Modern multiloop calculations

Lee Roman

Seminar series on motives and period integrals in Quantum field theory and String theory.
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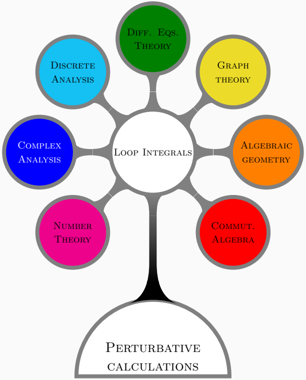
Budker Institute of Nuclear Physics

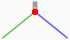
- High-precision theoretical description of Standard Model processes is of crucial importance. In particular, the New Physics — new particles and interactions — is likely to appear as small deviations from SM and therefore can be detected only with high precision of theoretical predictions at hand.
- From the computational point of view, our ability to obtain high-precision results depends crucially on multiloop calculation techniques. Complexity grows both qualitatively and quantitatively in an explosive way with the number of loops and/or scales.
- Besides these practical purposes, multiloop calculations provide a perfect polygon for trying the methods from various mathematical fields: differential equations, complex analysis, number theory, algebraic geometry etc.

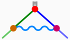


Example: $g-2$ at one loop [Schwinger, 1948]

- Electron scattering in electromagnetic field is described by two form factors $F_{1,2}$:

$$j_\mu = \overline{u}(p')\left[\gamma_\mu F_1(q^2) - \frac{\sigma_{\mu\nu}q^\nu}{2m}F_2(q^2)\right]u(p), \quad p' = p+q, \quad \sigma^{\mu\nu} = \tfrac{1}{2}[\gamma^\mu,\gamma^\nu]\,.$$

- $F_1(0) = 1$ and it can be shown that $F_2(0) = \frac{1}{2}(g-2)$ is the **anomalous magnetic moment (AMM)**.
- In the leading approximation $j_\mu^{(0)} =$  $= \overline{u}(p')\gamma_\mu u(p) \implies$ $F_1^{(0)} = 1,\, F_2^{(0)} = 0.$
- In the next-to-leading (NLO) approximation we have

$j_\mu^{(1)} =$ 

$$= -ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\overline{u}(p')\gamma^\nu(\hat{p}'-\hat{k}+m)\gamma_\mu(\hat{p}-\hat{k}+m)\gamma_\nu u(p)}{[(p'-k)^2-m^2+i0][k^2+i0][(p-k)^2-m^2+i0]}$$

Each element of the diagram corresponds to a specific factor in the expression.
The expression is already somewhat complicated, but we still can treat it manually if we use Feynman parametrization

$$\frac{1}{[(p'-k)^2-m^2+i0][k^2+i0][(p-k)^2-m^2+i0]} = 2\int_0^1\int_0^1\frac{dxxdz}{[k^2-2k\cdot(zp+\bar{z}p')x+i0]^3}\,,$$

and make a shift $k\to k+(zp+\bar{z}p')x$.

After some γ-matrix algebra we get

$$j_\mu^{(1)} = -2ie^2 \int_0^1 \int_0^1 dxdz \int \frac{d^4k}{(2\pi)^4 \left[-k^2 + (zp + \bar{z}p')^2 x^2 - i0\right]^3}$$

$$\times \bar{u}(p') \left\{ \gamma_\mu \left[2x^2 \left(m^2 + z\bar{z}q^2\right) - 2\bar{x}\left(2m^2 - q^2\right) - k^2 \right] - \frac{\sigma_{\mu\nu} q^\nu}{2m} \left(4x\bar{x}m^2\right) \right\} u(p)$$

The highlighted parts contribute to $F_1^{(1)}$ and $F_2^{(1)}$, respectively.

Performing Wick rotation $k_0 \to i\tilde{k}_0$ and taking the integrals we obtain Schwinger's celebrated result: $F_2^{(1)}(0) = 2\pi^2 e^2 \int_0^1 \int_0^1 dxdz \int \frac{d\tilde{k}^2 \tilde{k}^2 \left(4x\bar{x}m^2\right)}{(2\pi)^4 \left[\tilde{k}^2 + m^2x^2 - i0\right]^3} = \frac{\alpha}{2\pi}$.

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Dimensional regularization

Note that already in one loop we will encounter problems when calculating

$$F_1^{(1)}(q^2) = 2e^2 \int_0^1 \int_0^1 dx dz \int \frac{d^4 \tilde{k}}{(2\pi)^4} \frac{\tilde{k}^2 - 2\bar{x} \left(2m^2 - q^2\right) + 2x^2 \left(m^2 + z\bar{z}q^2\right)}{\left[\tilde{k}^2 + (zp + \bar{z}p')^2 x^2 - i0\right]^3} .$$

F_1 diverges both at large (UV) and at small (IR) $\tilde{k}^2$.

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Dimensional regularization

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$$F_1^{(1)}(q^2) = 2e^2 \int_0^1 \int_0^1 dx dz \int \frac{d^d \tilde{k}}{(2\pi)^d} \frac{\tilde{k}^2 - 2\bar{x} \left(2m^2 - q^2 \right) + 2x^2 \left(m^2 + z\bar{z}q^2 \right)}{\left[\tilde{k}^2 + (zp + \bar{z}p')^2 x^2 - i0\right]^3} .$$

F_1 diverges both at large (UV) and at small (IR) $\tilde{k}^2$. Both UV and IR divergencies are regularized within *dimensional regularization* $\boxed{d = 4 - 2\epsilon}$.

Next corrections

At two loops the calculations of $g-2$ get much more involved [Sommerfield, 1957]. Starting from 3 loops it is practically impossible to do calculations by hand. Current world records for analytical $g-2$ is 4 loops [Laporta, 2017] (earlier calculated numerically with impressive efforts by Kinoshita and collaborators).

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Incomplete list of modern multi-loop methods and tools

- Parametric representations
  - alpha- (or Feynman) representation (also LP repr.).
  - Baikov representation.
  - Mellin-Barnes representation.
- Expansion by regions.
  - In momentum representation
  - In Feynman representation.
- IBP reduction.
  - In momentum representation.
  - In parametric representations.
- Differential equations.
  - Reduction to ϵ-form.
  - Frobenius expansion near singular point.
  - Using ϵ-regular basis.
- Recurrence relations
  - with respect to dimensionality d.
  - with respect to powers of denominators.

The most popular (arguably) calculation path

1. Diagram generation ✓

Generate diagrams contributing to the chosen order of perturbation theory.

Tools: `qgraf` [Nogueira, 1993], `FeynArts` [Hahn, 2001], `tapir` [Gerlach et al., 2022],...

2. IBP reduction

Setup IBP reduction, derive differential system for master integrals.

Tools: `FIRE6` [Smirnov and Chuharev, 2020], `Kira2` [Klappert et al., 2021], `LiteRed` [RL, 2012], ...

3. DE Solution

Reduce the system to ϵ-form, write down solution in terms of polylogarithms. Fix boundary conditions by auxiliary methods.

Tools: `Fuchsia` [Gituliar and Magerya, 2017], `epsilon` [Prausa, 2017], `Libra` [RL, 2021]

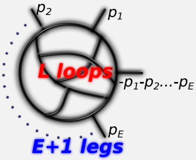
NB: 3rd step is not always doable.

IBP reduction

Given a Feynman diagram, consider a family

$$j(\boldsymbol{n}) = j(n_1, \ldots, n_N) = \int d\mu_L \boldsymbol{D}^{-\boldsymbol{n}} = \int \prod_{i=1}^{L} d^d l_i \prod_{k=1}^{N} D_k^{-n_k},$$

$l_1, \ldots l_L$ –loop momenta, $p_1, \ldots p_E$ — external momenta.



IBP identities [Chetyrkin and Tkachov, 1981]

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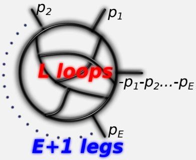


There are $N = \underbrace{L(L+1)/2}_{\#\text{ of } l_i \cdot l_j} + \underbrace{L \cdot E}_{\#\text{ of } l_i \cdot p_j}$ scalar products involving loop momenta.

$D_1, \ldots, D_M$ — denominators of the diagram, $D_{M+1}, \ldots, D_N$ — irreducible numerators, such that $D_1, \ldots, D_N$ form a basis, i.e. any scalar product can be uniquely expressed via linear function of D_k.

IBP identities

In dim. reg. the integral of divergence is zero (no surface terms):

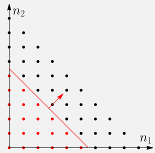
$$0 = \int d\mu_L \frac{\partial}{\partial l_i} \cdot q_j \boldsymbol{D}^{-\boldsymbol{n}} = \sum_s c_s(\boldsymbol{n}) j(\boldsymbol{n} + \boldsymbol{\delta}_s).$$

Explicitly differentiating, we obtain relations between integrals.

