Higher genus modular graph functions

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Introduction

- **String theory naturally generalizes real-analytic Eisenstein series**
  - for genus one surfaces,
    - ⋆ multiple Kronecker-Eisenstein sums,
    - ⋆ multiple integrations of Green function on the torus,

  $\Rightarrow$ modular graph functions invariant under $SL(2,\mathbb{Z})$.

(Green, Russo, Vanhove 2008; ED, Green, Vanhove 2015; ED, Green, Gurdogan, Vanhove 2015)

- **String theory includes contributions from surfaces of all genera**

  $\Longrightarrow$ **expect modular graph functions for higher genus surfaces.**

- **Focus of this talk is on genus two and higher**
  - simplest is Kawazumi-Zhang invariant (to be explained below);
  - genus-two string theory predicts an infinite number of higher invariants;
  - genus greater than two string theory offers no predictions,

    but mathematical constructions produce higher invariants.

- **I will give an account of what we know and do not know to date.**

Bibliography

Based on

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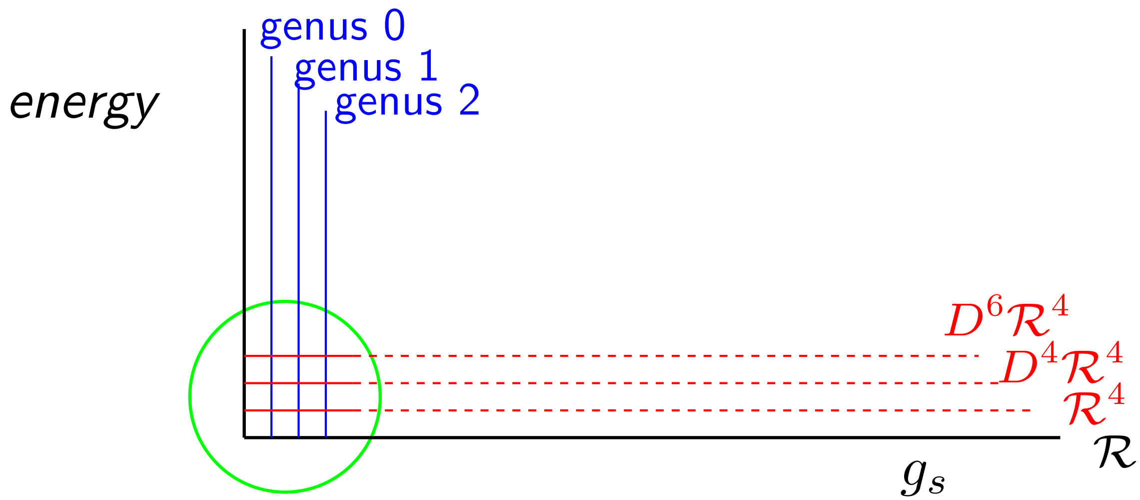
  Higher genus modular graph functions, string invariants, and their exact asymptotics
- ED, Michael Green and Boris Pioline, (in preparation)
  Asymptotics of the $D^8\mathcal{R}^4$ genus-two string invariant

and earlier work

- ED, Michael Green, arXiv:1308.4597, Journal of Number Theory, Vol 144 (2014) 111-150,
  Zhang-Kawazumi invariants and Superstring Amplitudes
- ED, Michael Green, Boris Pioline, Rudolfo Russo, arXiv:1405.6226, JHEP 1501 (2015) 031,
  Matching the $D^6\mathcal{R}^4$ interaction at two-loops
- Boris Pioline, arXiv:1504.04182, Journal of Number Theor. **163**, 520 (2016),
  A Theta lift representation for the Kawazumi-Zhang and Faltings invariants of genus-two Riemann surfaces

Expansion in genus and low energy



- **Supertring Perturbation Theory in** $g_s^{(2-2h)}$ **with** $h \geq 0$ **is the genus**
  - holds for small string coupling $g_s \ll 1$
  - but for all energies
- **Supergravity** $\mathcal{R}$
  - leading low energy expansion of string theory
  - holds for all couplings g_s
- **String induced effective interactions** $\mathcal{R}^4, D^4\mathcal{R}^4, D^6\mathcal{R}^4$
  - Evaluated in superstring perturbation theory
  - Accessible via the four-graviton scattering amplitude

Effective Interactions

- **Four-graviton amplitude in Type II at genus 0,**

$$\mathcal{A}^{(0)}(s_{ij}) = \mathcal{R}^4 \frac{1}{stu} \frac{\Gamma(1-s)\,\Gamma(1-t)\,\Gamma(1-u)}{\Gamma(1+s)\,\Gamma(1+t)\,\Gamma(1+u)}$$

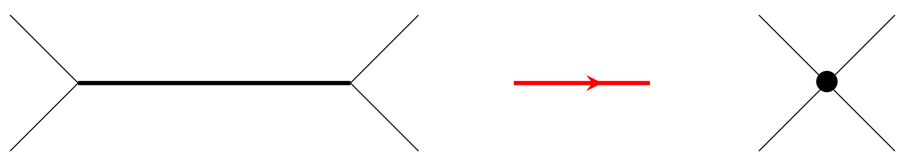
  - $\mathcal{R}^4 =$ unique maximally supersymmetric contraction of 4 Weyl tensors
  - External momenta k_i for $i = 1, 2, 3, 4$ with $k_i^2 = 0$ and $\sum_i k_i = 0$
  - Introduce dimensionless Lorentz-invariants $s_{ij} = -\alpha' k_i \cdot k_j / 2$
  - $s = s_{12} = s_{34},\ t = s_{13} = s_{24},\ u = s_{14} = s_{23}$ with $s + t + u = 0$

- **Low energy expansion corresponds to** $|s|,\ |t|,\ |u| \ll 1$

$$\frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2 + t^2 + u^2) + 2\zeta(3)^2 stu + \cdots$$

massless $\quad \mathcal{R}^4 \quad D^4\mathcal{R}^4 \quad D^6\mathcal{R}^4$

Exchange of massive string states produces local effective interactions.



