701.04614 [hep-ph].

General strategy: physical problem $\hat{P}(x)$

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$n = 0, 1, \dots, 8000$

$\big\downarrow$ plug into $\hat{P}(x) = \sum_{n=0}^{\infty} P(n) x^n$

$$P(n) = \underbrace{\varepsilon^{-3} P_{-3}(n) + \varepsilon^{-2} P_{-2}(n) + \varepsilon^{-1} P_{-1}(n)}_{\text{numbers}} + \underbrace{\varepsilon^{0} P_{0}(n)}_{\text{numbers}} + \dots$$

$n = 0, 1, \dots, 8000$

coupled systems for $P(n)$

`SolveCoupledSystem.m` ↓

large no. of moments

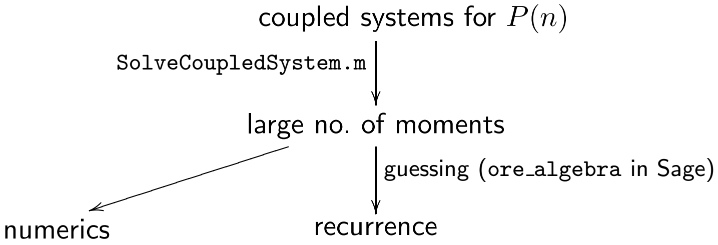
coupled systems for $P(n)$

↓ `SolveCoupledSystem.m`

large no. of moments

↙

numerics


coupled systems for $P(n)$
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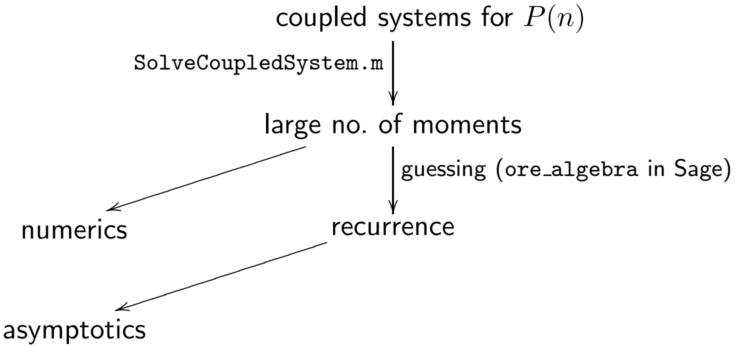
guessing (ore_algebra in Sage)
numerics
recurrence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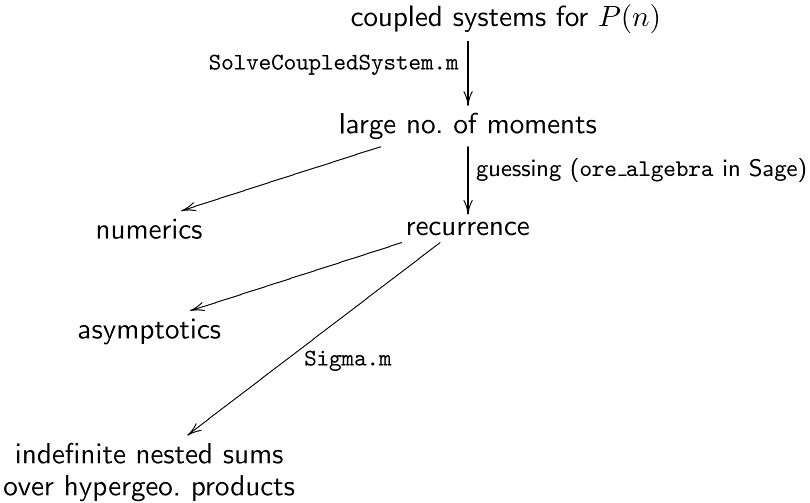
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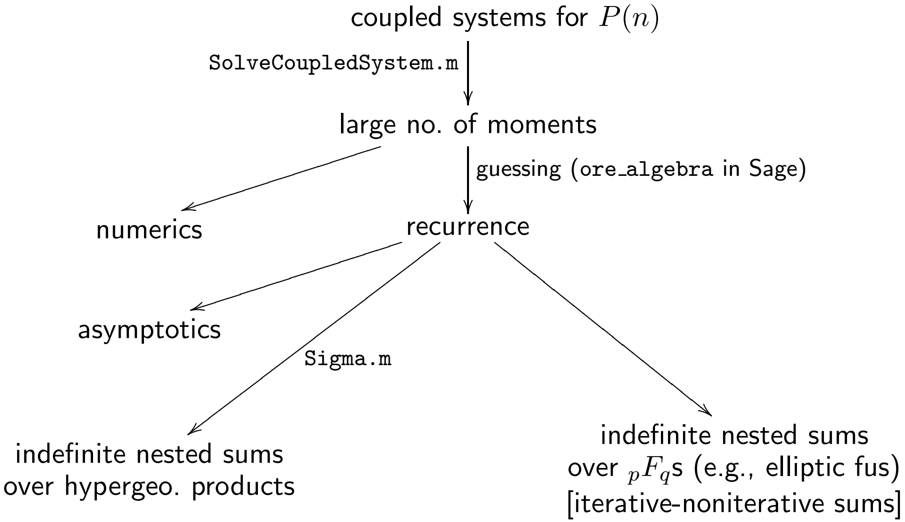
over hypergeo. products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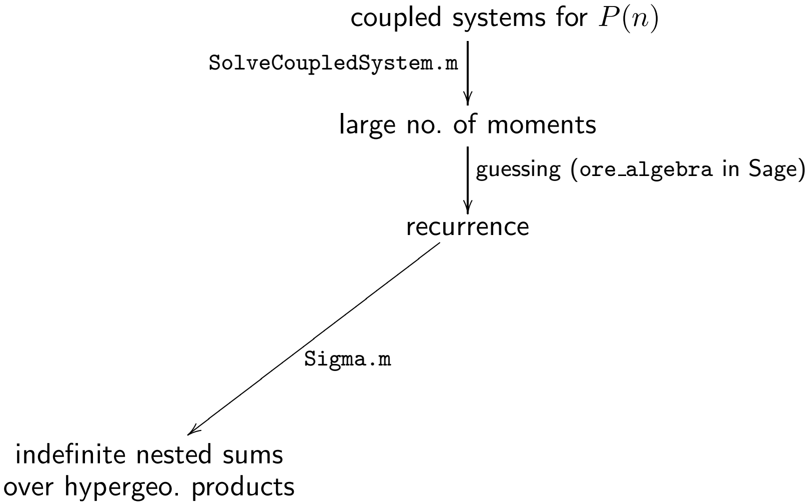
indefinite nested sums
over ${}_pF_q$s (e.g., elliptic fus)
[iterative-noniterative sums]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$n = 0, 1, \dots, 8000$

