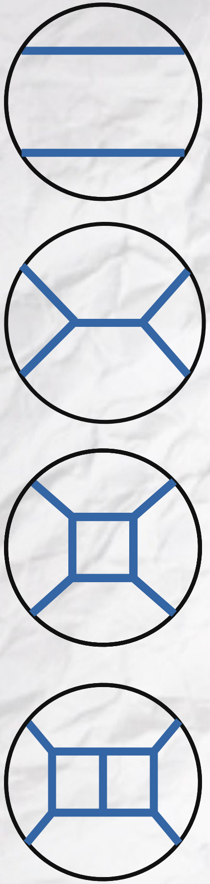


Two-loop supergravity on AdS_5 x S^5 from CFT

Hynek Paul

IPhT Saclay

See also: [2112.15174] by Huang and Yuan!

Based on:
[2204.01829] w/ James Drummond

Online seminar on motives and period integrals in QFT and ST, 08/02/2023

Motivation

Can we study (gravitational) scattering on curved space times?

- **Computing scattering amplitudes is already very hard already at tree-level!**
- **Even harder at loop-order (direct computations only in some toy-models)**

- **Let us use the AdS/CFT duality, which is an arena relating**
  - ✗ theories on curved space times (AdS)
  - ✗ QFT's at strong coupling

- **Can be used to describe strongly coupled gauge theory in terms of weakly coupled gravity theory**
- **Here: Thanks to analytic bootstrap methods even weakly coupled AdS gravity is more tractable from the dual CFT perspective!**

General setup

AdS/CFT correspondence

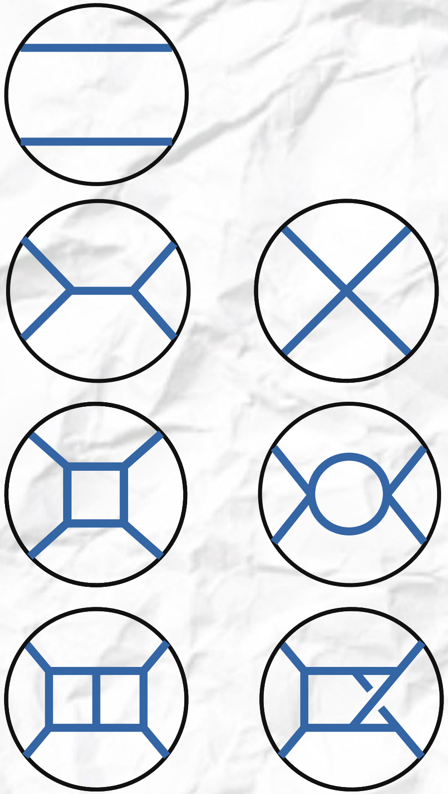
$\mathcal{N}=4$ SYM **with gauge group SU(N)**	$\Longleftrightarrow$	**supergravity on** $\mathrm{AdS}_5\times \mathrm{S}^5$
single-particle operators $\mathcal{O}_p$	$\Longleftrightarrow$	**single-particle states**
4pt correlation functions	$\Longleftrightarrow$	**AdS amplitudes** **(Witten diagrams)**
large N, strong coupling limit: $\mathrm{N}\to\infty,\ \lambda\to\infty$	$\Longleftrightarrow$	**supergravity limit:** $g_s\to 0,\ \alpha'\to 0$

't Hooft coupling
$\lambda = g_{\mathrm{YM}}^2 N$

Interested in corrections to supergravity in AdS:
loop-corrections (1/N) and string corrections (1/λ)

General setup

4pt-function of graviton multiplet: $\langle O_2 O_2 O_2 O_2 \rangle$



N^0 : **Generalised free fields** ✓

$1/N^2$: **Tree-level supergravity** ✓ [Arutyunov,Sokatchev,D'Hoker,Rastelli,...] [Rastelli-Zhou'16'17]

$1/N^4$: **One-loop supergravity** ✓ [Alday-Bissi'17] [Alday-CaronHuot'17] [Aprile-Drummond-Heslop-HP'17-'19]

$1/N^6$: **Two-loop supergravity** *this talk*

$\vdots$

+ higher derivative corrections $O(1/\lambda)$

Outline

(1) **The object of interest: the $\langle O_2 O_2 O_2 O_2 \rangle$ correlator**

(2) **The large N strong coupling expansion**

(3) **Consider loops: the leading log to any loop order**

(4) **Review of tree-level & one-loop correlators**

(5) **Bootstrapping the two-loop correlator**

(6) **Extracting the two-loop anomalous dimension**

(7) **Summary & Outlook**

The $\langle O_2 O_2 O_2 O_2 \rangle$ correlator

Simplest operator to consider: $\mathcal{O}_2(x,y) = y^i y^j \, \mathrm{Tr}\big(\Phi_i(x)\Phi_j(x)\big)$ $\qquad y^2 = 0$

- **half-BPS single-trace operator, protected dimension** $\Delta = 2$
- **transforms in [0,2,0] representation of su(4) R-symmetry**
- **dual to supergraviton multiplet**
- **higher-charge operators** $\mathcal{O}_p$ **dual to KK-modes on** S^5

Two & three-point functions protected: study four-point functions

$$\langle \mathcal{O}_2(x_1,y_1)\mathcal{O}_2(x_2,y_2)\mathcal{O}_2(x_3,y_3)\mathcal{O}_2(x_4,y_4)\rangle = g_{12}^2 g_{34}^2 \, \mathcal{G}(u,v;\sigma,\tau)$$

Dependence on conformal and R-symmetry cross-ratios:

$$u = x\bar{x} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = (1-x)(1-\bar{x}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2},$$
$$\frac{1}{\sigma} = y\bar{y} = \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2}, \qquad \frac{\tau}{\sigma} = (1-y)(1-\bar{y}) = \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2}.$$

The $\langle O_2 O_2 O_2 O_2 \rangle$ correlator

[Eden-Petkou-Schubert-Sokatchev, Nirschl-Osborn]