IBP reduction

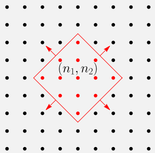
Laporta algorithm (FIRE, Kira, Reduze, ...)

- generate identities for many numeric $\boldsymbol{n} \in \mathbb{Z}^N$.
- use Gauss elimination and collect reduction rules to database.
- twist: mapping to finite fields $\mathbb{F}_p$ + reconstruction. $\Longleftarrow$ naturally parallelizable



Heuristic search (LiteRed)

1. Generate identities for shifts around $\boldsymbol{n}$ with *symbolic* entries.
2. Use Gauss elimination until acceptable rule is found.
3. **Solve Diophantine equations to derive applicability condition.**



Operators A and B

$$A_k f(n_1, \dots, n_k, \dots, n_N) = n_k f(n_1, \dots, n_k + 1, \dots, n_N)\,,$$
$$B_k f(n_1, \dots, n_k, \dots, n_N) = f(n_1, \dots, n_k - 1, \dots, n_N)\,,$$

It is easy to check that $[A_k, B_m] = \delta_{km}$, i.e., these operators[a] implement (a representation of) N-th Weyl algebra $\mathbb{A}_N$.

[a]NB: these notations imply that operators act on function rather than on its value. So $A_k f = \tilde{f}$, such that $\tilde{f}(\dots, n_k, \dots) = n_k f(\dots, n_k + 1, \dots)$. Thus we will sometimes use braces, like in $(A_k f)(\boldsymbol{n})$.

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Using linearity of D_k in s_{ij} and completeness, we can write $q_j \cdot \frac{\partial}{\partial l_i} D_k = c_{km}^{(ij)} D_m + c_k^{(ij)}$, where $c_{km}^{(ij)}$ and $c_k^{(ij)}$ are some coefficients independent of loop momenta. Then IBP identities can be written as

IBP identities in terms of A and B operators

$$\int d\mu_L \underbrace{\frac{\partial}{\partial l_i}}_{\mathcal{O}^{(ij)}} \cdot q_j \boldsymbol{D}^{-\boldsymbol{n}} = -\underbrace{\left[c_{km}^{(ij)} A_k B_m + c_k^{(ij)} A_k - d\delta_{ij}\right]}_{\mathcal{P}^{(ij)}(A,B)} j(\boldsymbol{n}) = 0.$$

NB: by construction $c_{km}^{(ij)}$ and $c_k^{(ij)}$ are independent of $\boldsymbol{n}$ and d.

Operators $\mathcal{P}^{(ij)}(A, B)$ generate a left ideal

$$\mathbb{L} = \langle \mathcal{P}^{(11)}, \ldots, \mathcal{P}^{(L,\, L+E)} \rangle_{\text{left}} = \left\{ \sum_{ij} C_{ij}(A, B) \mathcal{P}^{(ij)}(A, B) \middle| C_{ij}(A, B) \in \mathbb{A}_N \right\}.$$

Informally, $\mathbb{L}$ consists of all linear combinations of IBP identities. Any combination of IBP identities can be written as $\boxed{\mathcal{L} j(\mathbf{1}) = 0, \qquad \mathcal{L} \in \mathbb{L}}$.

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Let us write the integral $j(\boldsymbol{n})$ in the form

$$\boxed{j(\boldsymbol{n}) = \boldsymbol{Y}^{\boldsymbol{n}} j(\mathbf{1}) = \prod_{k=1}^{N} Y_k^{n_k} j(\mathbf{1}), \qquad Y_k^{n_k} = \begin{cases} B_k^{1-n_k} & n_k \leqslant 0 \\ \frac{1}{(n_k-1)!} A_k^{n_k-1} & n_k > 0 \end{cases}}$$

One might think of reducing $j(\boldsymbol{n})$ by finding the decomposition $\boldsymbol{Y}^{\boldsymbol{n}} = \mathcal{L}(A, B) + \mathcal{M}(A, B)$, where $\mathcal{L} \in \mathbb{L}$ and the "remainder" $\mathcal{M}$ is simplest possible[1]. Finding this decomposition is algorithmically solved via construction of Groebner basis of $\mathbb{L}$ (implemented, e.g., in `Singular`). Substituting this decomposition and using the fact that $\mathcal{L} j = 0$, we have $\boxed{j(\boldsymbol{n}) = \mathcal{M} j(\mathbf{1})}$. Assuming $\mathcal{M}$ is simple enough, we might hope for the reduction.

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Unfortunately, a little experimenting shows that this reduction is not satisfactory, the quotient ring $\mathbb{A}_N/\mathbb{L}$ is not even finite-dimensional (the number of "master integrals" is infinite).

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Indeed, $(B_k A_k f)(\ldots 1_k \ldots) = (A_k f)(\ldots 0_k \ldots) = 0 \cdot f(\ldots 1_k \ldots) = 0$.

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Therefore, along with the **left** ideal $\mathbb{L} = \langle \mathcal{P}^{(11)}, \ldots, \mathcal{P}^{(L,\, L+E)} \rangle_{\text{left}}$ we have to consider also the **right** ideal $\mathbb{R} = \langle B_1 A_1, \ldots, B_N A_N \rangle_{\text{right}} = \left\{ \sum_k B_k A_k C_k(A, B) \middle| C_k(A, B) \in \mathbb{A}_N \right\}$ and try to find the decomposition

IBP reduction as reduction wrt $\mathbb{L} + \mathbb{R}$

$$\boldsymbol{Y}^{\boldsymbol{n}} = \mathcal{L}(A, B) + \mathcal{R}(A, B) + \mathcal{M}(A, B),$$

where $\mathcal{L} \in \mathbb{L}$, $\mathcal{R} \in \mathbb{R}$, and the "remainder" $\mathcal{M}$ is simplest possible.

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It is easy to understand that finding this $\mathcal{L} + \mathcal{R} + \mathcal{M}$ decomposition gives a full reduction. Suppose we have rule $j(\boldsymbol{n}) \to \mathcal{M} j(\boldsymbol{1})$, reducing $j(\boldsymbol{n})$ to master integrals. It means that there exists $\mathcal{L} \in \mathbb{L}$, such that $[\boldsymbol{Y}^{\boldsymbol{n}} - \mathcal{L} - \mathcal{M}] f(\boldsymbol{1}) = 0$ for **arbitrary** function f. We then claim that $\mathcal{R} = [\boldsymbol{Y}^{\boldsymbol{n}} - \mathcal{L} - \mathcal{M}]$ belongs to $\mathbb{R}$.

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Despite an apparent similarity to $\mathcal{L} + \mathcal{M}$ decomposition, there seem to be no known effective algorithm of finding $\mathcal{L} + \mathcal{R} + \mathcal{M}$ decomposition. In particular, Groebner bases can not help. The problem looks very similar to D-modules theoretical integration problem, so there maybe such an algorithm. One should be warned though that existing (implementations of) D-modules algorithms are extremely slow.

Feynman and Baikov representations

To derive Feynman representation for the integral

$$j(n_1 \ldots, n_M) = \int \frac{\prod_{i=1}^{L} d^d l_i}{\pi^{Ld/2} \prod_{k=1}^{M} D_k^{n_k}}$$

we use exponential parametrization $D_k^{-n_k} = \int_0^\infty \frac{dz_k z_k^{n_k-1}}{\Gamma(n_k)} e^{-z_k D_k}$ to obtain

$$j(n_1 \ldots, n_M) = \int \frac{\prod_{i=1}^{L} d^d l_i}{\pi^{Ld/2}} \int\limits_{\mathbb{R}_+^M} \prod_{k=1}^{M} \frac{dz_k z_k^{n_k-1}}{\Gamma(n_k)} e^{-\Sigma_{k=1}^{M} z_k D_k}$$

Since D_k are linear functions of $l_i \cdot l_j$ and $l_i \cdot p_j$, we can represent

$$\Sigma_k z_k D_k = a_{ij} l_i \cdot l_j + 2 b_i \cdot l_i + c,$$

where a, b, c are linear combinations of z_k. Taking the integrals over l_i, we obtain

$$j(n_1 \ldots, n_M) = \int\limits_{\mathbb{R}_+^M} \prod_{k=1}^{M} \frac{dz_k z_k^{n_k-1}}{\Gamma(n_k)} U(z)^{-d/2} e^{-F(z)/U(z)},$$

where $U = \det a$ and $F = [c - (a^{-1})_{ij} b_i \cdot b_j] U$ are Symanzik polynomials.

Note that both U and F are homogeneous polynomials of z_k of degree L and $L+1$, respectively.