Genus-one string amplitude

- **Effective $\mathcal{R}^4$-type interactions in Type II**
  - Generated by a multiple integral over a torus $\Sigma_1 = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, of modulus $\tau \in \mathbb{H}$, namely $\tau = \tau_1 + i\tau_2$ with $\tau_1, \tau_2 \in \mathbb{R}$ and $\tau_2 > 0$,

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \prod_{i=1}^{N} \int_{\Sigma_1} \frac{d^2 z_i}{\tau_2} \, \exp\Big\{ \sum_{1 \leq i < j \leq N} s_{ij}\, g(z_i - z_j|\tau) \Big\}$$

- **Mathematically, one may consider this integral for arbitrary N**
  - $g(z|\tau)$ is the translation-invariant Green function on Σ_1,

$$\tau_2 \partial_{\bar z} \partial_z g(z|\tau) = -\pi \delta^{(2)}(z) + \pi \qquad\qquad \int_{\Sigma_1} d^2 z\, g(z|\tau) = 0$$

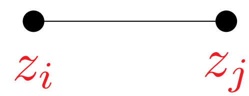
  - Integrals absolutely convergent for $|s_{ij}| < 1$; analytic near $s_{ij} = 0$;
  - $\mathcal{B}^{(1)}(s_{ij}|\tau)$ is invariant under the modular group $SL(2,\mathbb{Z})$.

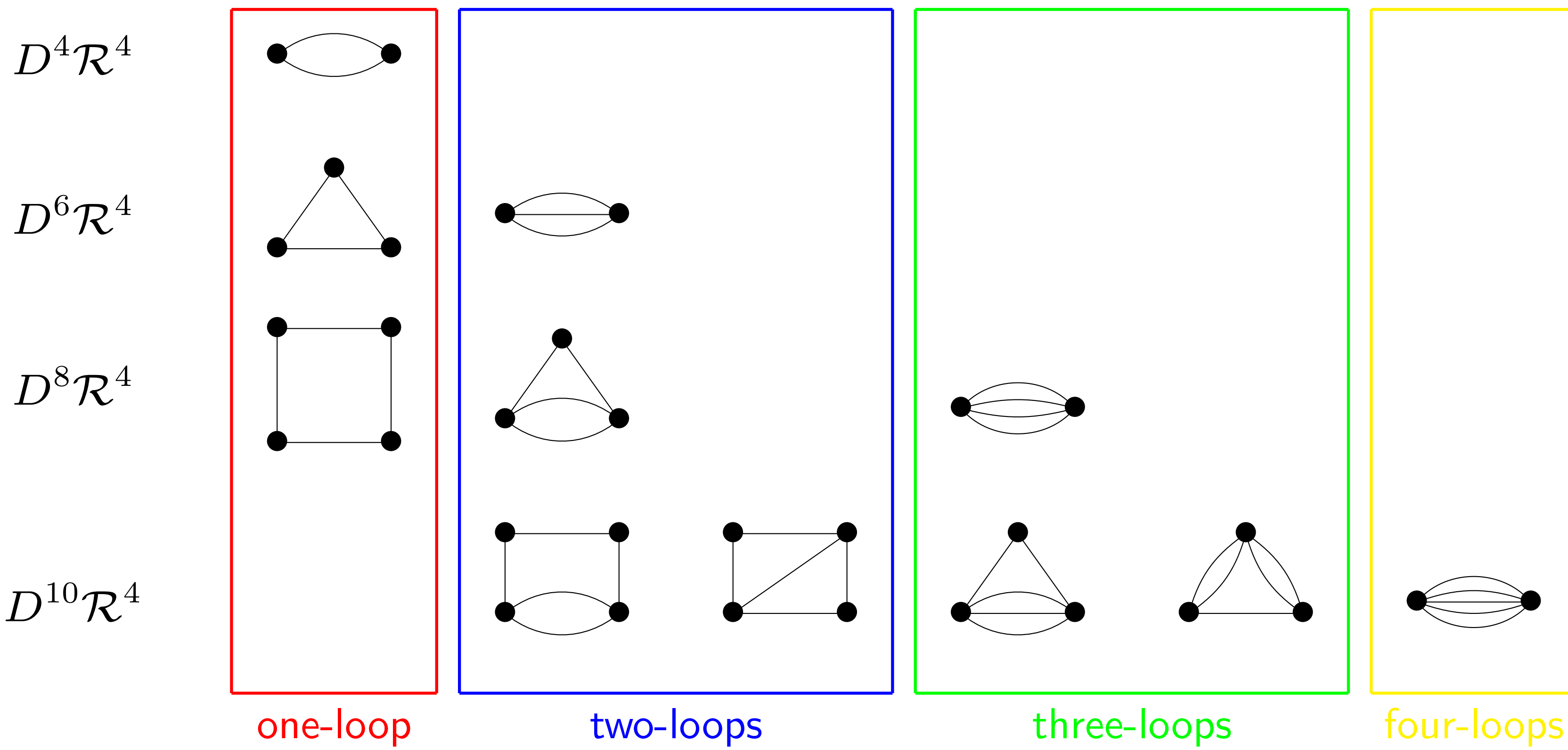
- **String amplitude obtained by integral over modulus of the torus,**

$$\mathcal{A}^{(1)}(s_{ij}) = \int_{\mathbb{H}/SL(2,\mathbb{Z})} \frac{d^2\tau}{\tau_2^2} \mathcal{B}^{(1)}(s_{ij}|\tau)$$

  - requires analytic continuation in s_{ij} (ED, Phong 1994).

