GKZ approach to Feynman integrals and beyond

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Contents

What Feynman integrals evaluate to?

In 1963 Regge conjectured that the convergent Feynman integrals form a new class of special functions satisfying certain differential equations. He called them generalized hypergeometric functions. His justification for this

V A Golubeva, 1976 Russ.Math.Surv.31 139

- Answer from Mellin-Barnes[Boos-Davydychev, '90] techniques: ${}_pF_q$, Appell, Lauricella, etc. Horn-type multivariable hypergeometric functions [Kalmykov-Bytev-Kniehl-Ward-Yost, '09]
- IBPs[Chetyrkin-Tkachov, '81]+Differential Equations[Gehrmann-Remiddi, '99]+ϵ-form [Henn, '13]. Goncharov polylogarithms and related elliptic structures
- Gel'fand-Kapranov-Zelevinsky (GKZ) observation: A-hypergeometric functions

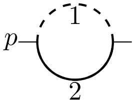
with integral representations. We only note that among the Euler type integrals associated with systems of the form (0.2) there are the integrals $\int \Pi P_i(t_1, \ldots, t_n)^{\alpha_i} t_1^{\beta_1} \ldots t_n^{\beta_n} dt_1 \ldots dt_n$, where P_i are polynomials, i.e., practically all integrals which arise in quantum field theory.

Remarks

- All the above answers expressible as infinite sums
- State of the art based on the most popular answer $\rightarrow$ efficiency
- Precise answer to this question not only of mathematical relevance

Invitation to GKZ: Bubble

- Suppose we want to study the Feynman integral



$$I_{\text{bubble}} = \int_{\mathbb{R}} \left(\frac{\mathrm{d}^d k}{\pi^{d/2}}\right) \frac{1}{[(k)^2]^{\alpha_1}[(k-p)^2+m^2]^{\alpha_2}}$$

- Lee-Pomeransky ($\beta = d/2$)

$$I_{\text{bubble}}/\xi_{\Gamma_\alpha} := I_g(\alpha,\beta) = \int_{\mathbb{R}_+^2} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{(z_1+z_2+(m^2+s)z_1z_2+m^2z_2^2)^\beta} \frac{\mathrm{d}z_1}{z_1}\frac{\mathrm{d}z_2}{z_2}$$

- GKZ approach: consider the more general version of this integral

$$I_g(\alpha,\beta,c) = \int_{\Omega} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{(c_1z_1+c_2z_2+c_3z_1z_2+c_4z_2^2)^\beta} \frac{\mathrm{d}z_1}{z_1}\frac{\mathrm{d}z_2}{z_2}$$

- Consider the matrix of exponents of $g(c,z)$

$$g(c,z) = (c_1z_1+c_2z_2+c_3z_1z_2+c_4z_2^2) \Longleftrightarrow \mathsf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

Invitation to GKZ: Bubble

- Integral satisfies the system of PDEs ($\partial_i := \partial/\partial c_i$)

$$\begin{aligned}(\partial_2\partial_3 - \partial_1\partial_4)I(\alpha,\beta,c) =&0,\\ (c_1\partial_1 + c_2\partial_2 + c_3\partial_3 + c_4\partial_4 + \beta)I(\alpha,\beta,c) =&0,\\ (c_1\partial_1 + c_3\partial_3 + \alpha_1)I(\alpha,\beta,c) =&0,\\ (c_2\partial_2 + c_3\partial_3 + 2c_4\partial_4 + \alpha_2)I(\alpha,\beta,c) =&0.\end{aligned}$$

- Canonical series solutions (Saito-Sturmfels-Takayama) ($\beta := d/2$)

$$\begin{aligned}\phi_1 =& c^{\gamma_1}{}_2F_1\left(\alpha_1, \alpha_1+\alpha_2-\beta; 2\alpha_1+\alpha_2-2\beta+1; \frac{c_2c_3}{c_1c_4}\right),\\ \phi_2 =& c^{\gamma_2}{}_2F_1\left(2\beta-\alpha_1-\alpha_2, \beta-\alpha_1; 2\beta-2\alpha_1-\alpha_2+1; \frac{c_2c_3}{c_1c_4}\right).\end{aligned}$$

- γ_i roots of a system of polynomial equations obtained from A
- General solution is

$$I(\alpha,c) = K_1\phi_1 + K_2\phi_2$$

- Feynman integral is the restriction of $I(\alpha,c) \to c_1 = c_2 = 1$, $c_3 = (s+m^2)$, $c_4 = m_2$