Consequence of superconformal symmetry ('partial non-renormalisation theorem'):

$$\mathcal{G}(u,v;\sigma,\tau) = \mathcal{G}_{\text{free}}(u,v;\sigma,\tau) + \mathcal{I}\,\mathcal{H}(u,v)$$

$$\mathcal{I} = \frac{(x-y)(x-\bar{y})(\bar{x}-y)(\bar{x}-\bar{y})}{(y\bar{y})^2}$$

➔ **Free-theory correlator given by Wick-contractions:**

$$\mathcal{G}_{\text{free}}(u,v;\sigma,\tau) = 4\left(1 + u^2\sigma^2 + \frac{u^2\tau^2}{v^2}\right) + 16a\left(u\sigma + \frac{u\tau}{v} + \frac{u^2\sigma\tau}{v}\right)$$

➔ **All non-trivial dynamics (i.e. coupling dependence) is captured by the 'interacting part`** $\mathcal{H}(u,v)$

Note:

(1) independent of R-symmetry cross-ratios

(2) is fully crossing symmetric: $\mathcal{H}(u,v) = \frac{1}{v^2}\mathcal{H}(u/v,1/v) = \frac{u^2}{v^2}\mathcal{H}(v,u)$

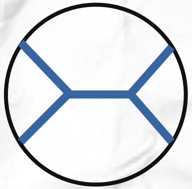
(3) admits a large N expansion $\mathcal{H}(u,v) = a\,\mathcal{H}^{(1)}(u,v) + a^2\,\mathcal{H}^{(2)}(u,v) + a^3\,\mathcal{H}^{(3)}(u,v) + O(a^4)$

large N expansion parameter

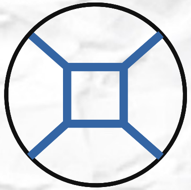
$$a = \frac{1}{N^2-1}$$

The large N, strong coupling expansion

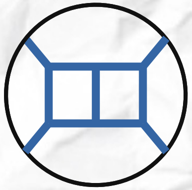
	$\lambda^{\frac{3}{2}}$	λ	$\lambda^{\frac{1}{2}}$	λ^0 (SUGRA)	$\lambda^{-\frac{1}{2}}$	λ^{-1}	$\lambda^{-\frac{3}{2}}$	λ^{-2}	$\lambda^{-\frac{5}{2}}$	λ^{-3}	$\lambda^{-\frac{7}{2}}$	λ^{-4}	$\dots$
a				$\mathcal{H}^{(1)}$									
a^2				$\mathcal{H}^{(2)}$									
a^3				$\mathcal{H}^{(3)}$									



tree-level



one-loop



two-loops

The large N, strong coupling expansion

tower of higher derivative corrections

				SUGRA			$\mathcal{R}^4$		$\partial^4\mathcal{R}^4$	$\partial^6\mathcal{R}^4$	$\partial^8\mathcal{R}^4$	$\partial^{10}\mathcal{R}^4$	
	$\lambda^{\frac{3}{2}}$	λ	$\lambda^{\frac{1}{2}}$	λ^0	$\lambda^{-\frac{1}{2}}$	λ^{-1}	$\lambda^{-\frac{3}{2}}$	λ^{-2}	$\lambda^{-\frac{5}{2}}$	λ^{-3}	$\lambda^{-\frac{7}{2}}$	λ^{-4}	$\ldots$
a				$\mathcal{H}^{(1)}$									
a^2				$\mathcal{H}^{(2)}$									
a^3				$\mathcal{H}^{(3)}$									

The large N, strong coupling expansion

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	$\lambda^{\frac{3}{2}}$	λ	$\lambda^{\frac{1}{2}}$	λ^0	$\lambda^{-\frac{1}{2}}$	λ^{-1}	$\lambda^{-\frac{3}{2}}$	λ^{-2}	$\lambda^{-\frac{5}{2}}$	λ^{-3}	$\lambda^{-\frac{7}{2}}$	λ^{-4}	$\ldots$
a				$\mathcal{H}^{(1)}$			$\mathcal{H}^{(1,3)}$		$\mathcal{H}^{(1,5)}$	$\mathcal{H}^{(1,6)}$	$\mathcal{H}^{(1,7)}$	$\mathcal{H}^{(1,8)}$	
a^2				$\mathcal{H}^{(2)}$									
a^3				$\mathcal{H}^{(3)}$									

$\rightarrow$ **Virasoro-Shapiro amplitude in** $\mathrm{AdS}_5 \times \mathrm{S}^5$

[Binder,Chester,Pufu,Wang,Aprile,Drummond, HP,Rigatos,Santagata,Alday,Hansen,Silva,...]

[see T. Hansen's talk from November]

The large $\mathcal{N}$, strong coupling expansion

tower of higher derivative corrections

				SUGRA			$\mathcal{R}^4$		$\partial^4\mathcal{R}^4$	$\partial^6\mathcal{R}^4$	$\partial^8\mathcal{R}^4$	$\partial^{10}\mathcal{R}^4$	
	$\lambda^{\frac{3}{2}}$	λ	$\lambda^{\frac{1}{2}}$	λ^0	$\lambda^{-\frac{1}{2}}$	λ^{-1}	$\lambda^{-\frac{3}{2}}$	λ^{-2}	$\lambda^{-\frac{5}{2}}$	λ^{-3}	$\lambda^{-\frac{7}{2}}$	λ^{-4}	$\ldots$
a				$\mathcal{H}^{(1)}$			$\mathcal{H}^{(1,3)}$		$\mathcal{H}^{(1,5)}$	$\mathcal{H}^{(1,6)}$	$\mathcal{H}^{(1,7)}$	$\mathcal{H}^{(1,8)}$	
a^2				$\mathcal{H}^{(2)}$									
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→ **Virasoro-Shapiro amplitude in** $\mathrm{AdS}_5 \times \mathrm{S}^5$

[Binder,Chester,Pufu,Wang,Aprile,Drummond, HP,Rigatos,Santagata,Alday,Hansen,Silva,...]

[see T. Hansen's talk from November]

follows from corresponding flat space amplitude:

low energy expansion of massless 4-particle amplitude of 10d type IIB string amplitude

[D'Hoker,Green,Russo,Vanhove, Gutperle,Phong,Gomez,Mafra,Pioline,...]