Genus-one modular graph functions

- **Taylor series expansion of $\mathcal{B}^{(1)}(s_{ij}|\tau)$ for fixed τ in powers of s_{ij}**
  - An integration point z_i is represented by a vertex •
  - A Green function is represented by an edge  $= \; g(z_i - z_j|\tau)$



Properties of genus-one modular graph functions

- **One-loop graphs with k vertices give real analytic Eisenstein series E_s**

$$\prod_{i=1}^{k} \int_\Sigma \frac{d^2 z_i}{\tau_2} g(z_i - z_{i+1}|\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^k}{\pi^k |m\tau+n|^{2k}} = E_k$$

  – convergent sums for $\mathrm{Re}(s) > 1$; modular $SL(2,\mathbb{Z})$-invariant
  – Laplace-eigenvalue equation, $(\Delta - s(s-1))\, E_s = 0$ with $\Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$

- **Two-loop graphs evaluate to the series**

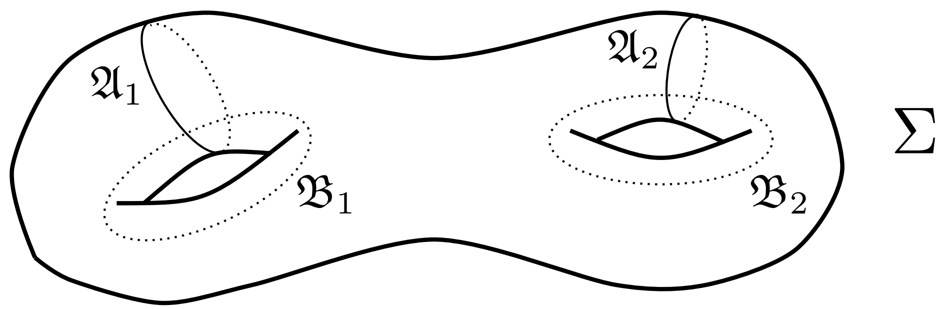
$$C_{s_1,s_2,s_3}(\tau) = \sum_{(m_r,n_r)\neq(0,0)} \prod_{r=1}^{3} \left(\frac{\tau_2}{\pi |m_r \tau + n_r|^2} \right)^{s_r} \delta(\sum_r m_r)\, \delta(\sum_r n_r)$$

– convergent sums for $\mathrm{Re}(s_r) \geq 1$; modular $SL(2,\mathbb{Z})$-invariant;
– satisfy inhomogeneous Laplace-eigenvalue equations, e.g.

$$\begin{aligned} \Delta C_{1,1,1} &= 6E_3 \\ (\Delta - 2) C_{2,1,1} &= 9E_4 - E_2^2 \\ (\Delta - 6) C_{3,1,1} &= 3C_{2,2,1} + 16 E_5 - 4E_2 E_3 \end{aligned} \qquad (1)$$

Genus-two surfaces

- Σ **is a compact Riemann surface of genus two**
  - Key difference with genus-one: no translation symmetry



- **Homology and cohomology**
  - One-cycles $H_1(\Sigma, \mathbb{Z}) \approx \mathbb{Z}^4$ with intersection pairing $\mathfrak{J}(\cdot,\cdot) \to \mathbb{Z}$
  - Canonical basis $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{A}_J) = 0, \ \mathfrak{J}(\mathfrak{B}_I, \mathfrak{B}_J) = 0$ with $I, J = 1, 2$

$$\mathfrak{J}(\mathfrak{A}_I, \mathfrak{B}_J) = \delta_{IJ}, \ \mathfrak{J}(\mathfrak{B}_I, \mathfrak{A}_J) = -\delta_{IJ}$$

  - Canonical dual basis of holomorphic one-forms ω_I in $H^{(1,0)}(\Sigma)$

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad\qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

  - Period matrix Ω obeys Riemann relations $\Omega^t = \Omega, \ \mathrm{Im}(\Omega) > 0$

Modular transformations and geometry

- **Transformation** $Sp(4,\mathbb{Z}) : H_1(\Sigma,\mathbb{Z}) \to H_1(\Sigma,\mathbb{Z})$ **leaves** $\mathfrak{J}(\cdot,\cdot)$ **invariant**
  - action on basis cycles given by

$$\begin{pmatrix} \mathfrak{B}_I \\ \mathfrak{A}_I \end{pmatrix} \to \sum_J M_{IJ} \begin{pmatrix} \mathfrak{B}_J \\ \mathfrak{A}_J \end{pmatrix} \qquad\qquad M^t \mathfrak{J} M = \mathfrak{J}$$

  - action on 1-forms ω_I and periods Ω_{IJ} given by

$$\begin{aligned} \omega &\to \omega\,(C\Omega + D)^{-1} \\ \Omega &\to (A\Omega + B)\,(C\Omega + D)^{-1} \end{aligned} \qquad\qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

- **Siegel upper half space** $\mathcal{S}_2$

$$\mathcal{S}_2 = \Big\{ \Omega_{IJ} \in \mathbb{C} \text{ with } \Omega^t = \Omega \text{ and } Y = \mathrm{Im}(\Omega) > 0 \Big\}$$

  - $\mathcal{S}_2 = \frac{Sp(4,\mathbb{R})}{SU(2)\times U(1)} = \frac{SO(3,2)}{SO(3)\times SO(2)}$ is Kähler with invariant metric

$$ds_2^2 = \sum_{I,J,K,L=1,2} Y^{-1}_{IJ}\, d\bar\Omega_{JK}\, Y^{-1}_{KL}\, d\Omega_{LI}$$

  - Moduli space of genus-two surfaces is $\mathcal{S}_2/Sp(4,\mathbb{Z})$ (minus diagonal Ω)

