Fibering out the mirror quintic

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work in progress with Vasily Golyshev and Albrecht Klemm

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Important note:

The first half of my presentation consists of results already published by Doran and Malmendier in 2019.

$$
\begin{array}{ccc}
\text{Calabi-Yau threefold} & \xleftrightarrow{\text{correspondence}} & \text{family of K3 surfaces} \\
\Big\downarrow H^3 & & \Big\downarrow H^2 \\
\text{periods} & \xleftarrow[\int]{} & \text{period functions}
\end{array}
$$

Key ingredient: elementary identities like

$$\frac{(5k)!}{k!^5} = \left[\frac{1}{1-t} \, \frac{(4k)!}{k!^4} \left(\frac{1}{t\,(1-t)^4} \right)^k \right]_{t^0}$$

A family of quintic hypersurfaces

A family of quartic hypersurfaces

Fibering out

Modularity of the conifold fiber

Physical application: growth of instanton numbers

Let X_z be the hypersurface

$$z\,x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - x_0\,x_1\,x_2\,x_3\,x_4 \;=\; 0$$

in $\mathbb{P}^4$. For $z \neq 0, 1/5^5$ this defines a Calabi-Yau threefold with Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}\,.$$

The group

$$G \;=\; \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \;\in\; (\mu_5)^5 \,|\, \alpha_0\,\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4 \;=\; 1\}$$

acts on X_z by $x_i \mapsto \alpha_i\,x_i$. This gives a $(1\,1\,1\,1)$-variation of Hodge structures V over $\mathbb{P}^1 \setminus \{0, 1/5^5, \infty\}$ defined by $V_z = H^3(X_z)^G$.

Properties of V:

- V_z can be seen as the complete middle cohomology of a mirror quintic (a suitable resolution of the quotient X_z/G).
- A trivialization Ω of F^3V is given by the residue of

$$\frac{\sum_{i=0}^{4} x_i \, \mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_4}{z\, x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - x_0\, x_1\, x_2\, x_3\, x_4} \, .$$

- Ω is annihilated by the Picard-Fuchs operator

$$\mathcal{L} \;=\; \Theta^4 - 5^5\, z\, (\Theta + 1/5)\, (\Theta + 2/5)\, (\Theta + 3/5)\, (\Theta + 4/5)$$

  where $\Theta = z\frac{\mathrm{d}}{\mathrm{d}z}$.
- For $|z| < 1/5^5$ one of the periods of Ω is given by

$$(2\pi i)^3 \, {}_4F_3\left({\frac{1}{5} \;\; \frac{2}{5} \;\; \frac{3}{5} \;\; \frac{4}{5} \atop 1 \;\; 1 \;\; 1} ; 5^5 z \right) \;=\; (2\pi i)^3 \sum_{k=0}^{\infty} \frac{(5k)!}{k!^5}\, z^k \, .$$

Let Y_z be the hypersurface

$$z\,x_0^4 + x_1^4 + x_2^4 + x_3^4 - x_0\,x_1\,x_2\,x_3 \;=\; 0$$

in $\mathbb{P}^3$. For $z \neq 0, 1/2^8$ this defines a K3 surface with Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}.$$

The group

$$G \;=\; \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \;\in\; (\mu_4)^4 \,|\, \alpha_0\,\alpha_1\,\alpha_2\,\alpha_3 \;=\; 1\}$$

acts on Y_z by $x_i \mapsto \alpha_i\,x_i$. This gives a $(1\,1\,1)$-variation of Hodge structures V over $\mathbb{P}^1 \setminus \{0, 1/2^8, \infty\}$ defined by $V_z = H^2(Y_z)^G$.

Properties of V:

- A trivialization Ω of $F^2 V$ is given by the residue of

$$\frac{\sum_{i=0}^{3} x_i \, \mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_3}{z\, x_0^4 + x_1^4 + x_2^4 + x_3^4 - x_0\, x_1\, x_2\, x_3} \,.$$

- Ω is annihilated by the Picard-Fuchs operator

$$\Theta^3 - 2^8\, z\, (\Theta + 1/4)\, (\Theta + 1/2)\, (\Theta + 3/4)$$

  where $\Theta = z\frac{\mathrm{d}}{\mathrm{d}z}$.
- For $|z| < 1/2^8$ one of the periods of Ω is given by

$$(2\pi i)^2 \, {}_3F_2\!\left({\tfrac{1}{2} \;\; \tfrac{1}{4} \;\; \tfrac{3}{4} \atop 1 \;\; 1} ; 2^8 z \right) \;=\; (2\pi i)^2 \sum_{k=0}^{\infty} \frac{(4k)!}{k!^4}\, z^k \,.$$