Feynman representation II

Now we insert $1=\int_0^\infty ds\delta\left(s-\widetilde{\Sigma}_k z_k\right)$, where $\widetilde{\Sigma}_k$ denotes any nonempty partial sum (i.e., k runs over any nonempty subset of $\{1,\ldots,M\}$). After rescaling $z_k\to sz_k$, we pull out s from the argument of δ-function and then take the integral over s. We obtain

Feynman representation (aka alpha-representation, parametric representation)

$$j(n_1\ldots,n_M)=\Gamma[\Sigma_k n_k-L\tfrac{d}{2}]\int\limits_{\mathbb{R}_+^M}\prod_{k=1}^{M}\frac{dz_k z_k^{n_k-1}}{\Gamma(n_k)}\frac{F^{Ld/2-\Sigma_k n_k}}{U^{(L+1)d/2-\Sigma_k n_k}}\delta\left(1-\widetilde{\Sigma}_k z_k\right)$$

Inserting instead $1=\frac{1}{\Gamma\left((L+1)\frac{d}{2}-\Sigma_k n_k\right)}\int ds\, s^{(L+1)\frac{d}{2}-\Sigma_k n_k-1}e^{-s}$ and rescaling $z_k\to sz_k$ we obtain a modified representation [RL and Pomeransky, 2013]:

Lee-Pomeransky representation

$$j(n_1\ldots,n_M)=\frac{\Gamma(d/2)}{\Gamma[(L+1)d/2-\Sigma_k n_k]}\int\limits_{\mathbb{R}_+^M}\prod_{k=1}^{M}\frac{dz_k z_k^{n_k-1}}{\Gamma(n_k)}G^{-d/2},\quad G=U+F$$

To prove equivalence, insert $1=\int_0^\infty ds\delta\left(s-\widetilde{\Sigma}_k z_k\right)$, rescale $z_k\to sz_k$ and take the integral $\int ds\, s^{\Sigma_k n_k-Ld/2-1}(U+sF)^{-d/2}$.

The integrand in the loop integral depends on loop momenta via scalar products $s_{i,j} = l_i \cdot q_j$. Therefore, we may think of “integrating out” other integration variables. Indeed, it appears to be possible [Baikov, 1997]. Let us write

$$j(n_1 \ldots, n_N) = \int \frac{\prod_{i=1}^{L} d^d l_i}{\pi^{Ld/2}} f(\boldsymbol{n}, s_{ij}), \qquad f(\boldsymbol{n}, s_{ij}) = \prod_{k=1}^{N} D_k^{-n_k}$$

We start from the integral over l_1. The integrand depends on l_1 via scalar products $s_{1,1}, \ldots, s_{1,K}$, where $K = L + E$. We write

$$\frac{d^d l_1}{\pi^{\frac{d}{2}}} = \frac{d^{K-1} l_{1\|}}{\pi^{\frac{K-1}{2}}} \frac{d^{d-K+1} l_{1\perp}}{\pi^{\frac{d-K+1}{2}}} = \frac{ds_{12} \ldots ds_{1K}}{\pi^{\frac{K-1}{2}} V^{\frac{1}{2}}(q_2, \ldots q_K)} \frac{\overbrace{ds_{11}}^{dl_1^2 = dl_{1\perp}^2} \left(\overbrace{\frac{V(q_1, \ldots q_K)}{V(q_2, \ldots q_K)}}^{l_{1\perp}^2}\right)^{\frac{d-K-1}{2}}}{\Gamma(d-K+1)/2]}$$

where $l_{1\|}$ ($l_{1\perp}$) denote the components in (the orthogonal complement of) the linear subspace spanned by $q_2 = l_2, \ldots, q_L = l_L, q_{L+1} = p_1, \ldots, q_{L+E} = p_E$.

Here $V(q_1, \ldots q_K) = \det\{q_i \cdot q_j|_{i,j=1\ldots K}\}$ is the *Gram determinant* = square of volume of the parallelepiped constructed on $q_1, \ldots q_K$. Respectively, the matrix $\widehat{V}(q_1, \ldots q_K) = \{q_i \cdot q_j|_{i,j=1\ldots K}\}$ is called the *Gram matrix*.

Repeating the same transformation for $l_2, \ldots, l_L$, we obtain

$$\frac{\pi^{(L-N)/2}}{\Gamma\left[\frac{d-K+1}{2}, \ldots, \frac{d-E}{2}\right]} \int \prod_{i=1}^{L} \prod_{j=i}^{K} ds_{ij} \frac{\left[V\left(q_1, \ldots, q_K\right)\right]^{(d-K-1)/2}}{\left[V\left(p_1, \ldots, p_E\right)\right]^{(d-E-1)/2}} f\left(\boldsymbol{n}, s_{ij}\right)$$

Since $D_1, \ldots D_N$ are linear in s_{ij} and form a basis, we have $\prod_{i=1}^{L} \prod_{j=i}^{K} ds_{ij} = J \prod_{k=1}^{N} dD_N$, where $J = \left(\det \frac{\partial D_k}{\partial s_{ij}}\right)^{-1}$ (here ij should be understood as index running over N distinct values). Also $V\left(q_1, \ldots, q_K\right) = P(D_1, \ldots, D_N)$ is polynomial in D_k (called *Baikov polynomial*). Finally we have

Baikov representation

$$j(\boldsymbol{n}) = \frac{\pi^{(L-N)/2} J}{\Gamma\left[\frac{d-K+1}{2}, \ldots, \frac{d-E}{2}\right]} \int_{\mathcal{D}} \prod_{k=1}^{N} \frac{dD_k}{D_k^{n_k}} \frac{\left[P\left(D_1, \ldots D_N\right)\right]^{(d-K-1)/2}}{\left[V\left(p_1, \ldots, p_E\right)\right]^{(d-E-1)/2}}$$

With some reservations, the integration region is

$$\mathcal{D} = \left\{\left(D_1, \ldots, D_N\right) \in \mathbb{R}^N \middle| P\left(D_1, \ldots D_N\right) > 0\right\}.$$

IBP reduction in parametric representations

Both Lee-Pomeransky and Baikov representations depend on one polynomial raised to the power, depending on d. It is obvious that if we act on the integrand with a random differential operator, the power of this polynomial will be shifted. Therefore, we will get relations between integrals not only with shifted indices, but also with shifted dimension. If we don't want this, we have to choose the differential operator very carefully.

Let us consider the operator $\partial_m Q_m = \frac{\partial Q_m}{\partial D_m} + Q_m \frac{\partial}{\partial D_m}$, where Q_m are some polynomials of D_k. If we act with this operator on the integrand of Baikov representation $j(\boldsymbol{n}) \propto \int \prod_k \frac{dD_k}{D_k^{n_k}} P^{\frac{d-K-1}{2}}$, we have

$$\int d\boldsymbol{D} P^{\frac{d-K-1}{2}} \left[\frac{\partial \left(\boldsymbol{D}^{-\boldsymbol{n}} Q_m \right)}{\partial D_m} + \frac{d-K-1}{2} \boldsymbol{D}^{-\boldsymbol{n}} Q_m \frac{\partial P}{P \partial D_m} \right]$$

Here we used notations $d\boldsymbol{D} = \prod_k dD_k$, $\boldsymbol{D}^{-\boldsymbol{n}} = \prod_k D_k^{-n_k}$. Extra power of P in the denominator may appear due to the term $Q_m \frac{\partial P}{P \partial D_m}$. However, if we choose Q_m s.t. $\boxed{Q_m \partial_m P = -QP}$, where Q is also some polynomial, the P in the denominator gets cancelled.

Let $\boldsymbol{p} = (p_1, \ldots p_n)$ be a vector of polynomials, then $\boldsymbol{Q} = (Q_1, \ldots, Q_n)$ is called syzygy of $(p_1, \ldots p_n)$ if the following relation holds

$$\boldsymbol{Q}\boldsymbol{p} = \sum_{m=1}^{n} Q_m p_m = 0\,.$$

Finding a basis of syzygy module is a classical task of commutative algebra. It is implemented in many CAS, including `Singular`, `Macaulay2`, `CoCoA`.

Thus, finding a syzygy basis of the set $\frac{\partial P}{\partial D_1}, \ldots, \frac{\partial P}{\partial D_N}, P$ we can construct IBP identities not shifting dimensions. Let $Q_m(\boldsymbol{D})\frac{\partial P}{\partial D_m} + QP = 0$ then we have

IBP identity from syzygy via A and B

$$[Q_m(B_1, \ldots, B_N)A_m + \tfrac{1}{2}(d - K - 1)Q(B_1, \ldots, B_N)]j(\boldsymbol{n}) = 0, \quad K = L + E$$

NB: In Ref. [Böhm et al., 2018] it was shown, that the syzygy module provides exactly the same information as momentum-space IBP identities. However, syzygy approach in Baikov representation provides a more flexible setup.

Note that $N = L(L+1)/2 + L \cdot E$ grows quadratically with L, while M, the # of lines in the diagram, grows only linearly. Parametric representation: only M indices. Therefore, the IBP reduction in LP representation might be more effective for higher L.

Integration region $\mathbb{R}_+^M$ has boundary, therefore surface terms are likely to appear.