Green function and volume form

- **How to generalize the genus-one formula to a genus-two formula ?**
  - recall the genus-one formula

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \prod_{i=1}^{N} \int_{\Sigma_1} \frac{d^2 z_i}{\tau_2} \, \exp\Big\{ \sum_{1 \leq i < j \leq N} s_{ij} \, g(z_i - z_j|\tau) \Big\}$$

- **Canonical metric and Kähler form for genus-two Σ**
  - modular invariant and smooth

$$\kappa = \frac{i}{4} \sum_{I,J} Y^{-1}_{IJ} \omega_I \wedge \overline{\omega_J} \qquad\qquad \int_\Sigma \kappa = 1$$

- **Natural "Arakelov Green function"** $\mathcal{G}(w,z|\Omega) = \mathcal{G}(z,w|\Omega)$
  - Inverse of scalar Laplace operator for canonical metric

$$\begin{aligned} \partial_{\bar w} \partial_w \mathcal{G}(w,z|\Omega) &= -\pi \delta(w,z) + \pi \kappa(w) \\ \int_\Sigma \kappa(w) \mathcal{G}(w,z|\Omega) &= 0 \end{aligned}$$

A natural genus-two candidate

- **A natural candidate formula for a string amplitude would be**

$$\mathcal{C}^{(2)}(s_{ij}|\Omega) = \prod_{i=1}^{N} \int_\Sigma \kappa(z_i) \ \exp\left\{ \sum_{1\leq i<j\leq N} s_{ij}\, \mathcal{G}(z_i, z_j|\Omega) \right\}$$

  - Integrals absolutely convergent for $|s_{ij}| < 1$; analytic near $s_{ij} = 0$,
  - Expanding in powers of s_{ij} gives *genus-two modular graph functions*.

- **But ... Genus-two string amplitudes are NOT given by $\mathcal{C}^{(2)}(s_{ij}|\Omega)$**

- **For integration over a single copy of Σ**
  - κ is the only natural modular invariant volume form.

- **For integration over multiple copies of Σ**
  - $Sp(4,\mathbb{Z})$ modular invariants other than $\prod_i \kappa(z_i)$ allowed.
  - For example, when $N = 2$ we can have $\kappa(z_1)\kappa(z_2)$ as well as

$$\sum_{I,J,K,L} Y^{-1}_{IL}\, Y^{-1}_{JK}\, \omega_I(z_1)\,\overline{\omega_J(z_1)}\,\omega_K(z_2)\,\overline{\omega_L(z_2)}$$

Genus-two string amplitude

- **Instead the $N=4$ graviton amplitude was calculated** (ED, Phong 2005)

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det Y)^2} \exp\left\{ \sum_{1\leq i<j\leq 4} s_{ij}\, \mathcal{G}(z_i,z_j) \right\}$$

- **The key difference with the candidate $\mathcal{C}^{(2)}$ is the structure of $\mathcal{Y}$**

$$\begin{aligned} 3\mathcal{Y} \quad = \quad & (t-u)\Delta(z_1,z_2)\wedge\Delta(z_3,z_4) && s = s_{12} = s_{34} \\ & +(s-t)\Delta(z_1,z_3)\wedge\Delta(z_4,z_2) && t = s_{13} = s_{24} \\ & +(u-s)\Delta(z_1,z_4)\wedge\Delta(z_2,z_3) && u = s_{14} = s_{23} \end{aligned}$$

– where Δ is a holomorphic $(1,0)_i \times (1,0)_j$ form on $\Sigma \times \Sigma$

$$\Delta(z_i,z_j) = \varepsilon^{IJ}\omega_I(z_i)\wedge\omega_J(z_j)$$

– The combination $\mathcal{Y}\wedge\bar{\mathcal{Y}}/(\det Y)^2$ is $Sp(4,\mathbb{Z})$-invariant,
– and produces a modular invariant $\mathcal{B}^{(2)}(s_{ij}|\Omega)$.

Low energy expansion

- **Contributions to local effective interactions**
  - Expand $\mathcal{B}^{(2)}(s_{ij}|\Omega)$ in powers of s_{ij} and integrate over $\mathcal{M}_2 = \mathcal{S}_2/Sp(4,\mathbb{Z})$

$\mathcal{R}^4, D^2\mathcal{R}^4$	zero, since $\mathcal{Y}$ vanishes for $s=t=u=0$
$D^4\mathcal{R}^4$	Siegel volume form on $\mathcal{M}_2$
$D^6\mathcal{R}^4$	one factor of $\mathcal{G}$ in expansion in powers of s_{ij}

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = 32(s^2+t^2+u^2) + 192\, stu\, \varphi(\Omega) + \mathcal{O}(s_{ij}^4)$$

$$\varphi(\Omega) = -\frac{1}{4} \sum_{I,J,K,L} Y^{-1}_{IL} Y^{-1}_{JK} \int_{\Sigma^2} \mathcal{G}(x,y)\omega_I(x)\overline{\omega_J(x)}\omega_K(y)\overline{\omega_L(y)}$$

- $\varphi(\Omega)$ **coincides with the Kawazumi-Zhang invariant** (ED, Green 2013)
  - introduced as a spectral invariant (Kawazumi 0801.4218 and Zhang 0812.0371)
  - related to the genus-two Faltings invariant (De Jong 2010)
  - formulated in terms of modular tensors (Kawazumi OIST lecture notes 2016)

$$\mathcal{A}_{IJ;KL} = \int_{\Sigma^2} \mathcal{G}(x,y)\, \omega_I(x)\overline{\omega_J(x)}\, \omega_K(y)\overline{\omega_L(y)}$$