A-philosophy I

- GKZ (Discriminants, Resultants and Multidimensional Determinants): "*The study of many problems becomes more transparent if we consider not individual polynomials but polynomials with indeterminate coefficients*"
- Multi-index notation ($\alpha \in \mathbb{Z}^N$)

$$z^\alpha := z_1^{\alpha_1} \cdots z_N{}^{\alpha_N}$$

- *Laurent* polynomials in N variables of the form

$$b_i(z) = \sum_{j=1}^{n_i} c_{ij} z^{\alpha_{ij}}, \qquad c_{ij} \in \mathbb{C}_* = \mathbb{C}\backslash\{0\}, \qquad i = 1, \ldots, q$$

- For each $b_i(z)$, we have the $N \times n_i$ configuration matrix

$$A_i = (\alpha_{i1} \ \cdots \ \alpha_{ik} \ \cdots \alpha_{in_i}), \qquad \alpha_{ik} \in \mathbb{Z}^N$$

- Each column of A_i associated with a monomial term in $b_i(z)$
- $n = n_1 + \cdots + n_q$ is the total number of monomials

A-philosophy II

- Product of Laurent polynomials

$$b(z) := b_1(z) \cdots b_q(z)$$

- Define the $(N + q) \times n$ matrix

$$\mathsf{A} := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ A_1 & A_2 & \dots & A_q \end{pmatrix}$$

- Here $0 = (0, \dots, 0)$ and $1 = (1, \dots, 1)$ are row vectors of length $|A_i|$.

$$\mathrm{co}(\mathsf{A}) := n - N - q.$$

Comment

One can associate a polytope to the matrix A and study its combinatorial properties. In this talk we will not follow this approach.

Example: ${}_2F_1$

- Single polynomial $n_1 = 4$, $N = 2$, $q = 1$

$$b(c, z) = c_1 + c_2 z_1 + c_3 z_2 + c_4 z_1 z_2 \iff A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\mathsf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

- Product of polynomials $n_1 = 2$, $n_2 = 2$, $N = 1$, $q = 2$

$$b(z)^\beta = b_1^{\beta_1} b_2^{\beta_2} = (c_1 + c_2 z)^{\beta_1} (c_3 + c_4 z)^{\beta_2} \iff \mathsf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Gel'fand-Kapranov-Zelevinsky systems

Defined by the following data

1. $(N+q)\times n$ matrix A such that the vector $(1,\dots,1)$ lies in its row span
$$b(z)=b_1(z)\cdots b_q(z).$$
2. A vector of parameters $\kappa=(\kappa_1,\dots,\kappa_{N+q})$, $\kappa_i\in\mathbb{C}$
3. A system of partial differential equations(PDEs) associated with A. Let $u,v\in\mathbb{N}^n$ and consider
$$\Big(\partial^u-\partial^v\Big)F(c)=0,\quad \text{where}\quad \mathrm{A}u=\mathrm{A}v,$$
$$\left(\sum_{j=1}^{n}a_{ij}\theta_j-\kappa_i\right)F(c)=0,\quad i=1,\dots,N+q$$
$\theta_j=c_j\partial/\partial c_j$ and $\partial^u=\partial_1^{u_1}\cdots\partial_n^{u_n}$

A-hypergeometric functions

A holomorphic function $F(c)$ or formal series is called A-hypergeometric if it satisfies the system of PDEs

GKZ and D-modules

- GKZ systems as holonomic ideals in the Weyl algebra D
- $D = \mathbb{C}\langle c_1, \dots, c_n, \partial_1, \dots, \partial_n \rangle$ modulo commutation rules beetwen c_i, ∂_j

$$D \ni p(z, \partial) = \sum_{\alpha,\beta} a_{\alpha\beta} c^\alpha \partial^\beta$$

- Toric ideal of A

$$I_{\mathsf{A}} := \langle \partial^u - \partial^v : \mathsf{A}u = \mathsf{A}v, \quad u, v \in \mathbb{N}^n \rangle \subset \mathbb{C}[\partial_1, \dots, \partial_n]$$

- Ideal generated by κ^T and $\theta = (\theta_1, \dots, \theta_n)^T$

$$\langle \mathsf{A}\theta - \kappa^T \rangle \subset \mathbb{C}[\theta_1, \dots, \theta_n]$$

- $H_{\mathsf{A}}(\kappa)$ is the left ideal on D generated by I_{A} and $\langle \mathsf{A}\theta - \kappa^T \rangle$