Let us write the LP representation in the form

$$j(\boldsymbol{n}) = \frac{\Gamma\left[d/2\right] \tilde{j}(\boldsymbol{n})}{\Gamma\left[(L+1)\,d/2 - \Sigma_k n_k\right]}, \qquad \tilde{j}^{(d)}(\boldsymbol{n}) = \boldsymbol{I}^{\boldsymbol{n}}\left[G^{-d/2}\right] = \prod_k I_k^{n_k}\left[G^{-d/2}\right],$$

where the functionals I_k^n are determined as

$$I_k^n[\phi(z_k)] = \begin{cases} \int_0^\infty \frac{dz_k z_k^{n-1}}{\Gamma(n)} \phi(z_k) & n > 0 \\ (-\partial_{z_k})^{|n|}\phi(0) & n \leqslant 0 \end{cases}$$

These functionals allow us to account also for negative n_k. It can be checked that

$$I_k^m[-\partial\phi(z_k)/\partial z_k] = I_k^{m-1}[\phi(z_k)], \qquad I_k^m[z_k\phi(z_k)] = m I_k^{m+1}[\phi(z_k)].$$

IBP reduction in LP representation II

Suppose now that we have a syzygy $QG + Q_k \partial_k G = 0$. Then we can transform $\boldsymbol{I}^{\boldsymbol{n}}\left[-\partial_k (Q_k/G^{d/2})\right]$ in two different ways.

- Using the first relation we get $\boldsymbol{I}^{\boldsymbol{n}}\left[-\partial_k (Q_k/G^{d/2})\right] = \sum_k \boldsymbol{I}^{\boldsymbol{n}-\boldsymbol{1}_k}\left[Q_k/G^{d/2}\right]$.
- Explicitly differentiating and using the syzygy relation, we get $\boldsymbol{I}^{\boldsymbol{n}}\left[-\partial_k (Q_k/G^{d/2})\right] = \boldsymbol{I}^{\boldsymbol{n}}\left[(\frac{d}{2}Q - \partial_k Q_k)/G^{d/2}\right]$.

Equating these two expressions and using the second relation, we get

IBP identity from syzygy in LP representation

$$[Q_k(A_1, \ldots, A_M)B_k + \tfrac{d}{2}Q(A_1, \ldots, A_M)]\tilde{j}(\boldsymbol{n}) = 0$$

Note that this derivation holds both for positive and non-positive indices.

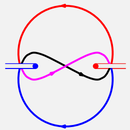
IBP reduction in Lee-Pomeransky representation is quite promising, but a fast algorithm for constructing a *minimal* (rather than Groebner) basis of syzygy module is very desirable.

NB: Special case of syzygy module needed for parametric reduction is closed wrt Lie bracket $[\boldsymbol{Q}^{(1)}, \boldsymbol{Q}^{(2)}] = [Q_m^{(1)}\partial_m, Q_n^{(2)}\partial_n]$. Maybe it can be used somehow?

IBP reduction with intersection theory? [Mastrolia and Mizera, 2019]

- Integral in LP representation is understood as bilinear pairing between integration cycle C and differential form ϕ.

$$\int_C G^{-\nu}\phi = \langle\phi|C] \ ,$$



Pochhammer contour.

- This bilinear form is invariant under $\phi \to \phi + \nabla_\nu \tilde{\phi}$ and/or $C \to C + \partial\tilde{C}$, where $\nabla_\nu = d - \nu G^{-1}dG$ is twisted differential. Therefore, $\langle\cdot|\cdot]$ is correctly defined on the elements of twisted de Rham cohomology and twisted homology — finite-dimensional spaces. Twisted cycles C can be understood as ordinary cycles on the Riemann surface of the function $G^{-\nu}$.
- Ref. [Cho and Matsumoto, 1995] introduced a pairing $\langle\phi_1|\phi_2\rangle$, correctly defined for ∇_ν and $\nabla_{-\nu}$ de Rham cohomologies. Then the IBP reduction is a basis expansion

$$\langle\phi|C] = \sum_i \langle\phi|\phi_i\rangle \langle\phi_i|C] \ ,$$

where $j_i = \langle\phi_i|C]$ are master integrals.
- Unfortunately, $\langle\phi_1|\phi_2\rangle$ is still very difficult to calculate in general. All examples considered so far correspond to integrals with only a few (1 or 2) indexes.

Dimension shifts and differentiation

Dimension shift relations [Tarasov, 1996; Derkachov et al., 1990]

Feynman representation in $d-2$ dimensions

$$j(d-2|\boldsymbol{n}) = \Gamma[\Sigma_k n_k + L - L\tfrac{d}{2}] \int\limits_{\mathbb{R}_+^M} \prod_{k=1}^{M} \frac{dz_k z_k^{n_k-1}}{\Gamma(n_k)} U \frac{F^{Ld/2-\Sigma_k n_k - L}}{U^{(L+1)d/2-\Sigma_k n_k - L}} \delta\left(1 - \widetilde{\Sigma}_k z_k\right)$$

Note an extra factor of U. Highlighted are the modifications which appeared due to the shift $d \to d-2$. Then it is easy to check that the following relation holds:

Dimension raising relation [Tarasov, 1996]

$$j(d-2|\boldsymbol{n}) = U(A_1, \ldots, A_N) j(d|\boldsymbol{n}) \qquad (U \text{ is 1st Symanzik polynomial})$$

Similarly, from Baikov representation we obtain

Dimension lowering relation [Derkachov, Honkonen, and Pis'mak, 1990]

$$j(d+2|\boldsymbol{n}) = \frac{2^L V^{-1}(p_1, \ldots, p_E)}{(d-K+1)_L} P(B_1, \ldots, B_N) j(d|\boldsymbol{n}) \quad (P \text{ is Baikov polynomial})$$

Note a remarkable correspondence:

Feynman parameters $z_k \Leftrightarrow A_k$ Baikov parameters $D_k \Leftrightarrow B_k$.

Differentiating wrt kinematic parameters.

Differentiating the integral $j(\boldsymbol{n})$ wrt to m^2 reduces to differentiating the **integrand**. Differentiating wrt some invariant $(p_i \cdot p_j)$ is trickier as the integrand depends on the scalar products of p_i, p_j with loop momenta. We have to express the derivative wrt $(p_i \cdot p_j)$ via derivatives wrt p_i and/or p_j

Differentiating wrt invariant [Remiddi, 1997]

$$\frac{\partial}{\partial (p_i \cdot p_j)} j(\boldsymbol{n}) = 2^{-\delta_{ij}} \left[\widehat{P}^{-1}\right]_{ik} p_k \cdot \partial_{p_j} j(\boldsymbol{n}) \ .$$

Here $\widehat{P} = \{p_i \cdot p_j | i, j = 1, \ldots, E\}$ is Gram matrix.
The derivative ∂_{p_j} can now be applied to the integrand of $j(\boldsymbol{n})$.

Alternatively, one might consider differentiation in Feynman or Baikov representations. Usually those also shift dimension, but this can be fixed as shown poreviously. E.g., using Lee-Pomeransky representation it is easy to obtain the following formula

Differentiating in Feynman representation

$$\frac{\partial}{\partial x} j(d-2|\boldsymbol{n}) = -\frac{\partial F(A_1, \ldots, A_N)}{\partial x} j(d|\boldsymbol{n}).$$

Here x is any kinematic parameter. Note the dimension shift in the lhs.

As a result of IBP reduction we express amplitudes via a finite set of master integrals $\boldsymbol{j} = (j_1, \ldots, j_K)^\intercal$. What is even more important, we can obtain closed equations for the master integrals. To obtain these equations we simply apply the dimensional shifts and/or differentiate the master integrals and then IBP-reduce the result. Then the dimension shifts and/or derivatives of the master integrals is expressed as linear combination of the same set of master integrals $\boldsymbol{j} = (j_1, \ldots, j_K)^\intercal$. We obtain

Differential equations

[Kotikov, 1991; Remiddi, 1997]

$$\partial_x \boldsymbol{j} = M(x,d)\boldsymbol{j}$$

Dimensional recurrences

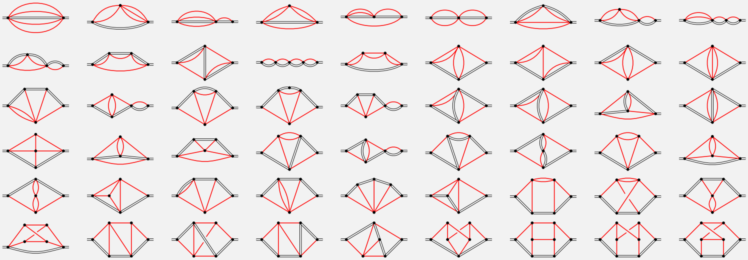
[Tarasov, 1996; Derkachov et al., 1990]

$$\boldsymbol{j}(d-2) = R(x,d)\boldsymbol{j}(d)$$

It appears that in multi-loop case it is often easier to solve these equations than to use direct methods for calculation of the master integrals.