Higher string-invariants

- **The KZ-invariant exists for all genera ≥ 2** (Kawazumi 2008; Zhang 2008)
  - unknown if correct object for string theory at genus ≥ 3.
- **But the Taylor expansion coefficients of** $\mathcal{B}^{(2)}(s_{ij}|\Omega)$
  - are *modular graph functions at genus-two*;
  - do naturally emerge from string theory at genus-two;
  - provide a string-motivated generalization of KZ-invariants at genus-two.
- **Higher string-invariants** (contribute to $D^8\mathcal{R}^4$ and $D^{10}\mathcal{R}^4$) (ED, Green 2013)

$$
\begin{aligned}
\mathcal{B}^{(2)}_{(2,0)} &= \int_{\Sigma^4} \frac{|\Delta(1,2)\Delta(3,4)|^2}{(\det Y)^2} \Big(\mathcal{G}(1,4) + \mathcal{G}(2,3) - \mathcal{G}(1,3) - \mathcal{G}(2,4) \Big)^2 \\
\mathcal{B}^{(2)}_{(1,1)} &= \int_{\Sigma^4} \frac{|\Delta(1,2)\Delta(3,4) - \Delta(1,4)\Delta(2,3)|^2}{(\det Y)^2} \Big(\mathcal{G}(1,2) + \mathcal{G}(3,4) \\
&\qquad + \mathcal{G}(1,4) + \mathcal{G}(2,3) - 2\mathcal{G}(1,3) - 2\mathcal{G}(2,4) \Big)^3 \\
\cdots &= \cdots
\end{aligned}
$$

Differential equations and Asymptotics

- **Genus-one modular graph functions satisfy**
  - System of inhomogeneous Laplace-eigenvalue equations
  - Laurent polynomial behavior near the cusp of degree bounded by weight

- **Genus-two KZ-invariant φ satisfies**
  - Characteristic class relations and role in Johnson homomorphism
    (Kawazumi 2008; Zhang 2008; DeJong 2010)
  - Eigenvalue equations for $Sp(4,\mathbb{R})$-invariant differential operators
    (ED, Green, Pioline, Russo 2014; Pioline 2015)
  - Theta-lift analogous to Borcherds for Igusa cusp form (Pioline 2015)
  - Laurent polynomial of degree $(1,1)$ near non-separating divisor
    (De Jong 2010; Pioline 2015; ED, Green, Pioline 2017)

- **Genus-two higher string-invariants satisfy**
  - Laurent polynomial of bounded degree near non-separating degeneration
    (ED, Green, Pioline 2017)
  - System of differential equations for invariant differential operators
    (in progress ED, Green, Pioline 201?)

Differential equations

- **Theorem 1** *Laplace eigenvalue equation* (ED, Green, Pioline, Russo 2014)

$$(\Delta - 5)\varphi = -2\pi\delta_{SN}$$

  - δ_{SN} has support on separating node (into two genus-one surfaces)
  - Δ is the Laplace-Beltrami operator on $\mathcal{S}_2$ with Siegel metric

$$\Delta = 4 \sum_{I,J,K,L} Y_{IK} Y_{JL} \bar\partial^{IJ} \partial^{KL} \qquad \partial^{IJ} = \frac{1}{2}(1+\delta^{IJ})\frac{\partial}{\partial \Omega_{IJ}}$$

  - proven by theory and methods of deformations of complex structures

- **Theorem 2** *Quartic differential operator eigenvalue equation* (Pioline 2015)

$$(32\, \Box_2^* \Box_0 - 15)\varphi = 0$$

  - $\Box_0, \Box_2$ are Maass-Siegel operators

$$\Box_0 = \varepsilon^{IJ} \varepsilon^{KL} \nabla_{IK} \nabla_{JL} \qquad \nabla_{IJ} = Y_{IK} Y_{JL} \partial^{KL}$$

- **System of differential equations satisfied by higher string invariants ?**

Asymptotics

For genus one, there is only one type of degeneration, as modulus $\tau \to i\infty$

- *Behavior of holomorphic Eisenstein series*

$$\mathbb{G}_{2k}(\tau) = -\frac{B_{2k}}{2k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \qquad q = e^{2\pi i \tau}$$

- *Behavior of non-holomorphic Eisenstein series,* $y = \pi \mathrm{Im}(\tau)$

$$E_k(\tau) = \frac{2\zeta(2k)}{\pi^{2k}} y^k + 4 \binom{2k-3}{k-1} \frac{\zeta(2k-1)}{(4y)^{k-1}} + \mathcal{O}(|q|)$$

- *Behavior of general genus-one modular graph functions.*
  - Finite degree Laurent polynomial in y plus exponentials,
  - eg two-loop modular graph functions (ED, Bill Duke 2017)

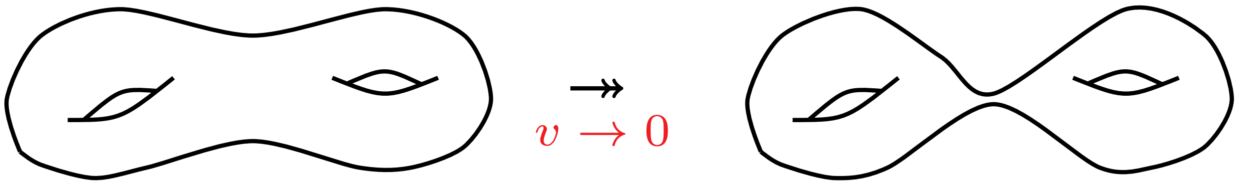
$$C_{a,b,c}(\tau) = c_w y^w + \frac{c_{2-w}}{y^{w-2}} + \sum_{k=1}^{w-1} \frac{c_{w-2k-1} \zeta(2k+1)}{y^{2k+1-w}} + \mathcal{O}(|q|)$$