For $|z| < 1/5^5$ we have the elementary identity

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix} \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ & 1 & 1 & 1 \end{matrix}; 5^5 z\right) \\
&= \sum_{k=0}^{\infty} \frac{(5k)!}{k!^5}\, z^k \\
&= \frac{1}{2\pi i} \oint_{|t|=1/5} \sum_{k=0}^{\infty} \sum_{n=-k}^{\infty} \frac{(5k+n)!}{k!^4\,(k+n)!}\, z^k\, t^n \ \frac{\mathrm{d}t}{t} \\
&= \frac{1}{2\pi i} \oint_{|t|=1/5} \frac{1}{1-t} \left(\sum_{k=0}^{\infty} \frac{(4k)!}{k!^4} \left(\frac{z}{t\,(1-t)^4} \right)^k \right) \frac{\mathrm{d}t}{t} \\
&= \frac{1}{2\pi i} \oint_{|t|=1/5} \frac{1}{1-t}\ {}_3F_2\left(\begin{matrix} \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ & 1 & 1 \end{matrix}; 2^8 \frac{z}{t\,(1-t)^4}\right) \frac{\mathrm{d}t}{t} \\
&=: \frac{1}{2\pi i} \oint_{\gamma_1} \frac{1}{t\,(1-t)}\, \phi_z^* \varpi(t)\, \mathrm{d}t\,.
\end{aligned}
$$

We have expressed a period of X_z as an integral of periods of $\phi_z^* Y$ over a path $\gamma_1 \in \pi_1(\mathcal{M}_z, t_0)$ with

$$\mathcal{M}_z = \phi_z^{-1}(\mathbb{P}^1 \setminus \{0, 1/2^8, \infty\}) \qquad \text{and} \qquad t_0 \gg 0 .$$

To get a complete 4×4 period matrix we do the following:

- We take derivatives with respect to z.
- We consider the monodromy with respect to z.

We can then express a complete period matrix

$$\begin{pmatrix} \int_{\gamma_1} \Omega & \int_{\gamma_1} \Omega' & \int_{\gamma_1} \Omega'' & \int_{\gamma_1} \Omega''' \\ \int_{\gamma_2} \Omega & \int_{\gamma_2} \Omega' & \int_{\gamma_2} \Omega'' & \int_{\gamma_2} \Omega''' \\ \int_{\gamma_3} \Omega & \int_{\gamma_3} \Omega' & \int_{\gamma_3} \Omega'' & \int_{\gamma_3} \Omega''' \\ \int_{\gamma_4} \Omega & \int_{\gamma_4} \Omega' & \int_{\gamma_4} \Omega'' & \int_{\gamma_4} \Omega''' \end{pmatrix}$$

in terms of integrals of periods of $\phi_z^* Y$. We say that we have fibered out the periods of the mirror quintic.

$$\oint_\gamma \frac{1}{t\,(1-t)}\,\phi_z^*\varpi\,\mathrm{d}t$$

$$\Big\downarrow \ \frac{\mathrm{d}}{\mathrm{d}z}$$

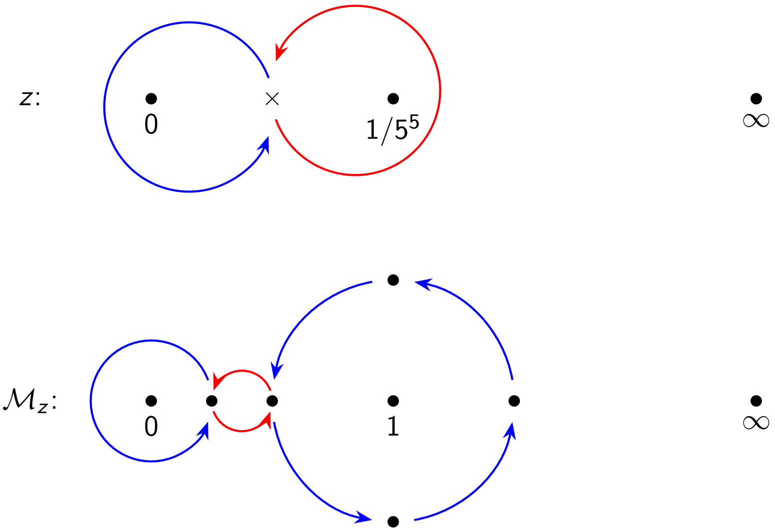
$$\frac{1}{z}\oint_\gamma \frac{5}{(1-5\,t)^2}\,\phi_z^*\varpi\,\mathrm{d}t$$

$$\Big\downarrow \ \frac{\mathrm{d}}{\mathrm{d}z}$$

$$\frac{1}{z^2}\oint_\gamma \frac{90\,t-150\,t^2}{(1-5\,t)^4}\,\phi_z^*\varpi\,\mathrm{d}t$$