Dimensional recurrence relations are especially useful for one-scale integrals, when the differential equations can not help. The approach is very effective when the matrix R in $\boldsymbol{j}(d-2) = R(d)\boldsymbol{j}(d)$ is triangular. Using analytical properties of integrals as functions of d to fix the arbitrary periodic functions, one can obtain the solution in the form of multiple sums with **factorized** summand. High-precision evaluation of these sums can be done with `SummerTime` package [RL and Mingulov, 2016]. Using PSLQ algorithm, one can turn the obtained numerical results into analytical expressions.

Four-loop HQET propagators with DRA method [RL and Pikelner, 2023]



Differential equations

- Differential equations for master integrals have the form

$$\partial_x \boldsymbol{j} = M(x,\epsilon)\boldsymbol{j}$$

- One can try to simplify the equation by transformation $\boldsymbol{j} = T\tilde{\boldsymbol{j}}$, so that

$$\partial_x \tilde{\boldsymbol{j}} = \widetilde{M}\tilde{\boldsymbol{j}}, \qquad \widetilde{M} = T^{-1}\left[MT - \partial_x T\right]$$

- [Henn, 2013]: there is often a "canonical" basis $\boldsymbol{J} = T^{-1}\boldsymbol{j}$ such that

$$\partial_x \boldsymbol{J} = \epsilon S(x)\boldsymbol{J} \qquad (\epsilon\text{-form})$$

- General solution for d.e. in ϵ-form is easily expanded in ϵ:

$$U(x,x_0) = \mathrm{Pexp}\left[\epsilon\int_{x_0}^{x} dx S(x)\right] = \sum_n \epsilon^n \iiint\limits_{x>x_n>\ldots>x_0} dx_n \ldots dx_1 S(x_n)\ldots S(x_1)$$

The algorithm of finding transformation to ϵ-form was devised in [RL, 2015]. It is implemented in 3 publicly available codes: `Fuchsia` [Gituliar and Magerya, 2017], `epsilon` [Prausa, 2017], and recently in `Libra` [RL, 2021].

General structure of reduction algorithm

Algorithm proceeds in three major stages, each involving a sequence of "elementary" transformations.

1. *Fuchsification*: Eliminating higher-order poles

Input: Rational matrix $M(x, \epsilon)$
Output: Rational matrix with only simple poles on the extended complex plane, $M(x, \epsilon) = \sum_k \frac{M_k(\epsilon)}{x - a_k}$.

2. *Normalization*: Normalizing eigenvalues

Input: Matrix from the previous step, $M(x, \epsilon) = \sum_k \frac{M_k(\epsilon)}{x - a_k}$.
Output: Matrix of the same form, but with the eigenvalues of all $M_k(\epsilon)$ being proportional to ϵ.

3. *Factorization*: Factoring out ϵ

Input: Matrix from the previous step.
Output: Matrix in ϵ-form, $M(x, \epsilon) = \epsilon S(x) = \epsilon \sum_k \frac{S_k}{x - a_k}$.

Both reduction to Fuchsian form and normalization of the matrix residues can be done with the following

Balance transformation

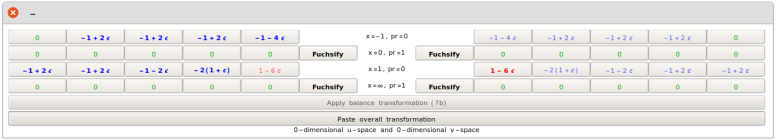
$T(x) = B(P, x_1, x_2|x) \stackrel{\text{def}}{=} \overline{P} + \frac{x-x_2}{x-x_1} P,$
$T^{-1}(x) = B(P, x_2, x_1|x) = \overline{P} + \frac{x-x_1}{x-x_2} P,$
where P is some projector and $\overline{P} = I - P$. When $x_1 = \infty$ or $x_2 = \infty$ omit denominator or numerator, respectively.
Balance transformation changes properties (pole order and eigenvalues of matrix residue) of the differential system at $x = x_1$ and $x = x_2$ without changing its properties at any other point.

The basic idea of the algorithm is to adjust the image and coimage of P to the properties of the system at x_1 and x_2, respectively, so as to improve those at x_1 without worsening them at x_2.

Libra program [RL, 2021]

- Libra is a *Mathematica* package useful for treatment of differential systems which appear in multiloop calculations.
- Tools for reduction to ϵ-form
  - Visual interface
  - Algebraic extensions
  - Birkhoff-Grothendieck factorization (for irreducibility criterion)

```
In[1]:  t=VisTransformation[M,x,ϵ];
```




```
In[2]:  t=FactorOut[M,x,ϵ,μ];
```

```
In[3]:  {L,d,R}=BirkhoffGrothendieck[T,x];
```

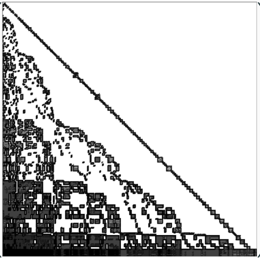
- Tools for constructing solution
  - Determining boundary constants via asymptotic coefficients.
  - Constructing ϵ-expansion of Pexp.
  - Constructing Frobenius expansion of Pexp.

```
In[3]:  U=PexpExpansion[{M,6},x];
```

One of many 4-loop massless vertex topologies with two off-shell legs.

- Differential system

$$\partial_x j = \underbrace{\left(\vphantom{\int} \quad \right)}_{374 \times 374 \text{ matrix}} j, \qquad \text{where } j = \left(\vphantom{\int} \quad \right)$$



- Maximum size of the diagonal blocks is "only" 11×11.
- No global rationalizing variable. Three algebraic extensions are needed for the reduction to ϵ-form:

$$x_1 = \sqrt{x}, \qquad x_2 = \sqrt{x - 1/4}, \qquad x_3 = \sqrt{1/x - 1/4}$$

Non-polylogarithmic integrals

- As we know from Bolybrukh counterexample to Hilbert's 21st problem[2], [Bolibrukh, 1989], the reduction algorithm may break already in the first step — reduction to global Fuchsian form. But with generic ϵ it is quite unlikely. I am not aware of any such case for multiloop integrals.
- The algorithm may and does sometimes break in the second step — in making all eigenvalues of all matrix residues proportional to ϵ. A strict criterion of irreducibility is devised in Ref. [RL and Pomeransky, 2017]. The question (ir)reducibility of the system is formulated as that of (non)triviality of a certain holomorphic vector bundle and thus can be decided via Birkhoff-Grothendieck factorization.

> With some reservations, the (ir)reducibility to ϵ-form corresponds to the (non)polylogarithmic integrals.

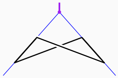
[2] Hilbert's 21st pr.: Proof of the existence of linear differential equations having a prescribed monodromic group.

Non-polylogarithmic integrals: “Systematic” approach

1. “Systematic” approach.
   - Reduce the system to $(A+\epsilon B)$-form:
     $$\partial_x j = (A+\epsilon B) j.$$
   - “Integrate out” the ϵ^0 form: make substitution $j = U_0 J$, where U_0 is a fundamental matrix for the unperturbed system $\partial_x U_0 = A U_0$.
   - The system for J is in ϵ-form:
     $$\partial_x J = \epsilon \widetilde{B} J, \qquad \widetilde{B} = U_0^{-1} B U_0.$$
   - The general solution $U_1 = \mathrm{Pexp}\left[\epsilon \int dx \widetilde{B}(x)\right]$ is expanded in terms of iterated integrals with weights being the elements of $\widetilde{B}$.
     NB: irreducibility to ϵ-form means that elements of $\widetilde{B}$ are transcendental functions. In particular, the weights might be possible to represent in terms of modular forms.
   - Pros: to some extent decouples the solution of unperturbed equation and ϵ-expansion.
   - Cons: Iterated integrals with transcendental weights are poorly investigated as compared to polylogarithms. When it comes to numerical evaluation, it is often necessary to reside to some sort of Frobenius method anyway.
2. Meanwhile, the Frobenius method can be applied directly to the differential system. It seems to be the most effective approach for numerical evaluation. In particular, it works for 3-loop massive form factors [Fael, Lange, Schönwald, and Steinhauser, 2022].
3. For many cases of non-polylogarithmic integrals there exists a one-fold integral representation in terms of polylogarithms and algebraic functions.