A-hypergeometric functions

A holomorphic function $F(c)$ or formal series is A-hypergeometric of degree κ if

$$H_{\mathsf{A}}(\kappa) \bullet F(c) = 0, \qquad \mathrm{rank}(H_{\mathsf{A}}(\kappa)) \geq \mathrm{vol}(\mathsf{A})$$

Generic κ: $\mathrm{rank}(H_{\mathsf{A}}(\kappa)) = \mathrm{vol}(\mathsf{A}) = \mathrm{degree}(I_{\mathsf{A}})$

Euler type solutions

- Vector of parameters $\kappa := (-\beta, -\alpha)$, $\beta \in \mathbb{C}^q$, $\alpha \in \mathbb{C}^N$

$$I_b(\kappa) = \int_\Omega \frac{z^\alpha}{b(c,z)^\beta} \mathrm{d}\eta_N, \qquad \mathrm{d}\eta_N = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_N}{z_N},$$

- Cycle $\Omega \subset (\mathbb{C}_*)^N \backslash \mathcal{V}(b)$ usually assumed to be compact
- Set of zeros of $b(c,z)$ is the algebraic variety $\mathcal{V}(b)$
- GKZ, '90: If the integral $I_b(\kappa)$ converges and defines a germ of analytic functions in the variables z, then it represents a solution of the A-hypergeometric system $H_{\mathsf{A}}(\kappa)$
- Also GKZ, '90... if all b_i have real coefficients one can take the integral also over some connected component of $\mathbb{R}^N \setminus \mathcal{V}(b)$
- Non-compact cycles for A-hypergeometric functions (Coamoeba)

Non compact cycles: Euler-Mellin integrals

- Berkesh-Forsgård-Passare(BFP), '13 Euler-Mellin integral is an integral of the form of $I_b(\kappa)$ taken over a cycle $\Omega = \text{Arg}^{-1}(\theta)$, related to coameba of $\mathcal{V}(b)$

Amoeba and Coamoeba of an algebraic variety

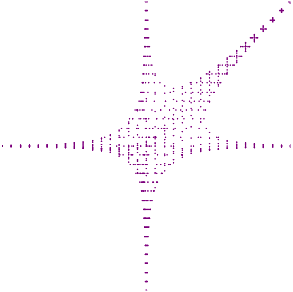
$$\mathcal{A}_b := \text{Log}(\mathcal{V}(b)), \qquad \text{Log}(z) = (\log|z_1|, \ldots, \log|z_N|)$$
$$\mathcal{A}'_b := \text{Arg}(\mathcal{V}(b)), \qquad \text{Arg}(z) = (\arg(z_1), \ldots, \arg(z_N))$$

- BFP consider the Euler-Mellin integral

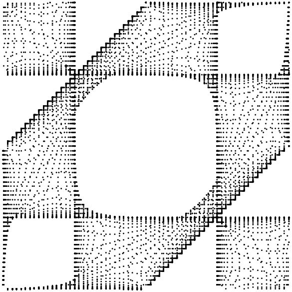
$$I_b(\kappa) = \int_{\mathbb{R}_+^N} \frac{z^\alpha}{b(z)^\beta} \mathrm{d}\eta_N = \int_{\mathbb{R}^N} \frac{e^{(\alpha,x)}}{b(e^x)^\beta} \mathrm{d}x$$

- Simplest case is $\Omega \in \mathbb{R}_+^N$ which covers Feynman integrals
- The convergence of these integrals is controlled by a theorem due to Berkesh, Forsgård and Passare (BFP).

Amoebas and coamoebas



Amoeba(shaded)$[-4,4]\times[-4,4]$



Coamoeba(shaded)$[-\pi,\pi]\times[-\pi,\pi]$

$b(z_1,z_2)=1+z_1^2+z_2^2+z_1z_2^2+z_2z_1^2=0$
[Credit: Jens Forsgård mathematica package]

- Noncompact cycle is given by a representative $\theta\in\Theta$ of a connected component of $\mathbb{R}^N\setminus\bar{\mathcal{A}}'_b$
$$\Omega=\text{Arg}^{-1}\theta$$
- For a Feynman integral we should choose θ such that $\Omega\in\mathbb{R}_+^N$, in other words such that $0\in\mathbb{R}^N\setminus\bar{\mathcal{A}}'_b$