$\lambda^{-\frac{1}{2}} \sim \alpha'$ $\quad a \sim g_s^2(\alpha')^4$

The large $\mathcal{N}$, strong coupling expansion

tower of higher derivative corrections

	$\lambda^{\frac{3}{2}}$	λ	$\lambda^{\frac{1}{2}}$	λ^0	$\lambda^{-\frac{1}{2}}$	λ^{-1}	$\lambda^{-\frac{3}{2}}$	λ^{-2}	$\lambda^{-\frac{5}{2}}$	λ^{-3}	$\lambda^{-\frac{7}{2}}$	λ^{-4}	$\ldots$
				SUGRA			$\mathcal{R}^4$		$\partial^4\mathcal{R}^4$	$\partial^6\mathcal{R}^4$	$\partial^8\mathcal{R}^4$	$\partial^{10}\mathcal{R}^4$	
a				$\mathcal{H}^{(1)}$			$\mathcal{H}^{(1,3)}$		$\mathcal{H}^{(1,5)}$	$\mathcal{H}^{(1,6)}$	$\mathcal{H}^{(1,7)}$	$\mathcal{H}^{(1,8)}$	
a^2			$\mathcal{H}^{(2,-1)}$	$\mathcal{H}^{(2)}$	0	$\mathcal{H}^{(2,2)}$	$\mathcal{H}^{(2,3)}$	$\mathcal{H}^{(2,4)}$	$\ldots$				
a^3	$\mathcal{H}^{(3,-3)}$	$\mathcal{H}^{(3,-2)}$	$\mathcal{H}^{(3,-1)}$	$\mathcal{H}^{(3)}$	$\ldots$								

S-duality (modular invariance)

'finite counter-terms'

follows from corresponding flat space amplitude:

low energy expansion of massless 4-particle amplitude of 10d type IIB string amplitude

$$\lambda^{-\frac{1}{2}} \sim \alpha' \qquad a \sim g_s^2(\alpha')^4$$

[Aharony-Alday-Bissi-Perlmutter'16]

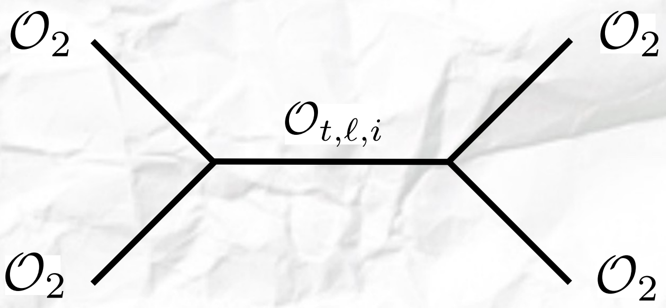
Ok, so far I have shown you the overall structure of the correlator...

... but how do you compute any of this?

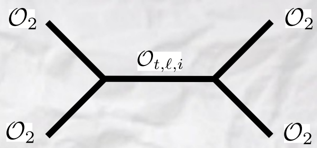
Ok, so far I have shown you the overall structure of the correlator...

... but how do you compute any of this?

→ Main tool: exploit consistency of the OPE



The OPE expansion and the double-trace spectrum



→ **correlator admits an expansion into (super)conformal blocks:**

$$\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle_{\text{long}} = g_{12}^2 g_{34}^2\, \mathcal{I} \sum_{t,\ell} A_{t,\ell}\, G_{t,\ell}(x,\bar{x})$$

conformal block

$$(-1)^\ell (x\bar{x})^t\, \frac{x^{\ell+1} F_{t+\ell+2}(x) F_{t+1}(\bar{x}) - \bar{x}^{\ell+1} F_{t+\ell+2}(\bar{x}) F_{t+1}(x)}{x-\bar{x}}$$

[Dolan,Osborn]

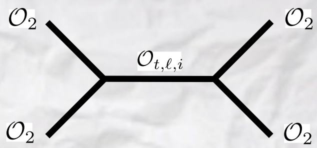
OPE coefficients $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_{t,\ell,i}\rangle^2$

sum over unprotected (long) operators $\mathcal{O}_{t,\ell,i}$

of twist $t = \frac{1}{2}(\Delta^{(0)} - \ell)$ **and spin** ℓ

What is the spectrum of exchanged operators?

The OPE expansion and the double-trace spectrum



Recall: N=4 SYM at large N & strong coupling $\Leftrightarrow$ supergravity limit

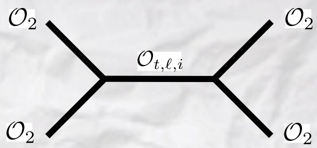
$\rightarrow$ long single-trace operators ('string states') decouple!

Remaining spectrum:

- **made from products of half-BPS operators**
- **at leading order in large N: only double-trace operators**

(correspond to bound, two-particle states in supergravity)

The OPE expansion and the double-trace spectrum



Recall: N=4 SYM at large N & strong coupling $\Leftrightarrow$ supergravity limit

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Remaining spectrum:

- made from products of half-BPS operators
- at leading order in large N: only double-trace operators

(correspond to bound, two-particle states in supergravity)

→ These operators are degenerate and they mix: $\mathcal{O}_2\Box^{t-2}\partial^\ell\mathcal{O}_2,\ \mathcal{O}_3\Box^{t-3}\partial^\ell\mathcal{O}_3,\ \ldots\ ,\ \mathcal{O}_t\Box^0\partial^\ell\mathcal{O}_t$

Good news: the mixing problem has been solved

- by considering many tree-level correlators, one can resolve the degeneracy
- leading-order three-point functions and anomalous dimensions are known!