Example: maximal cut of non-planar vertex

Consider one solution of the homogeneous differential system, $J_1 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2} + 2\epsilon, 1 + \epsilon | x\right)$. Integrating out ϵ^0 gives



$$J_1 = \sum_k \epsilon^k \sum_{i \in \{1,2\}^{k+1}} \frac{2\mathrm{K}(x_{i_0})}{\pi} \mathcal{I}(\Omega_{i_0 i_1}, \Omega_{i_1 i_2}, \ldots, \Omega_{i_{k-1} i_k}, \Omega_{i_k 1} | x),$$

where $\mathcal{I}$ denotes iterated integral, $x_1 = x, x_2 = \overline{x} = 1 - x$, and

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \widetilde{B}(x) dx = \begin{pmatrix} u(\overline{x}) v(x) & -u(\overline{x}) v(\overline{x}) \\ u(x) v(x) & -u(x) v(\overline{x}) \end{pmatrix} \frac{dx}{\pi x \overline{x}},$$

$$u(x) = \mathrm{K}(x) - 2\mathrm{E}(x), \quad v(x) = 2\overline{x}\mathrm{K}(x) - 2\mathrm{E}(x).$$

Ω can be expressed via modular forms. Meanwhile, there are much simpler representations in terms of one-fold integrals:

$$J_1 = \frac{\Gamma(\epsilon+1)}{\sqrt{\pi}\Gamma\left(\epsilon + \frac{1}{2}\right)} \sum_k \epsilon^k \int_0^1 \frac{dt}{\sqrt{t(1-t)(1-tx)}} \frac{\ln^k \frac{1-t}{(1-tx)^2}}{k!}$$

$$J_1 = \frac{1}{i\pi^2} \oint_{\sqrt{x}<|t|<1} \frac{dt\, \mathrm{K}(x/t^2)}{t(1-t^2)} \left[1 - 2\epsilon(1-2t) H_1 + 2\epsilon^2 \left[2H_{0,1} - (1-2t)\left(3H_{1,-1} + H_{1,1}\right)\right] + \ldots\right],$$

where $H_{\boldsymbol{n}} = H_{\boldsymbol{n}}(t)$ is harmonic polylogarithm.

- There is a nontrivial symmetry in all available examples of differential systems for multiloop integrals. It is closely related to the intersection theory for twisted (co)homology groups.
- In particular, this symmetry results in the nontrivial quadratic relations for the terms of ϵ-expansion of non-polylogarithmic integrals.

Rational equivalence

We will say that two systems are rationally equivalent if $\exists$ a rational transformation T which maps the first system to the second, $M_2 = T^{-1}(M_1 T - \partial_x T)$. We will write $M_2 \overset{R}{\sim} M_1$ for matrices of rationally equivalent systems.

- The monodromy groups of rationally equivalent systems are isomorphic.
- Thanks to dimensional recurrences we have $M(\epsilon - 1) \overset{R}{\sim} M(\epsilon)$.
- The ϵ-reducibility means the rational equivalence to the system $\partial_x \boldsymbol{J} = \epsilon S(x) \boldsymbol{J}$ with S being independent of ϵ.

Using the same tools (e.g. `Libra`) as for the reduction to ϵ-form it is easy to check whether the two systems are rationally equivalent.

The differential systems for master integrals have a block-triangular form, with each block corresponding to the master integrals of the specific sector. The corresponding homogeneous systems are satisfied by the maximally cut master integrals of the sector. Our observation concerns homogeneous differential systems corresponding to each diagonal block.

Observation

Let

$$\partial_x \boldsymbol{j} = M(\epsilon, x)\boldsymbol{j} \tag{DE}$$

be such a homogeneous differential system corresponding to some block irreducible to block-triangular form. Then we observe on many examples that the differential system

$$\partial_x \tilde{\boldsymbol{j}} = -M^{\intercal}(-\epsilon, x)\tilde{\boldsymbol{j}} \tag{DE$\star$}$$

is *rationally equivalent* to the original system (DE).

Can this statement be strictly proved from twisted intersection theory?

Remarks

1. Note that if $M(\epsilon) \overset{R}{\approx} -M^{\intercal}(-\epsilon)$ holds for $d = d_0 - 2\epsilon$ then it necessarily holds for any $d = d_0 + k - 2\epsilon$ ($k \in \mathbb{Z}$ can be both even and **odd**) — thanks to dimensional recurrences.
2. Note that for $-M^{\intercal}(-\epsilon) \overset{R}{\approx} M(\epsilon)$ to be possible, the eigenvalues of the matrix residues should be either integer or half-integer at $\epsilon = 0$. The latter is a widely known observation.

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Let us now consider the general solution of $\partial \boldsymbol{j} = M\boldsymbol{j}$ in the form of path-ordered exponent, $U(x, x_0|\epsilon) = Pexp\left[\int_{x_0}^{x} M\,dx\right]$. It is easy to see that $U^{-1\intercal}(x, x_0|-\epsilon) = Pexp\left[-\int_{x_0}^{x} M^{\star}\,dx\right]$, so, is a general solution of ϵ-conjugated differential system $\partial \boldsymbol{j} = M^{\star}\boldsymbol{j}$.

Now we use our observation: $M = T^{-1}((-M^{\star})T - \partial T)$. For path-ordered exponents it translates to

$$U(x, x_0|\epsilon) = T^{-1}(x, \epsilon)U^{-1\intercal}(x, x_0|-\epsilon)T(x_0, \epsilon)$$

Therefore, we have

$$U^{\intercal}(x, x_0|-\epsilon)T(x, \epsilon)U(x, x_0|\epsilon) = T(x_0, \epsilon)$$

Quadratic constraints (outcome)

Let $\boldsymbol{j}_1(x,\epsilon)$ and $\boldsymbol{j}_2(x,\epsilon)$ be any two (possibly coinciding) solutions of the system $\partial_x \boldsymbol{j} = M(x,\epsilon)\boldsymbol{j}$. Then it is possible to find (using the available techniques) a rational matrix $T(x,\epsilon)$, such that

$$\boldsymbol{j}_1^{\intercal}(x,-\epsilon)T(x,\epsilon)\boldsymbol{j}_2(x,\epsilon) = \mathrm{const}$$

is independent of x. (The right-hand side can be found by considering some suitable asymptotics.)

Note that the opposite sign of ϵ in $j_1^{\intercal}(x,-\epsilon)$, so this relation concerns the solutions of two **different** differential systems (related via $\epsilon \to -\epsilon$). But within dimensional regularization we are interested in the coefficients of ϵ expansion, which are the same, up to alternating sign, for $j_1(x,-\epsilon)$ and $j_1(x,\epsilon)$. Thus, expanding the above relation in ϵ, we obtain an infinite set of quadratic relations for the expansion coefficients of the solution of the original differential system.

These quadratic relations seem to be the same as those which come from the intersection theory.

Two-loop equal mass cut sunrise in $d = 2 - 2\epsilon$ dimensions can be expressed via hypergeometric functions ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1-\epsilon; y\right)$ and ${}_2F_1\left(\frac{4}{3}, \frac{2}{3}; 1-\epsilon; y\right)$ [Tarasov, 2006]. The quadratic constraint reads

$$
\begin{aligned}
{}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1-\epsilon; y\right) {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \epsilon+1; y\right) + \tfrac{(y-1)}{3\epsilon}\, {}_2F_1\left(\tfrac{4}{3}, \tfrac{2}{3}; 1-\epsilon; y\right) {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \epsilon+1; y\right) \\
+ \tfrac{(1-y)}{3\epsilon}\, {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1-\epsilon; y\right) {}_2F_1\left(\tfrac{4}{3}, \tfrac{2}{3}; \epsilon+1; y\right) = 1\,.
\end{aligned}
$$

When expanded in ϵ it results to the following "shuffling-like" identities ($N = 0, 1, \ldots$):

$$
\sum_{n=0}^{N} (-)^n H_{\alpha,\mathbf{1}_n} H_{\alpha,\mathbf{1}_{N-n}} + [1 + (-)^N] y(1-y) \sum_{n=0}^{N+1} (-)^n \left(\partial H_{\alpha,\mathbf{1}_n}\right) H_{\alpha,\mathbf{1}_{N+1-n}} = \delta_{N0}\,.
$$

where

$$
H_{\alpha,\mathbf{1}_n}(y) = \sum_{j=0}^{\infty} \frac{(3j)!}{3^{3j}\,(j!)^3} y^j S_{\underbrace{1,\ldots,1}_{n}}(j)
$$

Multiloop sunrise integrals in $d = 2$ in coordinate space are expressed via functions

$$\mathrm{IKM}[\{a_0, b_0\}_{m_0}, \{a_1, b_1\}_{m_1}, \ldots, s] = \int dx\, x^s \prod_k [I_0(m_k x)]^{a_k} [K_0(m_k x)]^{b_k}\,.$$

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Our approach allows one to obtain quadratic relations for those functions. E.g., for two-loop sunrise we obtain

$$\mathrm{IKM}\left[\{2,1\}_m, \{0,1\}_1, 1\right] \mathrm{IKM}\left[\{3,0\}_m, \{0,1\}_1, 3\right] - \mathrm{IKM}\left[\{2,1\}_m, \{0,1\}_1, 3\right] \mathrm{IKM}\left[\{3,0\}_m, \{0,1\}_1, 1\right] = \frac{4\left(1-5m^2\right)}{(1-m^2)^2\,(1-9m^2)^2}\,.$$