$$c_{2-w} = \sum_{m=1}^{w-2} \gamma_m \, \zeta(2m+1) \, \zeta(2w-2m-3)$$

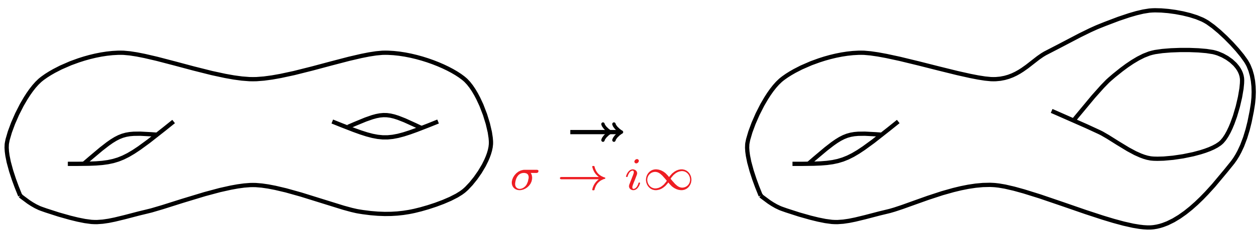
  - with $w = a + b + c$ and $c_w, \gamma_m, c_{w-2k-1} \in \mathbb{Q}$.

Degenerations of genus-two Riemann surfaces $\Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix}$

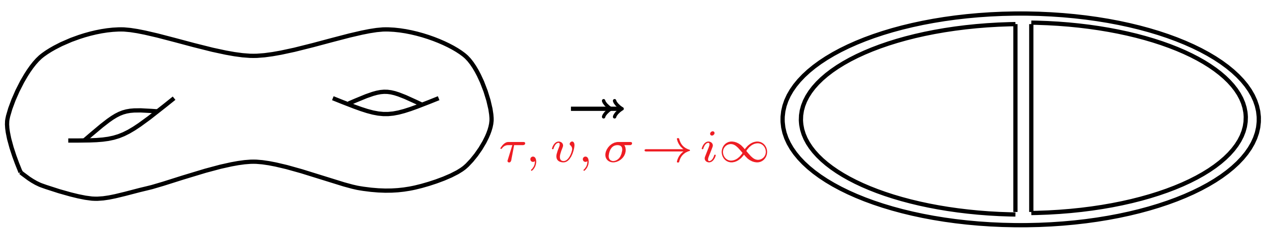
– *Separating degeneration*



– *Non-separating degeneration*



– *Maximal degeneration (or "tropical limit")*



Non-separating degeneration

- **A genus-two surface Σ degenerates to a torus Σ_1 with two punctures p_a, p_b**
  - keep the cycles $\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{A}_2$ fixed, and let $\mathfrak{B}_2 \to \infty$

$$\Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix} \qquad\qquad \mathrm{Im}(\sigma) \to \infty$$

  - τ is the modulus of Σ_1 and $v = \int_{p_a}^{p_b} \omega_1 = p_b - p_a$

- **The genus-two $Sp(4,\mathbb{Z})$ restricts to $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^3$** (Fourier-Jacobi group)
  - $SL(2,\mathbb{Z})$ subgroup of $M \in Sp(4,\mathbb{Z})$ such that $M\mathfrak{B}_2 = \mathfrak{B}_2$ is

$$M = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad\qquad \begin{cases} \tau' &= (a\tau+b)/(c\tau+d) \\ v' &= v/(c\tau+d) \\ \sigma' &= \sigma - cv^2/(c\tau+d) \end{cases}$$

  - $\mathbb{Z}^3$ subgroup shifts v by $\mathbb{Z}+\tau\mathbb{Z}$ and $\mathrm{Re}(\sigma)$ by $\mathbb{Z}$;
  - The degeneration parameter σ is not invariant under $SL(2,\mathbb{Z})$

Degeneration of Siegel modular forms

- **A genus-two ("rank two") Siegel modular form S of modular weight k**
  - is holomorphic and transforms under $Sp(4,\mathbb{Z})$ by,

$$S(\Omega') = \det(C\Omega + D)^k S(\Omega) \qquad\qquad \Omega' = (A\Omega + B)(C\Omega + D)^{-1}$$

  - Fourier expansion in powers of $e^{2\pi i\sigma}$ near non-separating node

$$S(\Omega) = \sum_{m=0}^{\infty} e^{2\pi i m\sigma} \phi_m(v|\tau)$$

  - $\phi_m(v|\tau)$ is a *Jacobi form* of modular weight k and index m
  - transforms under the residual modular group $SL(2,\mathbb{Z})$ by

$$\phi_m\left(\frac{v}{c\tau+d}\middle|\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k e^{2\pi i m c v^2/(c\tau+d)} \phi_m(v|\tau)$$

(Eichler and Zagier, "The theory of Jacobi forms", 1985)

Degeneration of the KZ-invariant

- **Non-separating degeneration of non-holomorphic modular functions**
  - is governed by a *real* $SL(2,\mathbb{Z})$-invariant parameter $t > 0$

$$t \equiv \frac{\det(\text{Im}\,\Omega)}{\text{Im}\,\tau} \qquad\qquad \Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix}$$

  - with the non-separating node characterized by $t \to \infty$

- **Theorem 3** Non-separating degeneration of KZ-invariant (Pioline 2015)