[Aprile-Drummond-Heslop-HP'17'18]

$$A^{(0)}_{t,\ell,i} \qquad \gamma_i^{(1)} = -\frac{2(t-1)_4(t+\ell)_4}{(\ell+2i-1)_6}$$

The structure of the leading log

$$\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle_{\text{long}} = g_{12}^2 g_{34}^2\, \mathcal{I} \sum_{t,\ell} A_{t,\ell}\, G_{t,\ell}(x,\bar{x})$$

$$\Delta_{t,\ell,i} = \Delta^{(0)} + 2\big(a\gamma_i^{(1)} + a^2\gamma_i^{(2)} + a^3\gamma_i^{(3)} + O(a^4)\big)$$
$$A_{t,\ell,i} = A_{t,\ell,i}^{(0)} + aA_{t,\ell,i}^{(1)} + a^2 A_{t,\ell,i}^{(2)} + a^3 A_{t,\ell,i}^{(3)} + O(a^4)$$

Combining the OPE decomposition with the large N expansion, one finds:

$$\begin{aligned}
\mathcal{H}^{(1)} =\;& \log^1(u)\left[A^{(0)}\gamma^{(1)}\right] G_{t,\ell}(x,\bar{x}) \\
&+ \log^0(u)\left[A^{(1)} + 2A^{(0)}\gamma^{(1)}\partial_\Delta\right] G_{t,\ell}(x,\bar{x})\,, \\
\mathcal{H}^{(2)} =\;& \log^2(u)\left[\tfrac{1}{2}A^{(0)}(\gamma^{(1)})^2\right] G_{t,\ell}(x,\bar{x}) \\
&+ \log^1(u)\left[(A^{(1)}\gamma^{(1)} + A^{(0)}\gamma^{(2)}) + 2A^{(0)}(\gamma^{(1)})^2\partial_\Delta\right] G_{t,\ell}(x,\bar{x}) \\
&+ \log^0(u)\left[A^{(2)} + 2(A^{(1)}\gamma^{(1)} + A^{(0)}\gamma^{(2)})\,\partial_\Delta + 2A^{(0)}(\gamma^{(1)})^2\partial_\Delta^2\right] G_{t,\ell}(x,\bar{x})\,, \\
\mathcal{H}^{(3)} =\;& \log^3(u)\left[\tfrac{1}{6}A^{(0)}(\gamma^{(1)})^3\right] G_{t,\ell}(x,\bar{x}) \\
&+ \log^2(u)\left[(\tfrac{1}{2}A^{(1)}(\gamma^{(1)})^2 + A^{(0)}\gamma^{(1)}\gamma^{(2)}) + A^{(0)}(\gamma^{(1)})^3\partial_\Delta\right] G_{t,\ell}(x,\bar{x}) \\
&+ \log^1(u)\left[(A^{(2)}\gamma^{(1)} + A^{(1)}\gamma^{(2)} + A^{(0)}\gamma^{(3)})\right. \\
&\qquad\left. + 2(A^{(1)}(\gamma^{(1)})^2 + 2A^{(0)}\gamma^{(1)}\gamma^{(2)})\partial_\Delta + 2A^{(0)}(\gamma^{(1)})^3\partial_\Delta^2\right] G_{t,\ell}(x,\bar{x}) \\
&+ \log^0(u)\left[A^{(3)} + 2(A^{(2)}\gamma^{(1)} + A^{(1)}\gamma^{(2)} + A^{(0)}\gamma^{(3)})\partial_\Delta\right. \\
&\qquad\left. + 2(A^{(1)}(\gamma^{(1)})^2 + 2A^{(0)}\gamma^{(1)}\gamma^{(2)})\partial_\Delta^2 + \tfrac{4}{3}A^{(0)}(\gamma^{(1)})^3\partial_\Delta^3\right] G_{t,\ell}(x,\bar{x})\,.
\end{aligned}$$

The structure of the leading log

$$\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle_{\text{long}} = g_{12}^2 g_{34}^2 \, \mathcal{I} \sum_{t,\ell} A_{t,\ell}\, G_{t,\ell}(x,\bar{x})$$

$$\Delta_{t,\ell,i} = \Delta^{(0)} + 2\left(a\gamma_i^{(1)} + a^2\gamma_i^{(2)} + a^3\gamma_i^{(3)} + O(a^4)\right)$$
$$A_{t,\ell,i} = A^{(0)}_{t,\ell,i} + aA^{(1)}_{t,\ell,i} + a^2A^{(2)}_{t,\ell,i} + a^3A^{(3)}_{t,\ell,i} + O(a^4)$$

Combining the OPE decomposition with the large N expansion, one finds:

$$\mathcal{H}^{(n)}(u,v)|_{\log^n(u)} = \frac{1}{n!}\sum_{t,\ell}\sum_{i=1}^{t-1} A^{(0)}_{t,\ell,i}\left(\gamma_i^{(1)}\right)^n G_{t,\ell}(x,\bar{x})$$

the leading log is determined by tree-level data only!

The structure of the leading log

$$\mathcal{H}^{(n)}(u,v)\big|_{\log^n(u)} = \frac{1}{n!}\sum_{t,\ell}\sum_{i=1}^{t-1} A^{(0)}_{t,\ell,i}\left(\gamma_i^{(1)}\right)^n G_{t,\ell}(x,\bar{x})$$

→ **one finds that these expressions resum to** $\dfrac{u^2\ f^{(n)}_{\log}(x,\bar{x})}{(x-\bar{x})^{8n-1}}$

HPL's up to weight n
with polynomial coefficients

The structure of the leading log

$$\mathcal{H}^{(n)}(u,v)|_{\log^n(u)} = \frac{1}{n!}\sum_{t,\ell}\sum_{i=1}^{t-1} A^{(0)}_{t,\ell,i}\left(\gamma^{(1)}_i\right)^n G_{t,\ell}(x,\bar{x})$$

→ one finds that these expressions resum to $\dfrac{u^2\ f^{(n)}_{\log}(x,\bar{x})}{(x-\bar{x})^{8n-1}}$

HPL's up to weight n with polynomial coefficients

Harmonic Polylogarithms (HPL's) are a class of transcendental functions of one real variable

- **generalisation of log's and polylog's**
- **labelled by and index: word w with letters** $a_i \in \{0,1\}$
- **iterative definition:**

$$H_{0w}(x) = \int_0^x dx' \frac{H_w(x')}{x'}$$

$$H_{1w}(x) = \int_0^x dx' \frac{H_w(x')}{1-x'}$$

- **weight := length of word**

Examples:

$H_0 = \log(x) \quad H_1 = -\log(1-x)$

$H_{0^m} = \frac{1}{m!}\log^m(x)$

$H_{0^n 1} = \mathrm{Li}_{n+1}(x)$

The structure of the leading log

[Aprile-Drummond-Heslop-HP'18]
[CaronHuot-Trinh'18]

One can considerably simplify the leading log by the use of a differential operator

$$\Delta^{(8)} = \frac{u^4}{(x-\bar{x})}\partial_{\bar{x}}^2(1-\bar{x})^2\partial_{\bar{x}}^2\partial_x^2(1-x)^2\partial_x^2(x-\bar{x})$$

→ **with symmetries**

$$\left(\Delta^{(8)}\right)^k \xrightarrow{x\to x'} \left(\Delta^{(8)}\right)^k$$

$$\left(\Delta^{(8)}\right)^2 \xrightarrow{x\to 1-x} \frac{u^4}{v^4}\left(\Delta^{(8)}\right)^2$$

- **its eigenvalue is the numerator of the anomalous dimension:** $\Delta^{(8)}\, u^2 G_{t,\ell}(x,\bar{x}) = (t-1)_4(t+\ell)_4\, u^2 G_{t,\ell}(x,\bar{x})$

- **one can therefore pull it out from the leading log!**

$$\mathcal{H}^{(n)}(u,v)|_{\log^n(u)} = \frac{1}{n!}\sum_{t,\ell}\sum_{i=1}^{t-1} A^{(0)}_{t,\ell,i}\left(\gamma_i^{(1)}\right)^n G_{t,\ell}(x,\bar{x})$$

$$\mathcal{H}^{(n)}(u,v)|_{\log^n(u)} = \frac{1}{u^2}\left(\Delta^{(8)}\right)^{n-1} g^{(n)}(x,\bar{x})$$

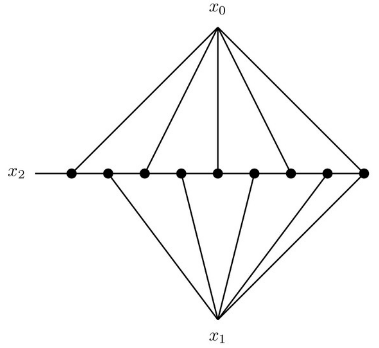
up to (n-1) times

much simpler expression!

there are further simplifications possible, but here is the main point:

The leading log can be explicitly computed (case by case) and its transcendental structure is entirely captured by zigzag-integrals and derivatives thereof!

The zigzag-integrals $Z^{(L)}$



[Drummond'12]

They are a special class of 4d loop-integrals:

- **they arise from a generalisation of the ladder integrals** $\phi^{(L)}$
- **determined by a differential equation**

$$x\bar{x}\partial_x\partial_{\bar{x}} Z^{(L)}(x,\bar{x}) = Z^{(L-1)}(1-x, 1-\bar{x})$$

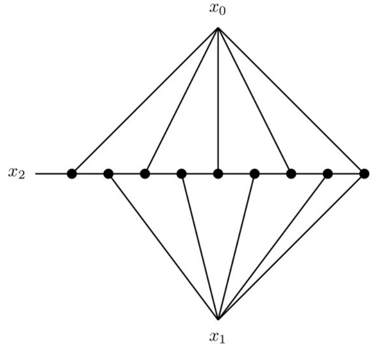
with

$$Z^{(1)}(x,\bar{x}) = \phi^{(1)}(x,\bar{x})$$
$$Z^{(2)}(x,\bar{x}) = \phi^{(2)}(x,\bar{x})$$

one- and two-loop ladder integrals

In terms of SVHPL's:

The zigzag-integrals $Z^{(L)}$



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$$Z^{(1)}(x,\bar{x}) = \phi^{(1)}(x,\bar{x})$$
$$Z^{(2)}(x,\bar{x}) = \phi^{(2)}(x,\bar{x})$$

one- and two-loop ladder integrals

In terms of SVHPL's:

$$Z^{(1)} = \mathcal{L}_2 - \mathcal{L}_{10}\,,$$
$$Z^{(2)} = \mathcal{L}_{200} - \mathcal{L}_{30}\,,$$
$$Z^{(3)} = \mathcal{L}_{2210} - \mathcal{L}_{2120} - 2\zeta_3(3\mathcal{L}_{20} + 2\mathcal{L}_{21})\,,$$
$$Z^{(4)} = \mathcal{L}_{2230} - \mathcal{L}_{2320} - 4\zeta_3(\mathcal{L}_{23} - \mathcal{L}_{220}) - 20\zeta_5\mathcal{L}_{20}\,,$$
$$Z^{(5)} = \mathcal{L}_{222120} - \mathcal{L}_{221220} + 4\zeta_3(\mathcal{L}_{2221} - \mathcal{L}_{2212}) + \zeta_5(4\mathcal{L}_{221} + 15\mathcal{L}_{220}) - 12\zeta_3^2\mathcal{L}_{22} - \frac{441}{8}\zeta_7\mathcal{L}_{20} + 18\zeta_3\zeta_5\mathcal{L}_2\,.$$

Single-valued HPL's:

$$\mathcal{L}_w(x,\bar{x}) = H_w(x) + \sum_{w_1,w_2} c_{w_1,w_2}H_{w_1}(x)H_{w_2}(\bar{x})$$

Next task: complete the leading log to a fully crossing symmetric function

Review of tree-level and one-loop

✔ **The tree-level correlator is given by** $\mathcal{H}^{(1)} = -16u^2\overline{D}_{2422}$

[Arutyunov-Frolov'00, Dolan-Osborn'01]

$$\mathcal{H}^{(1)} = \sum_i \frac{p_i(x,\bar{x})}{(x-\bar{x})^{d_i}} \mathcal{Q}_i(x,\bar{x})$$

denominator power 7

→ **SVHPL's up to weight 2:** $Z^{(1)}$

$\log(u) \quad \log(v)$

1

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- **One-loop correlator obtained by 'bootstrap method'**

[Aprile-Drummond-Heslop-HP'17]

$$\mathcal{H}^{(2)} = \sum_i \frac{p_i(x,\bar{x})}{(x-\bar{x})^{d_i}} \mathcal{Q}_i(x,\bar{x})$$

denominator power 15

→ **SVHPL's up to weight 4:** $3 \times Z^{(2)}$

$2 \times \log(u) Z^{(1)} \quad 3 \times \Psi^{(2)}$

$Z^{(1)}$

$\log(u) \quad \log(v)$

1

Nomenclature: these functions are built from the 'letters' $\{x, \bar{x}, 1-x, 1-\bar{x}\}$

Review of tree-level and one-loop

However, this can be considerably simplified using $\Delta^{(8)}$!