The right-hand side has been calculated from the limit $m \to 0$.
At 3 loops we, e.g. have

$$9\mathrm{IKM}\left(\{3,1\}_{\frac{1}{4}}, \{0,1\}_1, 1\right) \mathrm{IKM}\left(\{3,1\}_{\frac{1}{4}}, \{0,1\}_1, 3\right) - 16\mathrm{IKM}\left(\{3,1\}_{\frac{1}{4}}, \{0,1\}_1, 1\right)^2 = 20\,,$$

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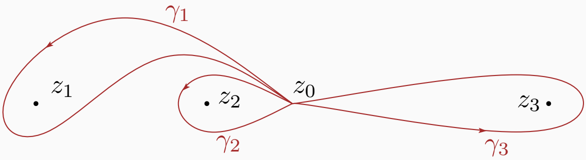
In Ref. [Broadhurst and Roberts, 2018] remarkable quadratic relations have been conjectured

$$\sum_{k,l} \mathrm{IKM}\left[\{\tilde{n}, N-\tilde{n}\}_1, k\right] D_{kl}(N, \tilde{n}, n) \mathrm{IKM}\left[\{n, N-n\}_1, l\right] = \pi^{N+1-\tilde{n}-n} B(N, \tilde{n}, n)\,,$$

where $D(N, \tilde{n}, n)$ and $B(N, \tilde{n}, n)$ are rational numerical matrices. Recently, these relations have been proved in Ref. [Fresán et al., 2020], except that matrix D was defined differently. Within our approach we have been able to do the same (with yet another definition of D).

Monodromy group



- Monodromy group $\mathcal{G}_{\circlearrowleft} \subset GL(n, \mathbb{C})$ of the differential system $\partial_z \boldsymbol{j} = M\boldsymbol{j}$ with $\boldsymbol{j} = (j_1, \ldots j_n)^\intercal$ determines how the solution space transforms under analytical continuation along nonequivalent closed paths[3]. It is generated by the monodromies around the loops encircling each singular point of the system.
- Monodromy group captures all nontrivial properties of the differential system while being blind to a specific realization (in particular, $\mathcal{G}_{\circlearrowleft}$ is invariant wrt rational transformations of the system).
  Hilbert's 21st problem: **Proof of the existence of linear differential equations having a prescribed monodromic group**.

---

[3]Reminder: Let $U(z)$ is a fundamental matrix, $\partial_z U = MU$ determined in the vicinity of a regular point z_0, and let $U(z)|_\gamma$ denotes its analytical continuation along the closed path γ starting and ending in this vicinity. Then $U(z)|_\gamma = U(z)g(\gamma)$, where $g(\gamma)$ is a complex $n \times n$ matrix (i.e. $g(\gamma) \in GL(n, \mathbb{C})$). In fact, this matrix depends only on homotopy class $[\gamma]$ (they form a fundamental group $\pi_1(\overline{\mathbb{C}})$). Thus the monodromy group $\mathcal{G}_{\circlearrowleft} = \{g([\gamma])| \, [\gamma] \in \pi_1(\overline{\mathbb{C}})\}$ is a representation of the fundamental group $\pi_1(\overline{\mathbb{C}})$.

Monodromy group at $\epsilon = 0$ and (ir)reducibility

The ϵ-reducible and ϵ-irreducible systems differ intrinsically by the type of their monodromy groups at $\epsilon = 0$:

- ϵ-reducible with rational transformations: monodromy group is trivial, $\mathcal{G}_{\circlearrowleft} = \{1\}$.
- ϵ-reducible with algebraic transformations: monodromy group is finite, $|\mathcal{G}_{\circlearrowleft}| < \infty$. Monodromy group becomes trivial on the corresponding covering space.
- ϵ-irreducible: monodromy group is (isomorphic to) a subgroup of $GL(n, \mathbb{Z})$?

In particular, for elliptic cases $\mathcal{G}_{\circlearrowleft}$ is a congruence subgroup of $SL(2, \mathbb{Z})$, see [Broedel et al., 2022] for the case of 2-loop sunrise and 3-loop banana graph. This fact allows one to express the integration kernels via modular forms.

Monodromies from Frobenius expansions

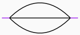
Monodromy group can be obtained numerically from Frobenius expansion, so it is not so easy to see the structure from, e.g.,

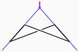
$$g_1 = \begin{pmatrix} 1 & 0. & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, g_2 = \begin{pmatrix} -2. & -5.6325 & -4.11456 \\ 0.618343 & 2.16094 & 0.84807 \\ -0.117344 & -0.220313 & 0.83906 \end{pmatrix},$$

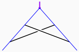
$$g_3 = \begin{pmatrix} -8. + 0.i & -16.8975 + 19.5116i & -12.3437 + 102.816i \\ 1.85503 - 0.296943i & 3.83906 - 4.57912i & -0.84807 - 21.5991i \\ -0.352031 + 0.406491i & 0.220313 + 1.52637i & 5.16094 + 4.57912i \end{pmatrix}.$$

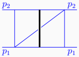
We need to find a matrix t such that $t^{-1} g_k t$ are all integer matrices. One needs some experimentation to find such a matrix. However, it appears to be possible! We find that $t = \begin{pmatrix} 1 & 0 & 3 \\ -3c - \frac{1}{32c} & \frac{i\left(1-96c^2\right)}{16\sqrt{3}c} & -3c - \frac{1}{32c} \\ c & \frac{2ic}{\sqrt{3}} & c \end{pmatrix}$ with $c = 0.11734382\ldots$ being some unrecognized constant, renders

$$t^{-1} g_1 t = \begin{pmatrix} -2 & 0 & -3 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}, t^{-1} g_2 t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, t^{-1} g_3 t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 6 \\ 0 & -4 & -5 \end{pmatrix}.$$

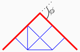

Two-loop sunrise[4]: $\mathcal{G}_{\circlearrowleft} \cong \Gamma_1(6) \subset SL(2,\mathbb{Z})$


Two-loop massive vertex [von Manteuffel and Tancredi, 2017]: $\mathcal{G}_{\circlearrowleft} \cong \Gamma(2) \subset SL(2,\mathbb{Z})$.


Two-loop EW vertex [Broedel, Duhr, Dulat, Penante, and Tancredi, 2019]: $\mathcal{G}_{\circlearrowleft} \cong \Gamma_1(6) \subset SL(2,\mathbb{Z})$.



3-loop forward box [Mistlberger, 2018]: $\mathcal{G}_{\circlearrowleft} \cong \Gamma_1(5) \subset SL(2,\mathbb{Z})$.



4-loop HQET vertex [Brüser, Dlapa, Henn, and Yan, 2020] : $\mathcal{G}_{\circlearrowleft} \cong \Gamma(3) \subset SL(2,\mathbb{Z})$.


3-loop equal-mass sunrise [Broedel, Duhr, and Matthes, 2022]:
$$\mathcal{G}_{\circlearrowleft} \cong \left\langle \begin{pmatrix} 1 & 6 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & -2 \\ 4 & 4 & -3 \end{pmatrix}, \begin{pmatrix} -3 & -10 & 7 \\ 12 & 31 & -21 \\ 16 & 40 & -27 \end{pmatrix} \right\rangle \subset GL(3,\mathbb{Z}).$$


3-loop HQET sunrise $\mathcal{G}_{\circlearrowleft} \cong \left\langle \begin{pmatrix} -2 & 0 & -3 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 6 \\ 0 & -4 & -5 \end{pmatrix} \right\rangle \subset GL(3,\mathbb{Z})$

[4] Here
$$\Gamma_1(N) = \left\{ g \in SL(2,\mathbb{Z}) \,\middle|\, g = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}, \qquad \Gamma(N) = \left\{ g \in SL(2,\mathbb{Z}) \,\middle|\, g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

Simplifications using symbol map

There is a standard approach to the simplification of the polylogarithmic expressions using symbol map. One might think of symbols as a cleaner way to represent iterated (or path-ordered) integrals with logarithmic weights (with some reservations, though):

$$I = \int\limits_{1>\tau_n>\ldots>\tau_1>0} \!\!\!\!\int\!\cdots\!\int d\ln p_n(\tau_n)\ldots d\ln p_1(\tau_1) \xrightarrow{\quad\quad\mathcal{S}\quad\quad} p_n\otimes\ldots\otimes p_1$$

Formal symbol manipulation rules then easily follow, e.g.

$$d\ln(pq) = d\ln p + d\ln q \qquad \Longrightarrow \qquad (\ldots\otimes pq\otimes\ldots) = (\ldots\otimes p\otimes\ldots) + (\ldots\otimes q\otimes\ldots)$$

Similarly, by ordering the integration variables in the product of integrals, we get $\mathcal{S}(I_1 I_2) = \mathcal{S}(I_1) ⧢ \mathcal{S}(I_2)$, where ⧢ denotes a shuffle product, e.g.

$$(a\otimes b)⧢(c\otimes d) = a\otimes b\otimes c\otimes d + a\otimes c\otimes b\otimes d + a\otimes c\otimes d\otimes b + c\otimes a\otimes b\otimes d + c\otimes a\otimes d\otimes b + c\otimes d\otimes a\otimes b$$