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$\downarrow$ IBP methods

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$

$\downarrow$ solver for $\hat{I}_i(x) = \sum_{n=0}^{\infty} I_i(n) x^n$

$$I_i(n) = \underbrace{\varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \varepsilon^{-1} F_{-1}(n) + \varepsilon^{0} F_{0}(n) + \ldots}_{\text{only numbers in } \mathbb{Q}}$$

$\downarrow$ plug into $\hat{P}(x) = \sum_{n=0}^{\infty} P(n) x^n$

$$P(n) = \underbrace{\varepsilon^{-3} P_{-3}(n) + \varepsilon^{-2} P_{-2}(n) + \varepsilon^{-1} P_{-1}(n)}_{\text{nice}} + \underbrace{\varepsilon^{0} P_{0}(n)}_{\text{partially nice}} + \ldots$$

all n solution

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

In[8]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[9]:= **initial** $=<<$ **iFile16**

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

In[8]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[9]:= **initial =<< iFile16**

Out[9]= {37, 34577/1296, 7598833/151875, 13675395569/230496000,

475840076183/7501410000, 1432950323678333/21965628762000,

21648380901382517/328583783127600,

52869784323778576751/802218994536960000,

49422862094045523994231/753773992230616156800,

33131879832907935920726113/509557943985299969760000,

5209274721836755168448777/80949984111854180459136,

56143711997344769021041145213/882589266383586456384353664,

453500433353845628194790025124807/7217228048879468556886950000000,

14061543374120479886110159898869387/226643167590350326435656036000000,

7155865226664919033249057851786199365711683703077002228078114958950300000000,

16286729046359273892841271257418854056836413/269396588055480390401343344736943104000000,

1428729642632302467951426905844691837805299/23940759575034122827861315961573673600000,

498938690219595294505102809199154550783080767/8468883667852979813171262304054002720000000, In[10]:= **rec =<< rFile16**

Out[10]=

$$
\begin{aligned}
&(n+1)^4(n+2)^2(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)\Big(309237645312n^{32} + 38256884318208n^{31} + \\
&2282100271087616n^{30} + 87428170197762048n^{29} + 2417273990256001024n^{28} + 51388547929265405952n^{27} + \\
&873862324676687036416n^{26} + 12209268055143308328960n^{25} + 142860861222820240162816n^{24} + \\
&1419883954103469621510144n^{23} + 12115561235109256405319680n^{22} + 89479384946084038000803840n^{21} + \\
&575561340618928527623274496n^{20} + 3239547818363227419971647488n^{19} + 16009805333085271423330779136n^{18} + \\
&69631814641718655426881659392n^{17} + 266892117418348771052573667328n^{16} + \\
&901901113782416884441719270144n^{15} + 2685821385767154471801366647296n^{14} + \\
&7038702625583766161604414471744n^{13} + 16195069575749412648646633248128n^{12} + \\
&32602540883321212533013752639288n^{11} + 57154680141624618025310553466704n^{10} + \\
&86710462147941775492301231896818n^{9} + 112917328975807075881545543668548n^{8} + \\
&124873767581470867343743078943772n^{7} + 115624836314544572769501784072647n^{6} + \\
&87938536330971046886456627610048n^{5} + 53481897815980319933589323279298n^{4} + \\
&25000430622737750756669804052204n^{3} + 8430930497463933665464836129855n^{2} + \\
&1825177817831282261293155379650n + 190428196025667395685609855000\Big)(2n+1)^4 P[n]
\end{aligned}
$$

$$
\begin{aligned}
&-(n+2)^3(2n+3)^3(2n+7)(2n+9)(2n+11)\Big(12369505812480n^{38} + 1613151061671936n^{37} + \\
&101748284195864576n^{36} + 4135139115563745280n^{35} + 121713599527855849472n^{34} + \\
&2765050919624810430464n^{33} + 50453046277771391664128n^{32} + 759760507477065230974976n^{31} + \\
&9628262076527899425374208n^{30} + 104191253579306374131613696n^{29} + 973595596739520084325171200n^{28} + \\
&7924537790312611436520013824n^{27} + 56571687381518195331462463488n^{26} + \\
&356133102136059681954436399104n^{25} + 1985507231916669869451824553984n^{24} + \\
&9836060321685410187563260035072n^{23} + 43406506634905372676489415905280n^{22} + \\
&170945808151999530921656848106496n^{21} + 601507760131008511164113355409920n^{20} + \\
&1892149418896523531194676203153920n^{19} + 5321173806292333448534132495165440n^{18} + \\
&13370912745727662541153592039812160n^{17} + 29987002021632029091547005084057760n^{16} + \\
&59921270253255984811455083696758912n^{15} + 106434458966741189159011567116493072n^{14} + \\
&167533688453539238956436945725341004n^{13} + 232781742346547554435545097479210510n^{12} + \\
&284125621128876904663642986868770746n^{11} + 302806836393712159148051277734975424n^{10} + \\
&279679164311116651162116055961513301n^{9} + 221781415386984655607595031093415136n^{8} + \\
&149214365004640710156345950062395186n^{7} + 83882523964213110328265187672574356n^{6} + \\
&38609679702395410742361774562392789n^{5} + 14149471988638475521561721269939086n^{4} + \\
&3963748138857399502678254252169734n^{3} + 795659668131014454843348852372480n^{2} + \\
&101701393436276172443717692853400n + 6204709909986751913151675960000\Big)\,P[n+1]
\end{aligned}
$$