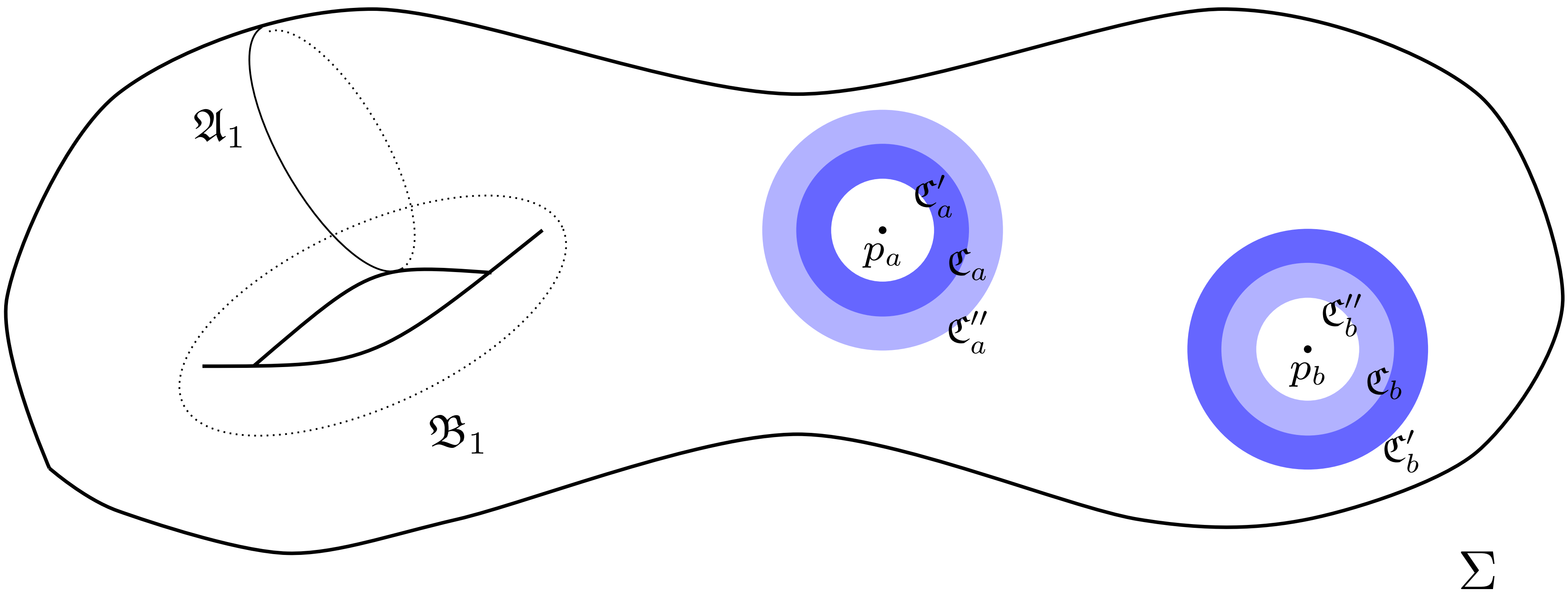
$$\varphi(\Omega) = \frac{\pi t}{6} + \frac{1}{2} g(v|\tau) + \frac{1}{4\pi t}\Big(E_2(\tau) - g_2(v|\tau)\Big) + \mathcal{O}(e^{-2\pi t})$$

  - $g(v|\tau)$ is the torus Green function;
  - E_2 is the non-holo Eisenstein series;
  - $g_2(v|\tau) = \int_{\Sigma_1} d^2z/\tau_2\, g(v-z|\tau) g(z|\tau)$;
  - Derived using the Laplace-eigenvalue equation for φ.

- **The Laurent polynomial is of finite degree in the variable** t
  - but it is not of finite degree in, say, $t+1$

Non-separating node from a punctured surface

- **Standard construction of a surface near a non-separating node** (Fay 1973)
  - start from a genus-one surface with two punctures p_a, p_b
  - local coordinates z_a, z_b which vanish respectively at p_a, p_b
  - identify points $z_a z_b = \mathfrak{t}$ between two annuli $\mathfrak{C}'_a = \mathfrak{C}'_b$ and $\mathfrak{C}''_a = \mathfrak{C}''_b$



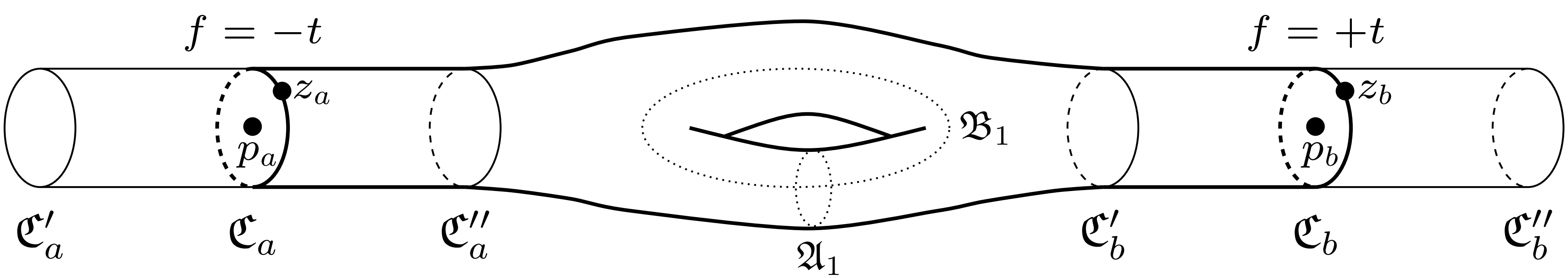
  - in practice not easy to implement, unless one can define cycles $\mathfrak{C}$ naturally

Non-separating node from a punctured surface (cont'd)

- **Key is the existence of a real single-valued harmonic function $f(z)$ on Σ**
  - such that in the degeneration limit
    $f(z) \to -\infty$ as $z \to p_a$ and $f(z) \to +\infty$ as $z \to p_b$
  - for large t cycles prescribed by $f(\mathfrak{C}_a) \to -2\pi t$ and $f(\mathfrak{C}_b) \to +2\pi t$

$$f(z) = -2\pi t + 4\pi \, \mathrm{Im} \int_{z_a}^{z} \Big(\omega_2 - \mathrm{Im}(v)\, \omega_1 / \mathrm{Im}(\tau) \Big)$$