We have, in particular, symbols for classical polylogarithms

$$\mathcal{S}(\mathrm{Li_n}(x)) = \underbrace{x\otimes\ldots\otimes x}_{n-1}\otimes(x-1)$$

Symbols are good for checking the identities, e.g., using $\mathcal{S}$ it is easy to establish[5]

$$7\mathrm{Li}_2\left(\tfrac{1+\varepsilon/z}{1-i\varepsilon}\right)-7\mathrm{Li}_2\left(\tfrac{1+\bar{\varepsilon}/z}{1+i\bar{\varepsilon}}\right)+7\mathrm{Li}_2\left(\tfrac{z+\bar{\varepsilon}}{\bar{\varepsilon}-i}\right)-7\mathrm{Li}_2\left(\tfrac{z+\varepsilon}{\varepsilon+i}\right)+11\mathrm{Li}_2\left(\tfrac{z+\varepsilon}{\varepsilon-i}\right)-11\mathrm{Li}_2\left(\tfrac{z+\bar{\varepsilon}}{\bar{\varepsilon}+i}\right)$$
$$+4\mathrm{Li}_2(1+z\varepsilon)-4\mathrm{Li}_2(1+z\bar{\varepsilon})+18\mathrm{Li}_2(-iz)-18\mathrm{Li}_2(iz)+11\mathrm{Li}_2\left(\tfrac{1+\bar{\varepsilon}/z}{1-i\bar{\varepsilon}}\right)-11\mathrm{Li}_2\left(\tfrac{1+\varepsilon/z}{1+i\varepsilon}\right)$$
$$=\tfrac{2i\pi^2}{5\sqrt{3}}-\tfrac{23}{3}i\pi\ln z+6i\pi\ln\left(2-\sqrt{3}\right)-\tfrac{i\psi'\left(\frac{1}{6}\right)}{5\sqrt{3}}-24iG,\qquad \text{where } \varepsilon=1/\bar{\varepsilon}=e^{2\pi i/3}.$$

However, strictly speaking, they are much less powerful in simplifying expressions. E.g., if we omit in the left-hand side a couple of dilogs with not so simple arguments, we could have failed to recognize in the symbol of the resulting expression that of the sum of the omitted dilogs.

Simplification algorithm idea

For a given expression:

1. find all possible arguments of Li_n which might enter the simplified form.
2. find equivalent form with the minimal number of polylogs.

[5] NB: This identity was used in real life (as well as some yet more complicated identities) for the simplification of the total cross section of Compton scattering @NLO [RL, Schwartz, and Zhang, 2021].

22-term relation for Li_3

Introducing

$$f(x) = \mathrm{Li}_3(x) + \frac{1}{24}\ln(1-x)\ln^2\left(x^2\right) - \frac{\pi^2}{12}\ln\left(x^2\right),$$

we can prove the identity

$$f(xyz) + 3f\left(\frac{\hat{x}}{\widehat{xyz}}\right) + 3f\left(\frac{xy\hat{z}}{\widehat{xyz}}\right) - 3f\left(\frac{-x\hat{y}\hat{z}}{\hat{x}\widehat{xyz}}\right) + 6f\left(\frac{-x\hat{y}}{\hat{x}}\right)$$
$$- 3f(xy) + 3f(x) + \frac{3}{2}\pi^2\ln x - 3\zeta_3 + \text{permutations} = 0,$$

where $\hat{a} = 1 - a$ and $x, y, z \in (0, 1)$.

This identity is probably equivalent to 22 term relation in [Goncharov, 1991].

NB: S_3 symmetry is explicit here.

- Each step towards increasing the # of loops and/or # of scales requires new methods. Those involve both technological advances and new algorithms coming from various fields of mathematics.
- IBP reduction still remains a bottleneck for some calculations. New ideas of IBP reduction appear, whether they will be successful is yet to find out.
- Differential equations method is already in a very good shape. However, there is still no regular approach to the computation of non-polylogarithmic integrals. From the practical point of view, there is always a Frobenius method which might be used to obtain numerical high-precision results.
- New ideas and approaches to multi-loop calculations are always very welcome.

Thank you!

DRA method

Let us briefly explain how DRA method works for triangular R. It is convenient to introduce $\boxed{\nu = d/2}$ and to consider all integrals as functions of ν. Then for the master integral $J = j_k$ we have the following inhomogeneous equation

$$J(\nu-1) = C(\nu)J(\nu) + D(\nu),$$

where $C(\nu) = c\frac{\prod_i (a_i-\nu)}{\prod_i (b_i-\nu)}$ is some rational function and $D(\nu)$ is a linear combination of simpler master integrals. We assume that simpler masters are already calculated at this stage by the same method (or evaluated explicitly in terms of Γ-functions).

Using the homogeneous solution $S^{-1}(\nu) = c^{-\nu}\frac{\prod_i \Gamma(a_i-\nu)}{\prod_i \Gamma(b_i-\nu)}$, we obtain

$$Y(\nu-1) = Y(\nu) + S(\nu-1)D(\nu), \qquad Y(\nu) = S(\nu)J(\nu)$$

The general solution of this equation reads

$$Y(\nu) = \omega(\nu) + \Sigma_{\pm} S(\nu-1)D(\nu),$$

depending on which of the two sums converges. Here $\omega(\nu) = \omega(\nu+1)$ is arbitrary periodic function and we have introduced notations

$$\Sigma_- f(\nu) = -\sum_{k=0}^{\infty} f(\nu-k) \qquad \Sigma_+ f(\nu) = \sum_{k=1}^{\infty} f(\nu+k)$$

$$Y(\nu) = \omega(\nu) + \Sigma_{\pm} S(\nu - 1) D(\nu),$$

Two questions are in order:

1. How does one fix $\omega(\nu)$.
2. Is it possible to calculate emerging multiple sums with high precision.

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Two questions are in order:

1. How does one fix $\omega(\nu)$.
2. Is it possible to calculate emerging multiple sums with high precision.

The answer to the first question is the essence of DRA method. First, let us introduce $z = e^{2i\pi\nu}$. Then the periodic function of ν shall be understood as function of z (since z does not change upon $\nu \to \nu + 1$) Let us write

$$\omega(z) = S(\nu) J(\nu) - \Sigma_{\pm} S(\nu - 1) D(\nu).$$

We know everything about the second term in the right-hand side, but $J(\nu)$ in the first term is the goal of our calculation, so we do not know much about it. However we can extract some information about analytical properties of $J(\nu)$, e.g., from parametric representation. Suppose that we've succeeded to prove that the whole right-hand side is analytic on some stripe $\mathrm{Re}\nu \in [\nu_0, \nu_0 + 1)$ and decays when $\mathrm{Im}\nu \to \pm\infty$. Then we can claim that $\omega(z)$ has no singularities and decays when $|z| \to \infty$. These mild restrictions lead to a very concrete form: $\omega = 0$ (Note that in real analysis the same restrictions would not say much about ω).

Comparison with Mellin-Barnes representation

Mellin-Barnes representation (not considered here) is a powerful tool which can provide expressions for the loop integrals in the form of multiple sums.

Form of the DRA results

The DRA results are expressed in terms of the multiple sums

$$\sum_{\infty > k_1 \geqslant \ldots \geqslant k_n} f_1(k_1)\ldots f_n(k_n)$$

The summand is factorized.

Complexity scales **linearly** with n.

for $k = 0..k_{max}$ **do**
  for $i = 0..n$ **do**
    $S_i = S_i + S_{s-1} f_i(k)$
  end
end
return S_n

Form of the MB results

The MB results are expressed in terms of the multiple sums

$$\sum_{k_1}\ldots\sum_{k_n} f(k_1 \ldots k_n)$$

The summand is not factorized.

Complexity scales **exponentially**.

for $k_1 = 0..k_{max}$ **do** ...//n-fold
  for $k_n = 0..k_{max}$ **do**
    $S = S + f(k_1, \ldots)$
  end
end
return S

It is easy to set up the following algorithm

1. Reduce both differential systems to normalized Fuchsian form with the same (but otherwise arbitrary) normalization conditions. Let T_1 and T_2 be the corresponding transformations.
2. If the reduced systems have different sets of singular points or different sets of eigenvalues of their matrix residues at least in one point (counting with multiplicities), the systems are not equivalent.
3. Otherwise, search for a constant (independent of x) **invertible** matrix $T_3(\epsilon)$ such that

   $$M_1 T_3 = T_3 M_2 \, .$$

   Here $M_{1,2}$ are the matrices for two normalized fuchsian forms obtained at step 1. This is just a system of linear equations for the elements of T_3.
4. If such a T_3 does not exist, the systems are not equivalent. Otherwise, they are equivalent by means of the transformation

   $$T = T_1 T_3 T_2^{-1}$$

Proof becomes a trivial exercise given the Proposition of [RL and Pomeransky, 2017].

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