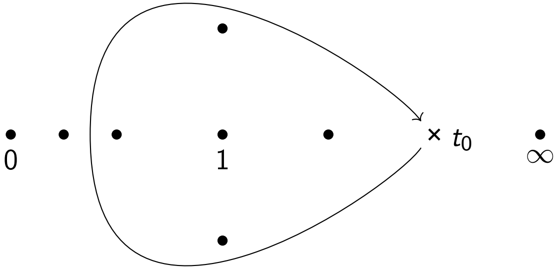
$$\Big\downarrow \ \frac{\mathrm{d}}{\mathrm{d}z}$$

$$\frac{1}{z^3}\oint_\gamma \frac{2730\,t^2-9300\,t^3+8250\,t^4}{(1-5\,t)^6}\,\phi_z^*\varpi\,\mathrm{d}t$$

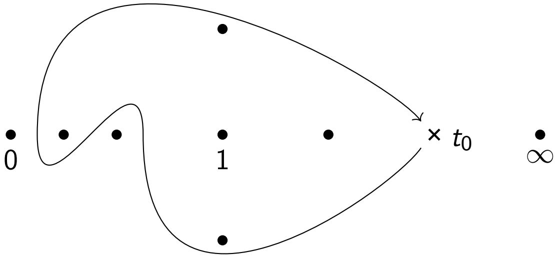


z:
0
1/5^5
∞
$\mathcal{M}_z$:
0
1
∞

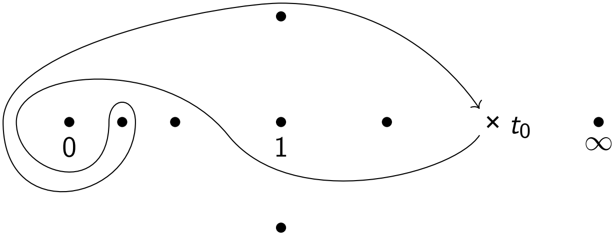
γ_1:



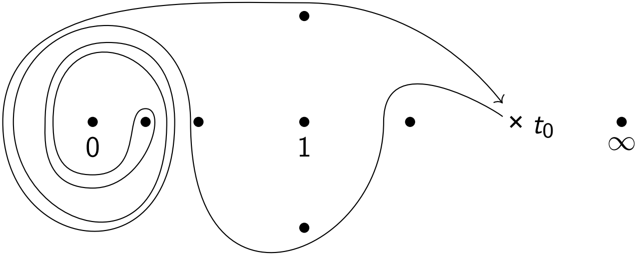
$M_{1/5^5}\,\gamma_1$:



$M_0 \, M_{1/5^5} \, \gamma_1$:



$M_0^2 \, M_{1/5^5} \, \gamma_1$:



We now consider the conifold point $z = 1/5^5$. The associated variety $X_{1/5^5}$ has 125 nodes as singularities. Blowing these up gives a smooth Calabi-Yau threefold $\widehat{X_{1/5^5}}$ with Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 25 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 25 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}.$$

It was shown by Schoen that the Galois representations on the third cohomology of $\widehat{X_{1/5^5}}$ come from the unique newform $f \in S_4(\Gamma_0(25))$ with Hecke eigenvalue $a_2 = 1$.

Does this modularity also manifest in the periods?

For $\delta = 1 - 5^5\, z \to 0$ a basis of periods has the expansion

$$\begin{pmatrix} -\sqrt{5}\, 2\pi i & b & d & c \\ 0 & w^+ & e^+ & a^+ \\ 0 & w^- & e^- & a^- \\ 0 & 0 & 0 & \sqrt{5}\,(2\pi i)^2 \end{pmatrix} \begin{pmatrix} \nu(\delta)\, \log\delta + O(\delta^3) \\ 1 + O(\delta^3) \\ \delta^2 + O(\delta^3) \\ \nu(\delta) \end{pmatrix}$$

with $\nu(\delta) = \delta + O(\delta^2)$.

In a paper with Klemm, Scheidegger and Zagier we numerically identify

$$\begin{pmatrix} w^+ & e^+ \\ w^- & e^- \end{pmatrix}$$

as a period matrix associated with f.

By fibering out we can now prove this and give similar relations for the other entries of the mixed period matrix.