BFP: From Euler-Mellin to GKZ

- Let us consider the weighted Newton polytopes of b_j $\tau\Delta_b = \sum_{j=1}^{q} \tau_j \Delta_{b_j}$.
- $I(\kappa)$ converges and defines and analytic funcion with parameters $\kappa = (-\beta, -\alpha)$ on the tube domain

$$\{(\alpha, \beta) \in \mathbb{C}^{N+q} | \tau := \text{Re } \beta \in \mathbb{R}^q_+, \qquad \sigma := \text{Re } \alpha \in \text{int}(\tau\Delta_b)\},$$

- If the polynomials $b(z)$ vanish on the positive orthant we can take a connected component Θ of $\mathbb{R}^N \backslash \overline{\mathcal{A}}'_b$, where $\overline{\mathcal{A}}'_b$ denotes the closure of the coamoeba of b and consider the integral

$$I_b(\kappa) = \int_{\text{Arg}^{-1}\theta} \frac{z^\alpha}{b(z)^\beta} \mathrm{d}\eta_N = \int_{\mathbb{R}^N} \frac{e^{\alpha\cdot(x+\mathrm{i}\theta)}}{b(e^{x+\mathrm{i}\theta})^\beta} \mathrm{d}x,$$

- Promoting the coefficients of $b(z)$ to indeterminates

$$I_b(c, \kappa) = \int_{\text{Arg}^{-1}\theta} \frac{z^\alpha}{b(c, z)^\beta} \mathrm{d}\eta_N$$

  represents an A-hypergeometric function (Theorem 4.2 in BFP)
- For generic parameters κ provides a basis of solutions of $H_{\mathsf{A}}(\kappa)$
- Each integral is evaluated on a representative of Θ for each connected component of $\mathbb{R}^N \backslash \overline{\mathcal{A}}'_b$

Formal series Solutions

- $\mathcal{L} := \{u \in \mathbb{Z}^n : \mathsf{A}u = 0\}$
- For $u \in \mathcal{L}$, we can write $u = u_+ - u_-$, where $u_\pm \in \mathbb{N}^n$ have disjoint support
- For $\gamma \in \mathbb{C}^n$ we define (falling factorials)

$$[\gamma]_{u_-} := \prod_{i:u_i<0} \prod_{j=1}^{-u_i} (\gamma_i - j + 1) = \prod_{i:u_i<0} (-1)^{-u_i} (\gamma_i)_{-u_i},$$

$$[\gamma + u]_{u_+} := \prod_{i:u_i>0} \prod_{j=1}^{u_i} (\gamma_i + u_i - j + 1) = \prod_{i:u_i>0} \prod_{j=1}^{u_i} (\gamma_i + j) = \prod_{i:u_i>0} (\gamma_i + 1)_{u_i}$$

$(a)_x$ are Pochhammer symbols

- Series solution

$$\phi_\gamma := \sum_{u \in \mathcal{L}} \frac{[\gamma]_{u_-}}{[\gamma + u]_{u_+}} c^{(\gamma+u)}$$

Canonical series algorithm (Saito-Sturmfels-Takayama)

Frobenious method

$$\frac{d^2y}{dx^2} + \omega^2 y = 0$$

1. Try series solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+k}, \ \ a_0 \neq 0$$

2. Indicial equation

$$k(k-1) = 0$$

Canonical Series

$$H_A(\kappa) \bullet F(c) = 0$$

1. Formal solution

$$\phi_\gamma := \sum_{u \in \mathcal{L}} \frac{[\gamma]_{u_-}}{[\gamma + u]_{u_+}} c^{(\gamma+u)}$$

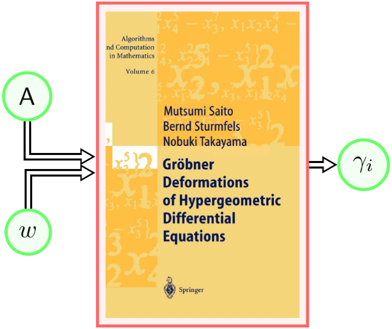
2. Fake indicial ideal

$$\text{find}_w(H_A(\kappa))$$

Convergence

Weight vector w chooses a region of convergence: representative of the Gröbner fan of I_A

Canonical series algorithm (Saito-Sturmfels-Takayama)