$$+2(n+3)^2(2n+5)^3(2n+9)(2n+11)\Big(24739011624960n^{40}+3317836466356224n^{39}+215508170284466176n^{38}+9032884062187945984n^{37}+$$

$$274636134389959884800n^{36}+6455501959255126179840n^{35}+122094572934385260036096n^{34}+1909387225793663151898624n^{33}+$$

$$25180108291969215434326016n^{32}+284171960705270647479074816n^{31}+2775794400720227034854326272n^{30}+$$

$$23677622163992853854566219776n^{29}+177624312783583749157935120384n^{28}+1178515602115604757944201871360n^{27}+$$

$$6947091965313419323781358354432n^{26}+36515023100308314818702129258496n^{25}+171621148571344894953594594017280n^{24}+$$

$$722837793013976317556258102507520n^{23}+2732534027077907914497042720534528n^{22}+9281028665970648470895368668485120n^{21}+$$

$$28337819215557708948254385336117248n^{20}+77786125749274632150536464583130752n^{19}+191877161455672780973502244537632256n^{18}+$$

$$424953221702140663089937921965135648n^{17}+843818276409975584824720931649555264n^{16}+$$

$$1499359936674956711935311062995422344n^{15}+2378007025570977662661938772843220240n^{14}+$$

$$3355671771434535852147325502571953770n^{13}+4196375762867184563407432891655585484n^{12}+$$

$$4627675779563752366067861596232781096n^{11}+4473175960511956000526499430851993603n^{10}+$$

$$3761696365025837909581516781307249585n^{9}+2726553473467254373993685951699145492n^{8}+$$

$$1683383212304999468664293798012773485n^{7}+871926653651504419744271839781064837n^{6}+$$

$$371307437598003570058538796122994147n^{5}+126427972742886389602285855482966072n^{4}+33048762330145623969058704448697313n^{3}+$$

$$6217924746857741077419160100404560n^{2}+748298077423337427195946099994100n+43181089548034246077698611794000\Big)P[n+2]$$

$$-2(n+4)^2(2n+5)(2n+7)^3(2n+11)\Big(24739011624960n^{40}+3322784268681216n^{39}+216160919414112256n^{38}+9074528155284275200n^{37}+$$
$$276348048819456311296n^{36}+6506479077331107315712n^{35}+123266585640616142569472n^{34}+1931040885785102661976064n^{33}+$$
$$25510503383281445462081536n^{32}+288418124175428279391485952n^{31}+2822442799033603081019326464n^{30}+$$
$$24120717233320712351821332480n^{29}+181295944719289040999116701696n^{28}+1205246297785423925076555694080n^{27}+$$
$$7119049557560114436136213413888n^{26}+37496933571993839665392189775872n^{25}+176616172467048982234270428880896n^{24}+$$
$$745539218875020737621728364206080n^{23}+2824909633156578132652259733712896n^{22}+9618101958268071244680677589035520n^{21}+$$
$$29441860528446423517613263360742912n^{20}+81033563306363873505877563416477312n^{19}+200454769103641040142838133702338304n^{18}+$$
$$445286624972461749049425309485328992n^{17}+887028447418790661018847407251573152n^{16}+$$
$$1581538101499869694224895701784875304n^{15}+2517550244392724509968791166585362672n^{14}+$$
$$3566593026520465155504695877897282630n^{13}+4479066125207404898722179511912639638n^{12}+$$
$$4962006990874351800791769650243464872n^{11}+4819992643914265990647887896664485209n^{10}+$$
$$4074895386694182240941538222230233221n^{9}+2970477229398746689186622534784613554n^{8}+$$
$$1845274131994015990683957902602775337n^{7}+962091291302144537393228847830431614n^{6}+$$
$$412595107814836563208757757032740146n^{5}+141540723940232563767779647013785485n^{4}+37292931812630561528276365992452010n^{3}+$$
$$7074865777225416725452872895397100n^{2}+858794112392644074221312049837000n+49997386738260112603615104780000\Big)\mathsf{P}[n+3]$$

$$+(n+5)^3(2n+5)(2n+7)(2n+9)^4\Big(12369505812480n^{38} + 1546355730284544n^{37} + 93441851805138944n^{36} +$$
$$3636063211393908736n^{35} + 102413434086873890816n^{34} + 2225107112182077718528n^{33} +$$
$$38808234188348931964928n^{32} + 558299807912629375074304n^{31} + 6755648626273815474733056n^{30} +$$
$$69769132238801205785001984n^{29} + 621900006220029229458259968n^{28} + 4826558182244413850688946176n^{27} +$$
$$32840774268722977511855751168n^{26} + 196981883700048989849717882880n^{25} +$$
$$1046061529031136798450810839040n^{24} + 4934888224954929426023144030208n^{23} +$$
$$20735286278224836075286873214976n^{22} + 77745549200390911029444008457216n^{21} +$$
$$260448286122609254214904458392064n^{20} + 780087654447729149285799146869248n^{19} +$$
$$2089276462852113795051294249728512n^{18} + 5001455921015163002705347586646080n^{17} +$$
$$10691068512696184477385875851523744n^{16} + 20374769440121072185247660725156544n^{15} +$$
$$34542976501702600883669655947085712n^{14} + 51947527795197316142253213880200764n^{13} +$$
$$69039779136078090572935768218052854n^{12} + 80712286124402599779679594199103258n^{11} +$$
$$82519759833385882007812859351392458n^{10} + 73248127158607338722648198918322201n^{9} +$$
$$55935262205790259307904762197107653n^{8} + 36322355479155199114489624391144238n^{7} +$$
$$19756597118002557191991191826327042n^{6} + 8822212911433711339358062994077203n^{5} +$$
$$3145597282374650512689680780380605n^{4} + 859907105684964990690798899478888n^{3} +$$
$$168963309995629650025632011492580n^{2} + 21205680751316222158938757272000n +$$
$$1274120732351744651125603886400\Big)P[n+4]$$