[Aprile-Drummond-Heslop-HP'19]

→ the entire correlator can be written with $\Delta^{(8)}$ pulled out:

$$\mathcal{H}^{(2)} = \frac{1}{u^2}\Delta^{(8)}\mathcal{L}^{(2)} + \mathcal{H}^{(1)}$$

much simpler 'preamplitude':

- denominator power 7
- same complexity as the tree-level correlator!

Note: the bootstrap conditions leave one free parameter: α

→ due to a one-loop divergence: counter-term ambiguity!

→ related to super-leading term $\mathcal{R}^4|_{\text{genus-1}}$

→ given by tree-level contact diagram $u^2\overline{D}_{4444}$,

which gives rise to a non-analytic contribution at spin 0:

$$\gamma^{(2)}_{2,\ell} = \frac{1344(\ell-7)(\ell+14)}{(\ell-1)(\ell+1)^2(\ell+6)^2(\ell+8)} - \frac{2304(2\ell+7)}{(\ell+1)^3(\ell+6)^3} - \frac{18\alpha}{7}\delta_{\ell,0}$$

supersymmetric localisation determines $\alpha = 60$ [Chester-Pufu]

Bootstrapping the two-loop correlator

Structure of leading log + intuition from tree-level & one-loop correlators

Natural proposal for a 'minimal ansatz' for $\mathcal{H}^{(3)}$:

$$\mathcal{H}^{(3)} = \frac{1}{u^2}(\Delta^{(8)})^2\,\mathcal{P}^{(3)} + a_2\mathcal{H}^{(2)} + a_1\mathcal{H}^{(1)}$$

with preamplitude $$\mathcal{P}^{(3)}(x,\bar{x}) = \sum_i \frac{p_i(x,\bar{x})}{(x-\bar{x})^{d_i}}\,\mathcal{Q}_i(x,\bar{x})$$

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Basis of functions $\mathcal{Q}_i(x,\bar{x})$:

- **SVHPL's up to weight 6**
- **no $\log^4$(u) in any channel**
- **$\log^3$(u) contributions from $Z^{(3)}$ and derivatives**
- **subtlety: need to include new letter $x-\bar{x}$ at weight 3:**

  (1) one-loop ambiguity α induces one-loop like contribution of the form of $\mathcal{H}^{(2,3)}$

  (2) from one-loop string corrections, we know new function $f^{(3)}(x,\bar{x})$ is required

Bootstrapping the two-loop correlator

this leads to the following 73 basis functions

w	$x \leftrightarrow \bar{x}$	$\mathcal{Q}_i(x,\bar{x})$	total
6	$-$	$6 \times Z^{(3)},\ A^{(6)},\ 3 \times B^{(6)},\ \zeta_3 f^{(3)},\ 2 \times \zeta_3 \log(u) Z^{(1)}$	13
	$+$	-	0
5	$-$	$6 \times \widetilde{\Psi}^{(3)},\ 3 \times \widetilde{\Pi}^{(5)},\ \zeta_3 Z^{(1)}$	10
	$+$	$6 \times \Psi^{(3)},\ 6 \times \Pi^{(5)},\ \Omega^{(5)},\ 2 \times \log(u)(Z^{(1)})^2,\ 3 \times \zeta_3 \log^2(u)$	18
4	$-$	$3 \times \log^2(u) Z^{(1)},\ 3 \times Z^{(2)}$	6
	$+$	$6 \times \Upsilon^{(3)},\ 2 \times \log^3(u)\log(v),\ (Z^{(1)})^2$	9
3	$-$	$f^{(3)},\ 2 \times \log(u) Z^{(1)}$	3
	$+$	$4 \times \log^3(u),\ 3 \times \Psi^{(2)}$	7
2	$-$	$Z^{(1)}$	1
	$+$	$3 \times \log^2(u)$	3
1	$-$	-	0
	$+$	$2 \times \log(u)$	2
0	$-$	-	0
	$+$	1	1

initial ansatz has 2308 free parameters

Bootstrapping the two-loop correlator

Impose bootstrap constraints:

(1) Preamplitude can be made fully crossing symmetric

$$\mathcal{P}^{(3)}(x,\bar{x}) = \mathcal{P}^{(3)}(x',\bar{x}') = \mathcal{P}^{(3)}(1-x,1-\bar{x})$$

(2) Matching the leading log, i.e. $log^3(u)$

(3) Pole cancellation: (Euclidean) correlator has no singularity at $x = \bar{x}$

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Constraints on $\mathcal{H}^{(3)}$

(4) Below twist 4 cancellation (this fixes $a_1 = -1$ **)**

$$\sum_{\ell}\left(\frac{1}{2}A^{(1)}_{2,\ell}\left(\gamma^{(1)}_{2,\ell}\right)^2 + A^{(0)}_{2,\ell}\,\gamma^{(1)}_{2,\ell}\,\gamma^{(2)}_{2,\ell} + A^{(0)}_{2,\ell}\left(\gamma^{(1)}_{2,\ell}\right)^3\partial_\Delta\right)G_{2,\ell}(x,\bar{x})$$

(5) Matching the log²(u) prediction at twist 4 using OPE data from previous orders

(this fixes $a_2 = 5$ **)**

(6) Matching the flat space correlator: two-loop supergravity in 10d

[Bissi-Fardelli-Georgoudis'20]