2F1 two integral

```
In[54]:= A = {{1, 1, 1, 1}, {0, 1, 0, 1}, {0, 0, 1, 1}};
         kerfromM2[A]
         finwIA[A, {0, 1, 1, 1}, {-β, -α[1], -α[2]}]
         fakeExp[%, Table[θ[i], {i, 1, 4}]]
         co1res[A, {0, 1, 1, 1}, {-β, -α[1], -α[2]}]
Out[55]= {{1}, {-1}, {-1}, {1}}
Out[56]= {θ[2] × θ[3], β + θ[1] + θ[2] + θ[3] + θ[4], α[1] + θ[2] + θ[4], α[2] +
Out[57]= {{-β + α[1], -α[1] + α[2], 0, -α[2]}, {-β + α[2], 0, α[1] - α[2],
Out[58]= {{-n, n, n, -n}, {-n}, {n}, {θ[2] × θ[3], β + θ[1] + θ[2] + θ[3] + θ
          {{-β + α[1], -α[1] + α[2], 0, -α[2]}, {-β + α[2], 0, α[1] - α[2
```

$$\left\{\begin{pmatrix} \frac{(\beta-\alpha(1))_n}{(-\alpha(1)+\alpha(2)+1)_n} & \frac{(\alpha(2))_n}{(1)_n} \end{pmatrix}, \begin{pmatrix} \frac{(\beta-\alpha(2))_n}{(1)_n} & \frac{(\alpha(1))_n}{(\alpha(1)-\alpha(2)+1)_n} \end{pmatrix} \right.$$

Interface: Macaulay2+Mathematica

Lee-Pomeransky representation of Feynman integrals

$$I_F(\alpha) = \xi_{\Gamma_\alpha} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} \frac{\mathrm{d}z_i}{z_i} z_i^{\alpha_i} \right) \frac{1}{(\mathcal{U}+\mathcal{F})^{d/2}} = \xi_{\Gamma_\alpha} \int_{\mathbb{R}_+^N} \frac{z^\alpha}{(\mathcal{U}+\mathcal{F})^{d/2}} \mathrm{d}\eta_N$$

- Factor independent of the kinematics

$$\xi_{\Gamma_\alpha} := \frac{\Gamma(d/2)}{\Gamma((L+1)d/2 - \sum_{i=1}^{N} \alpha_i) \prod_{i=1}^{N} \Gamma(\alpha_i)}$$

Symanzik polynomials

- Dimensionless (scaling in $\mathcal{F}$ assumed)
- Euclidean kinematics: invariants $-(p_i + p_j)^2 > 0$
- $\mathcal{U}$ homogeneous polynomial of degree L
- $\mathcal{F}$ homogeneous polynomial of degree $L+1$
- $\mathcal{U}$, $\mathcal{F}$ positive functions of their parameters
- kinematic dependence is in $\mathcal{F}$
- $\mathcal{U}$ and $\mathcal{F}$ can only vanish on the boundaries of the integration region

Proposal based on canonical series

Idea

Consider Feynman integrals as special points of A-hypergeometric functions. A Feynman integral is A-hypergeometric whenever we can compute its canonical series ϕ_i.

$$I_F(\kappa) = K_1\phi_1 + \cdots + K_r\phi_r$$

Feynman integrals

- Consider the coefficients in $g(z) = \mathcal{U} + \mathcal{F}$ as indeterminate

$$g(z) \to g(z,c) = \mathcal{U}(c) + \mathcal{F}(c) \Leftrightarrow \mathsf{A}$$

- In general we add a deformation to $g(z,c)$ to ensure canonical series when co(A) $= 0$, $g(z,c) \to g_r(z,c) := r(z) + g(z,c)$
- Feynman integrals are obtained from the restriction of canonical series to kinematics values.
- Designed for algorithmic evaluation of Feynman integrals

Proposal based on canonical series

Theorem

Let

$$g_r(c,z) = \sum_{i=1}^{n} c_i z^{a_i} \Longleftrightarrow \mathsf{A} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \end{pmatrix}.$$

The Euler-Mellin integral

$$I_{g_r}(\kappa) = \int_{\Omega} \frac{z^{\alpha}}{g_r(c,z)^{d/2}} \mathrm{d}\eta_N$$

is a solution of the A-hypergeometric system $H_{\mathsf{A}}(\kappa)$ of degree $\kappa = (-d/2, -\alpha)$. Noncompact cycles Ω considering the coamoeba of of $\mathcal{V}(g_r(c,z))$ and choosing representatives $\theta \in \Theta$ in $\mathbb{R}^N \backslash \overline{\mathcal{A}'}_{g_r}$.

Proof.