$$-(n+5)^2(n+6)^4(2n+5)(2n+7)(2n+9)^3(2n+11)^4\Big(309237645312n^{32} + 28361279668224n^{31} + 1249518729297920n^{30} + 35220794552352768n^{29} + 713726163159089152n^{28} + 11076866026783113216n^{27} + 136959486138712588288n^{26} + 1385658801437173350400n^{25} + 11691772665924577918976n^{24} + 83438339505976242995200n^{23} + 508989054278115477684224n^{22} + 2675508113418826174332928n^{21} + 12193213796145039633072128n^{20} + 48399020537651722726242304n^{19} + 167881257973769248139515904n^{18} + 510012482113388176546187776n^{17} + 1358662126092561923541267968n^{16} + 3174925021159974655053814528n^{15} + 6504205668151125355938798848n^{14} + 11663792381020901870157176128n^{13} + 18263581057905911985340656960n^{12} + 24881010123632244515458585528n^{11} + 29346856353503020415409305704n^{10} + 29775859546803351930591002266n^{9} + 25770328899499991754425455738n^{8} + 18817114309842270306167785140n^{7} + 11424980760825630752861027739n^{6} + 5656051955667821083952617134n^{5} + 2221448212382554437709999491n^{4} + 664859653803075491350122060n^{3} + 142190920852333874895041748n^{2} + 19313175036907229252501700n + 1248723341516324359641600\Big)\,P[n+5]==0$$

In[11]:= $\mathbf{recSol = SolveRecurrence[rec, P[n]]}$

In[11]:= $\mathbf{recSol = SolveRecurrence[rec, P[n]]}$

Out[11]= $\{\{0, \frac{(3+2n)(3+4n)}{(1+n)^2(1+2n)^2}\}$

$\{0, -\frac{(3+2n)(-8-9n+2n^2)}{(1+n)^2(1+2n)^2}\}$

$\{0, -\frac{(3+2n)(-5+8n^2)}{2(1+n)^2(1+2n)^2} + \frac{(3+2n)\sum_{i=1}^{n}\frac{1}{i}}{(1+n)(1+2n)} + \frac{2(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\}$

$\{0, \frac{(3+2n)(-513-2184n-2416n^2+768n^4)}{2(1+n)^3(1+2n)^3} + \frac{14(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)} + \Big(-\frac{2(3+2n)(3+44n+48n^2)}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i} + \frac{12(3+2n)\big(\sum_{i=1}^{n}\frac{1}{i}\big)^2}{(1+n)(1+2n)} + \frac{56(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} - \frac{4(3+2n)(3+44n+48n^2)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)(1+2n)}\}$

$$\Big\{0,\ \frac{1}{16(1+n)^4(1+2n)^4}\big(72519+572343n+1814716n^2+2918100n^3+2442240n^4+912896n^5+24576n^6-49152n^7\big)+\frac{16(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{3(1+n)(1+2n)}+\Big(-\frac{(3+2n)\big(29+307n+322n^2\big)}{4(1+n)^2(1+2n)^2}+\frac{44(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i^2}+$$

$$\Big(\frac{(3+2n)\big(91+259n+974n^2+1784n^3+1024n^4\big)}{4(1+n)^3(1+2n)^3}+\frac{22(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)}+\frac{24(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}-\frac{4(3+2n)\big(-13-4n+16n^2\big)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2}+\frac{16(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i}+\Big(-$$

$$\frac{(3+2n)\big(19+92n+80n^2\big)}{(1+n)^2(1+2n)^2}+\frac{40(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\Big(\sum_{i=1}^{n}\frac{1}{i}\Big)^2+\frac{20(3+2n)\big(\sum_{i=1}^{n}\frac{1}{i}\big)^3}{3(1+n)(1+2n)}+$$

$$\frac{64(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{3(1+n)(1+2n)}-\frac{3(3+2n)\big(63+209n+150n^2\big)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)^2(1+2n)^2}+$$

$$\Big(\frac{(3+2n)\big(347+1795n+4302n^2+4856n^3+2048n^4\big)}{2(1+n)^3(1+2n)^3}+\frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{-1+2i}-$$

$$\frac{4(3+2n)\big(19+92n+80n^2\big)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)^2(1+2n)^2}+\frac{32(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^3}{3(1+n)(1+2n)}-$$

$$\frac{8(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)^2}{i}}{(1+n)(1+2n)}-\frac{16(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)^2}{-1+2i}}{(1+n)(1+2n)}$$

$$-\frac{32(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{(1+n)(1+2n)}+\frac{64(3+2n)\displaystyle\sum_{i=1}^{n}\frac{\Big(\sum_{j=1}^{i}\frac{1}{j}\Big)\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}}{(1+n)(1+2n)}+$$

$$\frac{32(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{-1+2j}\big)^2}{i}}{(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{-1+2j}\big)^2}{-1+2i}}{(1+n)(1+2n)}\Big\},\ \{1,0\}\Big\}$$

In[12]:= $\mathbf{sol = FindLinearCombination[recSol, \{0, initial\}, n, 7, MinInitialValue \to 1]}$

In[12]:= $\mathbf{sol = FindLinearCombination[recSol, \{0, initial\}, n, 7, MinInitialValue \to 1]}$