Bootstrapping the two-loop correlator: results

Lo and behold, we are left with only 8 free parameters!

$$\mathcal{H}^{(3)} = \frac{1}{u^2}(\Delta^{(8)})^2\,\mathcal{P}^{(3)} + 5\,\mathcal{H}^{(2)} - \mathcal{H}^{(1)}$$

Due to complexity → ancillary file

- **Free parameters are of the form of tree-level contact diagrams:** $4 \times \overline{D}$ **and** $4 \times \zeta_3 \overline{D}$
- **They are the expected ambiguities from** $\partial^{10}\mathcal{R}^4|_{\text{genus-2}}$**, corresponding to:** $\sigma_2 - \frac{16}{7}$ $\sigma_3 + \frac{32}{7}$ $\sigma_2^2 - \frac{128}{7}$ $\sigma_2\sigma_3 - \frac{256}{7}$
- **Note they are part of the preamplitude only (just like in the one-loop case)**

Mellin amplitude '1' is not independent ambiguity any more: no $\mathcal{R}^4|_{\text{genus-2}}$

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Mellin amplitude '1' is not independent ambiguity any more: no $\mathcal{R}^4|_{\text{genus-2}}$

And further:

- We observe the absence of some functions at weight 6: $3 \times B^{(6)},\; 2 \times \zeta_3 \log(u) Z^{(1)}$
- Contribution from one-loop ambiguity α proportional to one-loop string correction $\mathcal{H}^{(2,3)}$

  i.e. can be written as part of the preamplitude → $\mathcal{H}^{(2,3)} = \frac{1}{u^2}(\Delta^{(8)})^2\,\mathcal{P}^{(2,3)}$

The large $\mathcal{N}$ strong coupling expansion

higher derivative corrections

				SUGRA			$\mathcal{R}^4$		$\partial^4\mathcal{R}^4$	$\partial^6\mathcal{R}^4$	$\partial^8\mathcal{R}^4$	$\partial^{10}\mathcal{R}^4$	
	$\lambda^{\frac{3}{2}}$	λ	$\lambda^{\frac{1}{2}}$	λ^0	$\lambda^{-\frac{1}{2}}$	λ^{-1}	$\lambda^{-\frac{3}{2}}$	λ^{-2}	$\lambda^{-\frac{5}{2}}$	λ^{-3}	$\lambda^{-\frac{7}{2}}$	λ^{-4}	$\dots$
a				$\mathcal{H}^{(1)}$			$\mathcal{H}^{(1,3)}$		$\mathcal{H}^{(1,5)}$	$\mathcal{H}^{(1,6)}$	$\mathcal{H}^{(1,7)}$	$\mathcal{H}^{(1,8)}$	
a^2			$\mathcal{H}^{(2,-1)}$	$\mathcal{H}^{(2)}$	0	$\mathcal{H}^{(2,2)}$	$\mathcal{H}^{(2,3)}$	$\mathcal{H}^{(2,4)}$	$\dots$				
a^3	$\mathcal{H}^{(3,-3)}$	$\mathcal{H}^{(3,-2)}$	$\mathcal{H}^{(3,-1)}$	$\mathcal{H}^{(3)}$	$\dots$								

finite counter-terms

S-duality
(modular invariance)

follows from corresponding flat space amplitude:

low energy expansion of massless 4-particle amplitude of 10d type IIB string amplitude

$$\lambda^{-\frac{1}{2}} \sim \alpha' \qquad a \sim g_s^2(\alpha')^4$$

Comparison with results of Huang-Yuan

We find agreement upon setting X=0:

$$\mathcal{H}^{(3)}_{\text{our}} - \mathcal{H}^{(3)}_{\text{HY}}|_{\mathcal{X}=0} = \tfrac{36}{7}\,\zeta_3\mathcal{H}^{(2,3)} - \left(\tfrac{5849}{1008} - \tfrac{\alpha}{240}\right)\mathcal{H}^{(2,3)} + \left(\overline{D}\text{- and } \zeta_3\overline{D}\text{-ambiguities}\right)$$

Note: X is a free parameter in their result, contributing with up to weight 6 functions and with infinite spin support in the OPE.

- **We find X=0 since it is sourced by a preamplitude containing weight 4 functions with letter x-xb!**

  (even though it does not appear in the full correlator, i.e. it is annihilated by the action of $(\Delta^{(8)})^2$)

- **Second reason: even if we include this extra letter at weight 4, we find that**

  (crossing symmetry of $\mathcal{P}^{(3)}$) + (matching leading log) $\to$ X=0

  in words: one can add things in the kernel of $(\Delta^{(8)})^2$ to make it crossing symmetric, but this introduces $\log^3(u)$ contributions!

Lastly, we fix the one-loop ambiguities in their result by carefully tracking the contribution of the one-loop ambiguity α to order a^3.

Comment on contributions from triple-trace operators

Now, what have we learned?

(1) We can see new operators (beyond the well understood double-trace sector) contributing to the OPE:

In particular, we find that (based on the consistency of the OPE) that new operators need to contribute to $\log^2(u)$ starting from twist 6!

→ twist of first triple-trace operators $\mathcal{O}_2 \partial^{\ell_1} \mathcal{O}_2 \partial^{\ell_2} \mathcal{O}_2$

This argument shows that they mix with double-trace operators and this information is already present in lower order correlators...

(2) Having obtained the two-loop correlator, we can extract from it new OPE data!

→ two-loop anomalous dimensions $\gamma^{(3)}$...