The calculation is as follows:

- The quartic family is modular, i.e.

$$t_2^* \varpi = E_{2,2}$$

  with a Hauptmodul t_2 of $\Gamma_0^*(2)$ and the Eisenstein series $E_{2,2} \in M_2(\Gamma_0(2))$.
- There is a modular function t_{50} under $\Gamma_0^*(50)$ satisfying $\phi_{1/5^5}(t_{50}(\tau)) = t_2(5\,\tau)$.
- We can pull back all period integrals at the conifold to integrals of modular forms on the upper half plane. In particular, we have

$$t_{50}^* \left(\frac{1}{2\pi i} \, \frac{1}{t\,(1-t)} \; \phi_{1/5^5}^* \varpi \, \mathrm{d}t \right) = \; 5 \, f_{50} \, \mathrm{d}\tau$$

  with $f_{50}(\tau) = f(\tau) - 4\, f(2\tau)$.

This gives the following identifications:

- $f_{50} \to$ periods $w^{\pm}$
- $F_{50} = \frac{t_{50}\,(7+13\,t_{50}+5\,t_{50}^2-25\,t_{50}^3)}{2\,(1-5t_{50})^4}\, f_{50} \to$ periods $e^{\pm}$
- $g_{50} = \frac{5\,t_{50}\,(1-t_{50})}{(1-5\,t_{50})^2}\, f_{50} \to$ periods $a^{\pm}$ and $\sqrt{5}\,(2\pi i)^2$ from residues
- b, c and d come from integrals over open contours, e.g.

$$\frac{b}{(2\pi i)^3} = \sum_{k=0}^{\infty} \frac{(5k)!}{k!^5} \left(\frac{1}{5^5}\right)^k = 5 \int_{\frac{1}{5}(-2+i/\sqrt{2})}^{\frac{1}{5}(2+i/\sqrt{2})} f_{50}(\tau)\,\mathrm{d}\tau\,.$$

The nature of f_{50} and F_{50} is well understood. Getting a better understanding of g_{50} and the mixed periods is work in progress. For example, we have the numerical identity

$$\tfrac{1}{(2\pi i\,\sqrt{5})^2}\det\begin{pmatrix} w^- & a^- \\ b & c \end{pmatrix} = \tfrac{w^-}{2\pi i\,\sqrt{5}}\left(1+\tfrac{5}{3}\log 5\right) + \tfrac{125}{6} L'\!\left(f \otimes \left(\tfrac{5}{\cdot}\right), 2\right).$$

A family of quintic hypersurfaces

A family of quartic hypersurfaces

Fibering out

Modularity of the conifold fiber

Physical application: growth of instanton numbers

In the following we work locally around $z = 0$. Let

$$f_0 = 1 + O(z)$$
$$f_1 = f_0 \log z + O(z)$$

be two solutions of the Picard-Fuchs operator $\mathcal{L}$. The instanton numbers $n_1, n_2, n_3, \ldots$ are defined by

$$t = \frac{f_1}{f_0}$$
$$q = e^t$$
$$5 + \sum_{d=1}^{\infty} n_d \, d^3 \, \frac{q^d}{1-q^d} = \frac{1}{f_0^2} \, \frac{5}{z^3 \, (1 - 5^5 \, z)} \left(\frac{\mathrm{d}t}{\mathrm{d}z} \right)^{-3} .$$

This gives the instanton numbers 2875, 609250, 317206375, ...

It was famously argued by Candelas et al. that n_k is the number of rational curves of degree k on a generic quintic hypersurface.

The instanton numbers asymptotically grow like

$$n_k \sim \left(\frac{b^2}{4\,\pi^2\,w^+}\right)^2 \frac{e^{-2\pi i\,k\,w^+/b}}{k^3\,\log^2 k}$$

and we can now identify

$$b = 5\,(2\pi i)^3 \int_{\frac{1}{5}(-2+i/\sqrt{2})}^{\frac{1}{5}(2+i/\sqrt{2})} \left(f(\tau) - 4\,f(2\tau)\right) \mathrm{d}\tau$$

$$w^+ = 5\,(2\pi i)^3 \int_0^\infty f(\tau)\,\mathrm{d}\tau\,.$$

Summary and outlook

- Simple manipulations of periods can lead to useful period identities. This does not restrict to hypergeometric periods.
- In some examples this allows to relate modularity of Calabi-Yau threefolds to modularity of families of K3 surfaces.
- The resulting identities involve:
  - Periods of holomorphic modular forms.
  - Periods of meromorphic modular forms with vanishing residues.
  - Periods of meromorphic modular forms with non-vanishing residues.
  - Integrals of these forms along open contours.
- More generally, one might get identities involving objects which are not quite modular forms. These suggest relations like

$$\text{periods of } \frac{E_4}{\sqrt{j}} = \text{periods of the newform in } S_4(\Gamma_0(24))\,.$$