Out[12]=

$$\frac{1}{3(1+n)^4(1+2n)^4}(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7)+\frac{32(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{9(1+n)(1+2n)}-$$

$$\frac{(3+2n)(-3+101n+126n^2)\sum_{i=1}^{n}\frac{1}{i^2}}{3(1+n)^2(1+2n)^2}-\frac{(3+2n)(115+921n+1967n^2+1524n^3+340n^4)\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)^3(1+2n)^3}+$$

$$\frac{44(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)(1+2n)}-\frac{(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{i})^2}{3(1+n)^2(1+2n)^2}+\frac{40(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^3}{9(1+n)(1+2n)}+$$

$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{9(1+n)(1+2n)}-\frac{4(3+2n)(77+261n+190n^2)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{3(1+n)^2(1+2n)^2}+\frac{16(3+2n)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}+$$

$$\frac{2(3+2n)(13-153n-303n^2+12n^3+172n^4)\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^3(1+2n)^3}+\frac{88(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}-$$

$$\frac{4(3+2n)(-41-53n+2n^2)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^2(1+2n)^2}+\frac{80(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^2\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}+$$

$$\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{(-1+2i)^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}-\frac{4(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)^2(1+2n)^2}+$$

$$\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{i})(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)(1+2n)}+\frac{64(3+2n)(\sum_{i=1}^{n}\frac{1}{-1+2i})^3}{9(1+n)(1+2n)}-\frac{16(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{i}}{3(1+n)(1+2n)}-$$

$$\frac{32(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{-1+2i}}{3(1+n)(1+2n)}-\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{3(1+n)(1+2n)}+$$

$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}}{3(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{i}}{3(1+n)(1+2n)}+$$

$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{-1+2i}}{3(1+n)(1+2n)}$$

In[13]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[14]:= **sol = TransformToSSums[sol];**

In[15]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

In[13]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[14]:= **sol = TransformToSSums[sol];**

In[15]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

Out[15]=
$$\frac{1}{3(1+n)^4(1+2n)^4}\big(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7\big)+\frac{64(3+2n)^2\mathtt{S}[1,n]}{3(1+n)(1+2n)^2}+\frac{64(3+2n)(2+3n)\mathtt{S}[1,n]^2}{3(1+n)(1+2n)^2}+\Big(-\frac{2(3+2n)\big(147+985n+1871n^2+1268n^3+212n^4\big)}{3(1+n)^3(1+2n)^3}+\frac{224(3+2n)\mathtt{S}[2,2n]}{3(1+n)(1+2n)}+\frac{128(3+2n)\mathtt{S}[-2,2n]}{3(1+n)(1+2n)}\Big)\mathtt{S}[1,2n]-\frac{4(3+2n)\big(23+123n+114n^2\big)\mathtt{S}[1,2n]^2}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)\mathtt{S}[1,2n]^3}{3(1+n)(1+2n)}+\frac{64(3+2n)\mathtt{S}[2,n]}{3(1+n)(1+2n)}-\frac{4(3+2n)\big(53+229n+190n^2\big)\mathtt{S}[2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)\mathtt{S}[3,2n]}{3(1+n)(1+2n)}+\Big(-\frac{64(3+2n)^2}{3(1+n)(1+2n)^2}-\frac{128(3+2n)(2+3n)\mathtt{S}[1,2n]}{3(1+n)(1+2n)^2}\Big)\mathtt{S}[-1,2n]-\frac{64(3+2n)(2+3n)\mathtt{S}[-1,2n]^2}{3(1+n)(1+2n)^2}-\frac{32(3+2n)\big(1+8n+8n^2\big)\mathtt{S}[-2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)\mathtt{S}[-3,2n]}{3(1+n)(1+2n)}-\frac{128(3+2n)\mathtt{S}[-2,1,2n]}{3(1+n)(1+2n)}$$

In[13]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[14]:= **sol = TransformToSSums[sol];**

In[15]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

In[16]:= **SExpansion[sol, n, 2]**

Out[16]=
$$\ln 2^2\left(\frac{64\mathrm{LG}[n]}{n}+\frac{160}{3n^2}-\frac{44}{n}\right)+\ln 2\left(\left(\frac{320}{3n^2}-\frac{88}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^2}{n}-\frac{430}{3n^2}+\frac{160\zeta_2}{3n}-\frac{14}{n}\right)+\zeta_2\left(\frac{160\mathrm{LG}[n]}{3n}+\frac{40}{n^2}-\frac{84}{n}\right)+\left(\frac{160}{3n^2}-\frac{44}{n}\right)\mathrm{LG}[n]^2+\left(-\frac{430}{3n^2}-\frac{14}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^3}{3n}+\frac{64\ln 2^3}{3n}+\frac{145}{2n^2}+\frac{32\zeta_3}{n}+\frac{41}{n}$$

In[13]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[14]:= **sol = TransformToSSums[sol];**

In[15]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

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Out[16]=
$$\ln 2^2\left(\frac{64\mathrm{LG}[n]}{n}+\frac{160}{3n^2}-\frac{44}{n}\right)+\ln 2\left(\left(\frac{320}{3n^2}-\frac{88}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^2}{n}-\frac{430}{3n^2}+\frac{160\zeta_2}{3n}-\frac{14}{n}\right)+\zeta_2\left(\frac{160\mathrm{LG}[n]}{3n}+\frac{40}{n^2}-\frac{84}{n}\right)+\left(\frac{160}{3n^2}-\frac{44}{n}\right)\mathrm{LG}[n]^2+\left(-\frac{430}{3n^2}-\frac{14}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^3}{3n}+\frac{64\ln 2^3}{3n}+\frac{145}{2n^2}+\frac{32\zeta_3}{n}+\frac{41}{n}$$