The two-loop anomalous dimension

focus on twist 4 double-trace operators O2dO2 (no mixing)

$\gamma^{(3)}$ appears in the log(u) part of the correlator:

$$\begin{aligned}\mathcal{H}^{(3)} = \quad & \log^3(u)\left[\tfrac{1}{6}A^{(0)}(\gamma^{(1)})^3\right]G_{t,\ell}(x,\bar{x}) \\ & + \log^2(u)\left[\left(\tfrac{1}{2}A^{(1)}(\gamma^{(1)})^2 + A^{(0)}\gamma^{(1)}\gamma^{(2)}\right) + A^{(0)}(\gamma^{(1)})^3\partial_\Delta\right]G_{t,\ell}(x,\bar{x}) \\ & + \log^1(u)\left[\left(A^{(2)}\gamma^{(1)} + A^{(1)}\gamma^{(2)} + A^{(0)}\gamma^{(3)}\right)\right. \\ & \qquad \left. + 2\left(A^{(1)}(\gamma^{(1)})^2 + 2A^{(0)}\gamma^{(1)}\gamma^{(2)}\right)\partial_\Delta + 2A^{(0)}(\gamma^{(1)})^3\partial_\Delta^2\right]G_{t,\ell}(x,\bar{x}) \\ & + \log^0(u)\left[A^{(3)} + 2\left(A^{(2)}\gamma^{(1)} + A^{(1)}\gamma^{(2)} + A^{(0)}\gamma^{(3)}\right)\partial_\Delta\right. \\ & \qquad \left. + 2\left(A^{(1)}(\gamma^{(1)})^2 + 2A^{(0)}\gamma^{(1)}\gamma^{(2)}\right)\partial_\Delta^2 + \tfrac{4}{3}A^{(0)}(\gamma^{(1)})^3\partial_\Delta^3\right]G_{t,\ell}(x,\bar{x})\,.\end{aligned}$$

Focus on twist 4 double-trace operators to avoid mixing: there is one unique operator for each spin

Analytic function of spin is most elegantly obtained using the Lorentzian inversion formula [CaronHuot'17]

$$\gamma^{(3)}_{2,\ell} = c_3\left(S_{-3} - S_3 - 2S_{1,-2} + 3\zeta_3\right) + c_2\,S_{-2} + c_1\,S_1 + c_0 + c_0^{(a)} + \alpha\,\tilde{\gamma}^{(2,3)}_{2,\ell}$$ for $\ell \geq 6$

The two-loop anomalous dimension

$$\gamma^{(3)}_{2,\ell} = c_3\left(S_{-3} - S_3 - 2S_{1,-2} + 3\zeta_3\right) + c_2\, S_{-2} + c_1\, S_1 + c_0 + c_0^{(a)} + \alpha\,\tilde{\gamma}^{(2,3)}_{2,\ell}$$

with coefficients being functions of $J^2 = (\ell+3)(\ell+4)$

$$c_3 = \frac{-221184J^2(J^2-2)(J^8-50J^6-653592J^4+30292416J^2+15169835520)}{5(J^2-6)^2(J^2-12)(J^2-20)(J^2-30)(J^2-42)(J^2-56)(J^2-72)},$$

$$c_2 = \frac{-18432\,q_2(J^2)}{(J^2-6)^2(J^2-12)^2(J^2-20)(J^2-30)^2(J^2-42)(J^2-56)^2(J^2-72)(J^2-90)(J^2-132)},$$

$$c_1 = \frac{-27648J^2(J^2-2)(J^8+525J^6+1730258J^4-79817784J^2-39925126080)}{(J^2-6)^2(J^2-12)(J^2-20)(J^2-30)(J^2-42)(J^2-56)(J^2-72)},$$

$$c_0 = \frac{384\,q_0(J^2)}{5J^2(J^2-6)^5(J^2-12)^2(J^2-20)^2(J^2-30)^2(J^2-42)(J^2-56)^2(J^2-72)(J^2-90)(J^2-132)},$$

- ✗ **Contains nested (alternating) harmonic sums**

$$S_{\vec{a}} \equiv S_{\vec{a}}(\ell+3)$$

$$S_{a_1,a_2,\ldots,a_n}(m) = \sum_{k=1}^{m} \frac{(\mathrm{sgn}(a_1))^k}{k^{|a_1|}} S_{a_2,\ldots,a_n}(k), \qquad S_\emptyset(m) = 1$$

- ✗ **The formula is analytic in spin down to l=6:**

  apparent poles at spins l=6,8 cancel non-trivially in the combination $c_2 S_{-2} + c_0$

- ✗ **pole at l=5: signals presence of non-analytic contributions for spins l=0,2,4 → consistent ambiguities from** $\partial^{10}\mathcal{R}^4|_{\text{genus-2}}$

- ✗ **obeys reciprocity symmetry:** $\ell \mapsto -\ell - 7$ **and its large spin expansion has only even powers in** $1/J^2$

Summary

- We have constructed two-loop supergravity (order $1/N^6$) correlator from an educated ansatz for the preamplitude
  - Only free parameters correspond to the expected tree-level ambiguities consistent with the $\partial^{10}\mathcal{R}^4|_{\text{genus-2}}$ correction
  - Our result is consistent with the one obtained by Huang-Yuan and we argued that their free parameter X=0

- Identified zigzag-integrals as basis for leading log at any loop order
- Revisited the bulk-point limit and understood the role of the extra letter $x-\bar{x}$ (not in this talk!)
- From the two-loop correlator, we extracted the two-loop anomalous dimension as a function of spin, which passes some non-trivial consistency conditions (analyticity in spin and reciprocity symmetry)

Open Questions

- Higher loop orders?
  - Analogous 'minimal' ansatz at any loop order reads $\mathcal{H}^{(n)} = \frac{1}{u^2}(\Delta^{(8)})^{n-1}\mathcal{P}^{(n)} + \sum_{i=1}^{n-1} a_i \mathcal{H}^{(i)}$
  - Complexity grows only in basis, not in coefficient functions!
  - Space of SVHPL's grows exponentially! → need better understanding of transcendental basis
  - Two-loop order was special due to enhanced crossing symmetry of $(\Delta^{(8)})^2$...

- It would be interesting to compute the Mellin space representation of our result.

- Higher external charges (KK-modes): leading log for any correlator predicted by 10d symmetry

- Can we bootstrap two-loop string corrections using similar ansatz?
  - However, corresponding flat space counterpart not known...

- A data point for exploration of double-copy in AdS

  Recent super-gluon result [2301.13240]

- Can we apply this to other holographic theories? Maybe yes for cases with hidden conformal symmetry → AdS_3, ...