Hans Jockers, The Limiting Value of D-brane Superpotentials

(joint work w/ D. Morrison, J. Walcher)

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1. Introduction

Mirror Pair (X,Y) of CY 3-folds:

Type IIB on X $\cong$ Type IIA on Y

c.s. moduli space of X — quantum Kähler moduli space of Y

$$\Pi_\Gamma(t) = \int_{\Gamma \in H_3(X,\mathbb{Z})} \Omega(t)$$ — quantum Periods of Y

c.f., Candelas, de la Ossa, Green, Park '91, ...

Homological Mirror Symmetry:

Cat. B-branes: $D^b(X)$ $\cong$ Cat. A-branes: $\mathrm{Fuk}(Y)$

@ LCS limit: subvarieties of singular CY 3-fold X_0 — @ LV limit: Lagrangian submanifolds L of Y

Goal: Mirror of certain B-branes @LCS $\rightarrow$ Construct A-branes & their properties

2. van Geemen lines of the (mirror) quintic

Quintic: Dwork pencil $X_t = \{ P = t(x_1^5 + \dots + x_5^5) + x_1 x_2 x_3 x_4 x_5 = 0 \} \subset \mathbb{CP}^4 / \mathbb{Z}_5^3$

van Geemen lines: $C_t \simeq \mathbb{P}^1 \hookrightarrow X_t \,;\; [u,v] \mapsto x_i = a_i^{1/5}(u + b_i v)$

Eqn's for a_i & b_i: $P(x_\ell(a_i,b_i)) = \sum \underbrace{c_n(a_i,b_i)}_{=0} u^n v^{5-n} = 0$

Result: 2 families of solutions: $C_{\pm,t,s}$

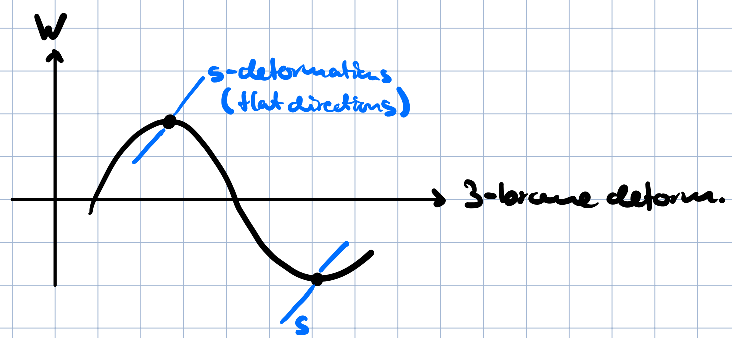
a) $C_{\pm,t,s} \sim_{\text{hom.}} C_{+,t,s'}$, i.e., $\exists$ 3-chain Γ s.t. $\partial\Gamma = C_{\pm,t,s} - C_{+,t,s'}$

b) $C_{+,t,s} \sim_{\text{rat.}} C_{+,t,s'}$, i.e., $[C_{+,t,s}] = [C_{+,t,s'}] \in CH^2(X_t)$

$C_{+,t,s} \not\sim_{\text{rat.}} C_{-,t,s'}$, i.e., $[C_{+,t,s} - C_{-,t,s'}] \neq 0$ in $CH^2(X_t)$

Physics picture: D3-branes on $\mathbb{R}^{1,1} \times C_\pm \subset \mathbb{R}^{1,3} \times X$

Soroush, HJ '08
Alim, Hecht, Mayr, Mertens '09
Li, Lian, Yau '09
(based on:
Lerche, Mayr, Warner '02
Mayr '01)



Domain wall tension:

Walcher '06

$$W(C_+) - W(C_-) = 2W(C_+) = -2W(C_-) = \int_\Gamma \Omega(t) = \nu_\Gamma(t)$$

w/ $\nu_\Gamma(t)$ a normal function of X_t

Computation of $\nu_\Gamma(t)$:

(1) Inhomog. P.F. diff. equations: $\mathcal{L}(t, t\frac{d}{dt})\,\nu_\Gamma(t) = f_\Gamma(t)$

(2) P.F. system for "open/closed" deformations for $W(t, \text{3-brane def.})$

Today: Evaluate $\int_\Gamma \Omega(t)$ in the limit $t\to 0$: Limiting value of D-brane Superpot.

$$X_t \text{ degenerates to } X_0 = \bigcup_{i=1}^{5} \mathbb{P}^3_i \quad w/ \quad \mathbb{P}^3_i := \{x_i = 0\} \cap \mathbb{CP}^4$$

Need to define: $CH^2(X_0)$ w/ X_0 singular

Green, Griffiths, Kerr '10:

Semi-stable degeneration:
$$\begin{array}{c} X_t \to \mathcal{X} \\ \downarrow \\ t\in\Delta \end{array} \quad w/ \quad \mathcal{X} \text{ smooth,} \quad (\mathcal{X} \text{ retracts to } X_0)$$

$$CH^2(X_0) := \mathbb{H}_{\delta+\partial_B}\left(Z^2\left(C^*\left(\coprod_{i=1}^{5} X_i\right), *\right)\right) \quad \text{(Hypercohomology, spectral sequence)}$$

w/ $Z^2(M,k)$: co-dim. 2-cycles in $M\times\Delta^k$ w/ $\Delta^k := \mathbb{C}[t_0,\dots,t_k]/(t_0+\dots+t_k-1)$

c.f. Bloch '86, Levine, ..., Kerr '03

$$\Rightarrow CH^2(X_0) \supset \bigoplus_{i=1}^{5} CH^2(pt_i, 3)$$

Result:

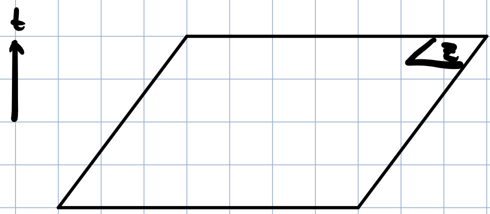
(1) $[C_+ - C_-] \in CH^2(\mathbb{Q}(\sqrt{-3}),3) \subset CH^2(pt,3) \subset CH^2(X_0)$

(2) $[C_+ - C_-] \in \mathcal{B}_2(\mathbb{C})$ "Bloch group"

Neumann, Zagier '85: Ideal triangulation of hyperbolic 3-manifold

3. Hyperbolic 3-manifolds & A-branes

$$\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{>0} \ni (z,t), \quad ds^2 = \frac{dz\,d\bar{z} + dt^2}{t^2}$$



$\mathrm{Isom}^+(\mathbb{H}_3) \simeq PSL(2,\mathbb{C})$

Orient. Hyperbolic 3-manifolds $\mathcal{H} \ni M = \mathbb{H}^3/G$ w/ G a discrete subgroup of $PSL(2,\mathbb{C})$ (Kleinian Group)

Scissor congruence group: $\mathcal{P}(\mathbb{H}^3) \ni P$ polytopes w/ geodesic edges

$P \sim P' \iff P$ is scissor congruent to P'

Invariants: $P \sim P' \Rightarrow \mathrm{vol}(P) = \mathrm{vol}(P')$

$$P \sim P' \Rightarrow \delta(P) = \sum_{\text{edges } e} \ell(e) \otimes \theta(e) = \delta(P') \quad \text{Dehn invariant}$$
$$\in \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$$

Set $D(\mathbb{H}^3) = \{\ker \delta\} \subset \mathcal{P}(\mathbb{H}^3)$

Conj: $\mathrm{vol}: D(\mathbb{H}^3) \to \mathbb{R}$ is injective (relates to Hilbert's 3^{rd} Problem on $\mathbb{H}^3$)

Fact: $\mathcal{H}_{\text{compact}} \to D(\mathbb{H}^3)$, fund. domain $\mapsto P$ (not injective)

$\cup$ M/G, G has no parabolic element

Bloch, Dupont, Sah, W. Neumann
W. Neumann, Yang

$$\begin{tikzcd} CH^2(pt,3) \arrow[r, "\text{"Mirror Sym."}"] \arrow[dr, "\mathrm{Reg}"'] & D(\mathbb{H}^3) \arrow[d, "\mathrm{vol}"] \\ & \mathbb{R} \end{tikzcd}$$

Kerr, Lewis, Müller-Stach '04; Kerr '03 (Ph.D.)

"Limit Superpotential Value"

Results:

c.f. textbook: Maclachlan, Reid

(1) $\mathrm{Reg}(C_{+,0,s}) = \frac{65}{4\pi^2} \left|\Delta_{\mathbb{Q}\sqrt{-3}}\right|^{3/2} \zeta_{\mathbb{Q}(\sqrt{-3})}(2)$

$\zeta_{\mathbb{Q}(\sqrt{-3})}(s) = \prod_{\mathfrak{P}} (1 - N(\mathfrak{P})^{-s})^{-1}$ $\quad \mathfrak{P}$: prime ideals of ring of integers $R_{\mathbb{Q}(\sqrt{-3})}$

(2) Classify compact arithmetic hyperbolic 3-manifolds M via quaternionic algebras ($\Rightarrow$ prefactor 65)

(3) Proposal: A-brane L is scissor congruent to M