Special function algorithms

Details

- HarmonicSums package
  Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]
  Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]
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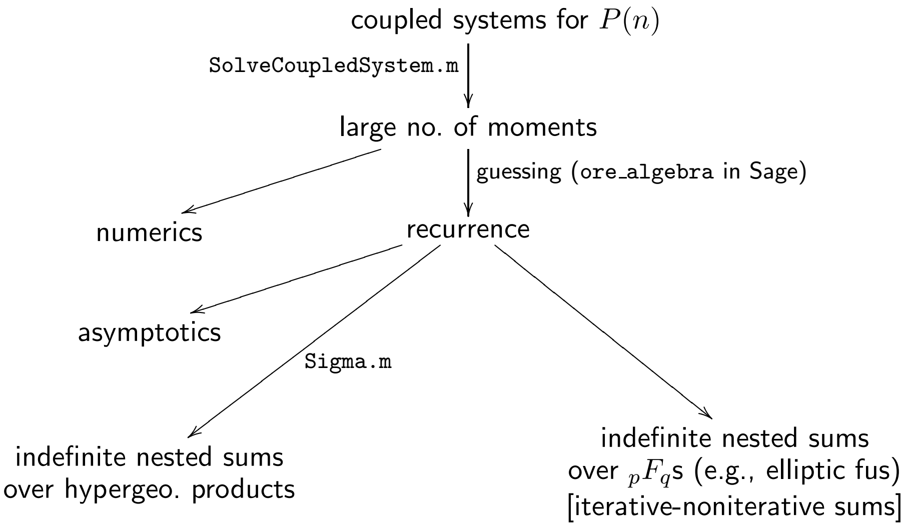
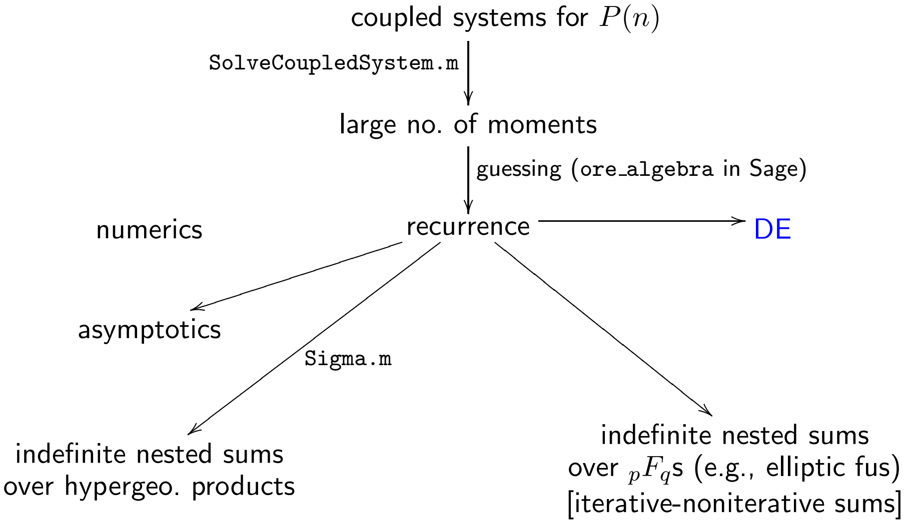
- RICA package
  Blümlein, Fadeev, CS. ACM Communications in Computer Algebra 57(2), pp. 31-34. 2023.


coupled systems for $P(n)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics
Sigma.m
indefinite nested sums
over hypergeo. products
indefinite nested sums
over ${}_pF_q$s (e.g., elliptic fus)
[iterative-noniterative sums]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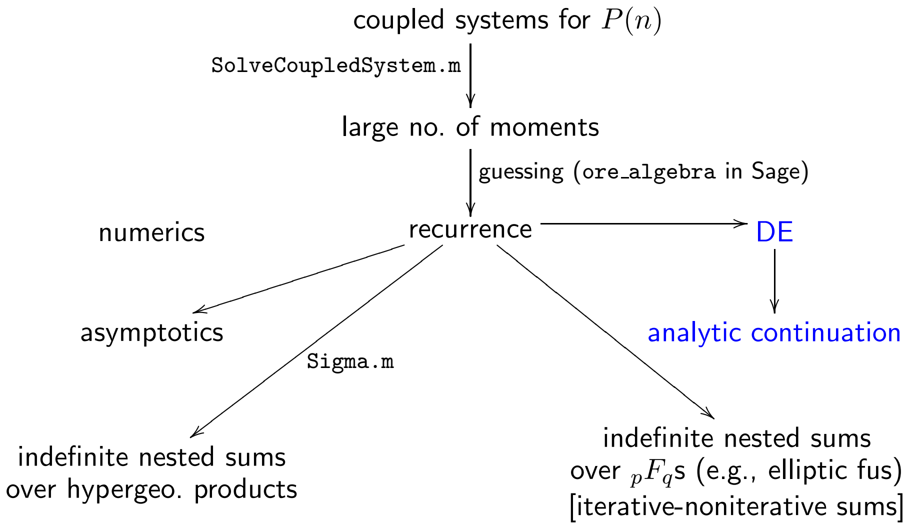
indefinite nested sums over ${}_pF_q$s (e.g., elliptic fus) [iterative-noniterative sums]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[iterative-noniterative sums]

Evaluate beyond 0

convergency
radius

$r = 1$

$$\sum_{n=0}^{\infty} f_n (x+1)^n$$

given $f_n \in \mathbb{Q}$

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

Matching evaluations at a common point x

convergency radius

$r = 1$

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{\infty} g_{j,n} x^n$$

$$\sum_{n=0}^{\infty} f_n (x+1)^n$$

find $g_{j,n} \in \mathbb{R}$

given $f_n \in \mathbb{Q}$

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

Matching evaluations at a common point x

$r = 0.078$

convergency radius

$r = 1$

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{\infty} g_{j,n} x^n$$

$$\sum_{n=0}^{\infty} f_n (x+1)^n$$

find $g_{j,n} \in \mathbb{R}$

given $f_n \in \mathbb{Q}$

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

Matching evaluations at a common point x

$r = 0.078$ | convergency radius | $r = 1$

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{\infty} g_{j,n} x^n$$

find $g_{j,n} \in \mathbb{R}$

$$\sum_{n=0}^{\infty} f_n (x+1)^n$$

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Matching evaluations at a common point x