- **Funnel construction of the non-separating degeneration**



  - The cycle $\mathfrak{A}_2$ is homologous to the cycles $\mathfrak{C}_a, \mathfrak{C}'_a, \mathfrak{C}''_a$ and $\mathfrak{C}_b, \mathfrak{C}'_b, \mathfrak{C}''_b$;
  - The cycles are pairwise identified by $\mathfrak{C}_a \approx \mathfrak{C}_b$, $\mathfrak{C}'_a \approx \mathfrak{C}'_b$ and $\mathfrak{C}''_a \approx \mathfrak{C}''_b$;
  - The points are pairwise identified by $z_a \approx z_b$;
  - The cycle $\mathfrak{B}_2$ may be chosen to be a simple curve connecting z_a to z_b.

Degeneration of higher string invariants

- **Consider the full genus-two string amplitude**

$$\mathcal{B}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{Y}}}{(\det Y)^2} \exp \left\{ \sum_{i<j} s_{ij} \mathcal{G}(z_i, z_j|\Omega) \right\}$$

  – Expanding in powers of s_{ij} and collecting all terms homogeneous of degree w
  – gives modular graph functions of weight w for genus-two surfaces Σ

$$\mathcal{B}_w(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{Y}}}{(\det Y)^2} \left(\sum_{i<j} s_{ij} \mathcal{G}(z_i, z_j|\Omega) \right)^w$$

- **Theorem 4** *Under non-separating degeneration of* Σ (ED, Green, Pioline 2017)
  – $\mathcal{B}_w(s_{ij}|\Omega)$ *has a Laurent polynomial of degree* (w, w) *in* t

$$\mathcal{B}_w(s_{ij}|\Omega) = \sum_{k=-w}^{w} \mathcal{B}_w^{(k)}(s_{ij}|v, \tau)\, t^k + \mathcal{O}(e^{-2\pi t})$$

  – $\mathcal{B}_w^{(k)}(s_{ij}|v, \tau)$ *are invariant under* $SL(2, \mathbb{Z}) \subset Sp(4, \mathbb{Z})$

$$\mathcal{B}_w^{(k)} \left(s_{ij} \middle| \frac{v}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) = \mathcal{B}_w^{(k)}(s_{ij}|v, \rho)$$

- Note – *similarity with the Laurent polynomial expansion for genus one;*
  – $\mathcal{B}_w^{(k)}(s_{ij}|v, \tau)$ generalize Jacobi forms to a non-holomorphic setting;
  – combine genus-one modular graph functions and *elliptic polylogarithms*.

Ingredients in proof of Theorem 4

- **Uniform asymptotics of the Green function**

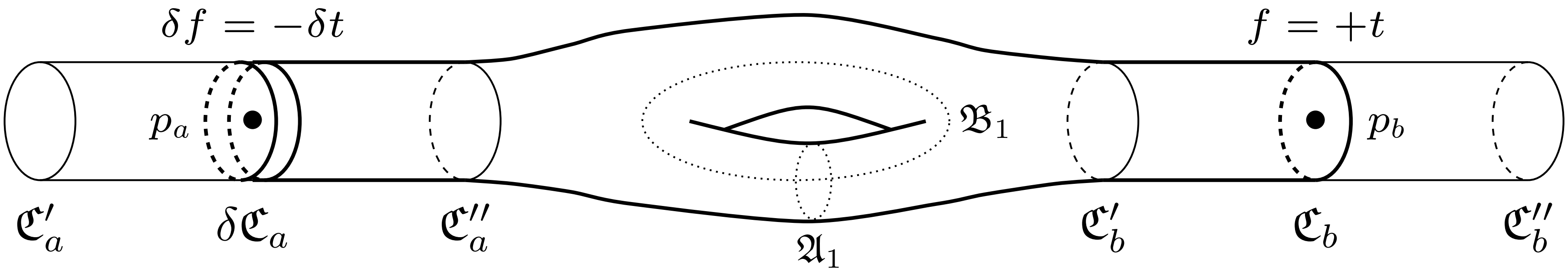
$$\mathcal{G}(x,y|\Omega) = \frac{\pi t}{12} + g(x-y|\tau) - \frac{f(x)f(y)}{4\pi t} + \cdots + \mathcal{O}(e^{-2\pi t})$$

  – see DGP 2017 for the complete expression (improving on Wentworth 1991)

- **Measure**

$$4\pi^2 \mathcal{Y} = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \sum_{i<j} s_{ij} \partial_{z_i} f(z_i) \partial_{z_j} f(z_j)$$

- **The cycles $\mathfrak{C}_a, \mathfrak{C}_b$ have exponentially vanishing coordinate radius**
  – $z \in \mathfrak{C}_a$ satisfies $|z-p_a|^2 \approx e^{-2\pi t}$
  – Power dependence in t arise from poles at p_a, p_b in the integrand
  – Extract variation in t-dependence (cfr RG with cut-off t)



Summary and outlook

- **Low energy expansion of string theory has revealed a rich structure of**
  - Modular graph functions for genus-one Riemann surfaces;
  - Kawazumi-Zhang and higher string invariants for genus-two surfaces.
- **Asymptotics for higher string invariants**
  - Remarkable structure in Laurent polynomial for non-sep degeneration
  - Tested versus maximal degeneration limit (ED, Green, Pioline 2018)
  - Similar result for separating degeneration (ED, Green, Pioline 2018)
  - Crucial for the discovery of relations between modular graph functions
- **Differential equations for higher string invariants**
  - In progress (ED, Green, Pioline)
  - Asymptotics is expected to be a key guide