$r = 0.078$ convergency radius $r = 1$

0 $x < 0.078$ 1

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{\infty} g_{j,n} x^n \qquad \sum_{n=0}^{500000} f_n (x+1)^n$$

find $g_{j,n} \in \mathbb{R}$ given $f_n \in \mathbb{Q}$

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

Matching evaluations at a common point x

$r = 0.078$

convergency
radius

$r = 1$

0 — $x < 0.078$ — 1

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{10000} g_{j,n} x^n$$

DE (order 48, deg 2800)

$$\sum_{n=0}^{500000} f_n (x+1)^n$$

find $g_{j,n} \in \mathbb{R}$

given $f_n \in \mathbb{Q}$

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

Matching evaluations at a common point x

$r = 0.078$

convergency radius

$r = 1$

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{10000} g_{j,n} x^n$$

DE (order 48, deg 2800) $\sum_{n=0}^{500000} f_n (x+1)^n$

find $g_{j,n} \in \mathbb{R}$
1400 digits precision

given $f_n \in \mathbb{Q}$

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.

Matching evaluations at a common point x

$r = 0.078$

convergency radius

$r = 1$

0 — $x = 0.005$ — 1

$$\sum_{j=0}^{6} \log(x)^j \sum_{n=-2}^{10000} g_{j,n} x^n$$

DE (order 48, deg 2800) $\sum_{n=0}^{500000} f_n (x+1)^n$

find $g_{j,n} \in \mathbb{R}$

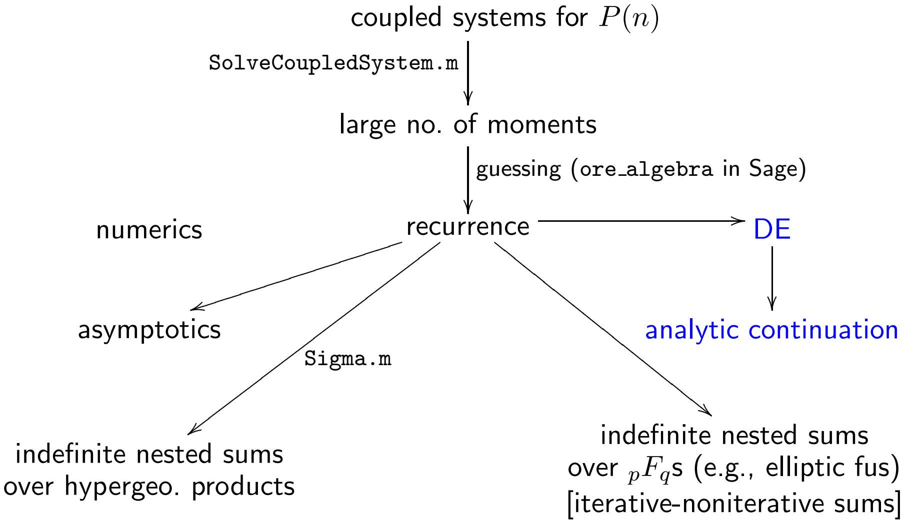
1400 digits precision

given $f_n \in \mathbb{Q}$

PSLP ↓

$g_{j,n} \in \mathbb{Q}(\pi, \zeta_3, \ldots)$

J. Blümlein, A. De Freitas, P. Marquard, n. Rana, C. Schneider. Analytic results on the massive three-loop form factors: quarkonic contributions. Physical Review D 108(094003), pp. 1-73. 2023.


coupled systems for $P(n)$
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[iterative-noniterative sums]

Calculations based on Tactic 4:

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- J. Blümlein, P. Marquard, C. Schneider, K. Schönwald. The three-loop unpolarized and polarized non-singlet anomalous dimensions from off shell operator matrix elements. Nucl. Phys. B 971, pp. 1-44. 2021.
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Conclusion (RISC–DESY cooperation)

Our calculations rely on

1. symbolic summation and integration methods to derive recurrences
2. flexible recurrence and DE solver
3. coupled systems solver
4. the large moment method

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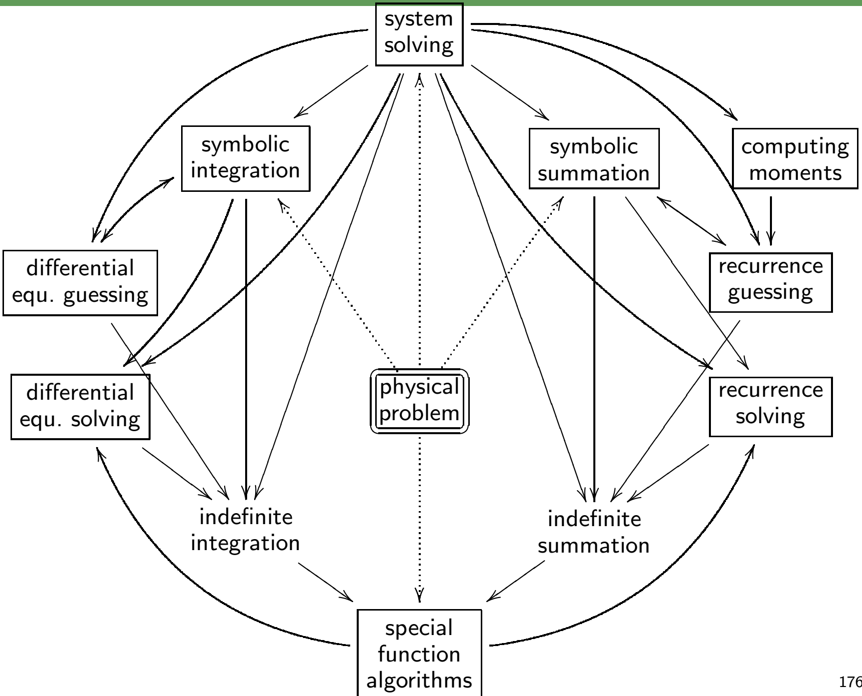
5. special function algorithms
   - to support the above calculations
   - to simplify the results further
   - to explore their mathematical structures and properties



system solving
symbolic integration
symbolic summation
computing moments
differential equ. guessing
recurrence guessing
differential equ. solving
physical problem
recurrence solving
indefinite integration
indefinite summation
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    - to support the above calculations
    - to simplify the results further
    - to explore their mathematical structures and properties
6. stable and efficient software packages

Main CA-packages

In[17]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[18]:= << **MultiIntegrate.m**

MultiIntegrate by Jakob Ablinger © RISC-Linz

In[19]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[20]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[21]:= << **SumProduction.m**

SumProduction by Carsten Schneider © RISC-Linz

In[22]:= << **OreSys.m**

OreSys by Stefan Gerhold (optimized by Carsten Schneider) © RISC-Linz

In[23]:= << **SolveCoupledSystem.m**

SolveCoupledSystem by Carsten Schneider © RISC-Linz

Conclusion (RISC–DESY cooperation)

Our calculations rely on

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Appendix: Symbolic tools for special functions

Symbolic tools for special functions

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,. . .)

$$\boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

Symbolic tools for special functions

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,...)

$$\boxed{\sum_{i=1}^{n}\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j}}$$

Integral representation:

$$=\int_0^1\frac{x^n-1}{1-x}\left(\int_0^x\frac{\int_0^y\frac{1}{1-z}dz}{y}dy-\zeta(2)\right)dx, \qquad \zeta(z):=\sum_{i=1}^{\infty}1/i^z$$

Symbolic tools for special functions

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,. . .)

$$\boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

Integral representation:

$$= \int_0^1 \frac{x^n - 1}{1-x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta(2) \right) dx, \qquad \zeta(z) := \sum_{i=1}^{\infty} 1/i^z$$

Asymptotic expansion:

$$= \left(\frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \ln(n)$$

$$- \tfrac{1}{100n^5} - \tfrac{1}{6n^4} + \tfrac{13}{36n^3} - \tfrac{1}{4n^2} - \tfrac{1}{n} + 2\zeta(3) + O(\tfrac{\ln(n)}{n^6}).$$

limit computations **numerical evaluation**

- Generalized algorithms for generalized harmonic sums

$$\sum_{k=1}^{n}\frac{2^k\sum_{i=1}^{k}\frac{2^{-i}\sum_{j=1}^{i}\frac{S_1(j)}{j}}{i}}{k}=-\frac{21\zeta(2)^2}{20n}+\frac{1}{8n^2}+\frac{295}{216n^3}-\frac{1115}{96n^4}+O(n^{-5})$$
$$+\Big(\frac{1}{2n}-\frac{3}{4n^2}+\frac{19}{12n^3}-\frac{5}{n^4}+O(n^{-5})\Big)\zeta(2)$$
$$+2^n\Big(\frac{3}{2n}+\frac{3}{2n^2}+\frac{9}{2n^3}+\frac{39}{2n^4}+O(n^{-5})\Big)\zeta(3)$$
$$+\Big(\frac{1}{n}+\frac{3}{4n^2}-\frac{157}{36n^3}+\frac{19}{n^4}+O(n^{-5})\Big)(\log(n)+\gamma)$$
$$+\Big(\frac{1}{2n}-\frac{3}{4n^2}+\frac{19}{12n^3}-\frac{5}{n^4}+O(n^{-5})\Big)(\log(n)+\gamma)^2)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]]

Back

- Generalized algorithms for cyclotomic harmonic sums

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{1+2i}}{j^2}}{(1+2k)^2}=\Big(-3+\frac{35\zeta(3)}{16}\Big)\zeta(2)-\frac{31\zeta(5)}{8}$$
$$+\frac{1}{n}-\frac{33}{32n^2}+\frac{17}{16n^3}-\frac{4795}{4608n^4}+O(n^{-5})$$
$$+\log(2)\Big(6\zeta(2)-\frac{1}{n}+\frac{9}{8n^2}-\frac{7}{6n^3}+\frac{209}{192n^4}+O(n^{-5})\Big)$$
$$+\Big(-\frac{7}{4}-\frac{7}{16n}+\frac{7}{16n^2}-\frac{77}{192n^3}+\frac{21}{64n^4}+O(n^{-5})\Big)\zeta(3)$$
$$+\Big(\frac{1}{16n^2}-\frac{1}{8n^3}+\frac{65}{384n^4}+O(n^{-5})\Big)(\log(n)+\gamma)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]]

Back

- Generalized algorithms for nested binomial sums

$$\sum_{j=1}^{n}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2} = 7\zeta(3)+\sqrt{\pi}\sqrt{n}\Bigg\{\Bigg[-\frac{2}{n}+\frac{5}{12n^2}-\frac{21}{320n^3}-\frac{223}{10752n^4}+\frac{671}{49152n^5}$$
$$+\frac{11635}{1441792n^6}-\frac{1196757}{136314880n^7}-\frac{376193}{50331648n^8}+\frac{201980317}{18253611008n^9}$$
$$+O(n^{-10})\Bigg]\ln(\bar{n})-\frac{4}{n}+\frac{5}{18n^2}-\frac{263}{2400n^3}+\frac{579}{12544n^4}+\frac{10123}{1105920n^5}$$
$$-\frac{1705445}{71368704n^6}-\frac{27135463}{11164188672n^7}+\frac{197432563}{7927234560n^8}+\frac{405757489}{775778467840n^9}$$
$$+O(n^{-10})\Bigg\}$$

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