Show that $I_{g_r}(\kappa)$ satisfies the GKZ system. Validity for non-compact cycles demonstrated by as we discussed before BFP, '13

Relation to other proposals and methods

- Maximal cuts and n-loop bananas, fixed dimensions, compact cycles:
  Vanhove, '18 Klemm-Nega-Safari, '19 Bönish-Fischbach-Klemm-Nega-Safari, '20
- Full massive sunset with emphasis on triangulations of polytopes: Klausen, '19
- Feynman Integrals satysfying GKZ differential equations also in
  Nasrollahpoursamami, '16

Remark

- Above approaches emphasize triangulations of Convex Polytopes $\rightarrow$ Gamma series representations
- Canonical series through Gröbner bases for some w and triangulations of polytopes are intimately connected
  Sturmfels, Gröbner bases and convex polytopes, '95

More recent developments

- Cohen-Macaulay property Tellander-Helmer'21

$$\mathrm{in}_{(w,-w)}(H_{\mathsf{A}}(\kappa)) = \langle A\theta - \kappa^T \rangle + \mathrm{in}_w(I_{\mathsf{A}})$$

- Choice of w can simplify sum representation
- Kinematic singularities of Feynman integrals through A-determinants Klausen '21, Mizera-Telen'21, Fevola-Mizera-Tellen, '23

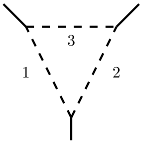
$$g(z) = \mathcal{U} + \mathcal{F}$$

- Banana Feynman integrals from series representation (Frobenius method) Bönisch-Duhr-Fischbach-Klemm-Nega'21
- Analytic continuation tool Ananthanarayan-Bera-Friot-Pathak'21

FeynGKZ

Ananthanarayan-Banik, Souvik Bera-Datta, '22 A Mathematica package for solving Feynman integrals using GKZ hypergeometric systems

Example: Triangle



$$s_1 = -p_1^2,\ s_2 = -p_2^2,\ s_3 = -(p_1 + p_2)^2$$

- Polynomial

$$g(z) = z_1 + z_2 + z_3 + s_3 z_1 z_2 + s_1 z_1 z_3 + s_2 z_2 z_3 \iff \mathsf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

- $\mathrm{co}(\mathsf{A}) = 2$. Two variable hypergeometric function

Example: Triangle

$$I_g(\kappa) = \int_\Omega \mathrm{d}\eta_3 \frac{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}}{(c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_1 z_2 + c_5 z_1 z_3 + c_6 z_2 z_3)^\beta}$$

Macaulay 2 + Mathematica

- Input: $w = (0,0,1,0,0,0)$, A
- $\mathrm{fin}_w(H_{\mathsf{A}}(\kappa)) = \langle \theta_2\theta_5, \theta_3\theta_4 \rangle + \langle \mathsf{A}\theta - \kappa^T \rangle$
- Output:

$$\{\gamma_i\} = \{(-\alpha_1, C-\beta, B-\beta, 0, 0, \beta - A), (\alpha_2 - \beta, C-\beta, 0, \beta - B, 0, -\alpha_3), (\alpha_3 - \beta, 0, B-\beta, 0, \beta - C, -\alpha_2), (A - 2\beta, 0, 0, \beta - B, \beta - C, \alpha_1 - \beta)\}$$

$A = \alpha_1 + \alpha_2 + \alpha_3$, $B = \alpha_1 + \alpha_2$, and $C = \alpha_1 + \alpha_3$.

Example: Triangle

Mathematica

$$\phi_1 = c^{\gamma_1} \sum_{m\geq 0, n\in\mathbb{Z}} \frac{(\alpha_1)_{m+n}\,(A-\beta)_{m+n}}{(-\beta+C+1)_n(-\beta+B+1)_m(1)_m(1)_n} x^m y^n,$$

$$\phi_2 = c^{\gamma_2} \sum_{m\geq 0, n\in\mathbb{Z}} \frac{(\beta-\alpha_2)_{m+n}\,(\alpha_3)_{m+n}}{(-\beta+C+1)_n(1)_m(\beta-B+1)_m(1)_n} x^m y^n,$$

$$\phi_3 = c^{\gamma_3} \sum_{m\geq 0, n\in\mathbb{Z}} \frac{(\beta-\alpha_3)_{m+n}\,(\alpha_2)_{m+n}}{(1)_n(-\beta+B+1)_m(1)_m(\beta-C+1)_n} x^m y^n,$$

$$\phi_4 = c^{\gamma_4} \sum_{m\geq 0, n\in\mathbb{Z}} \frac{(2\beta-A)_{m+n}\,(\beta-\alpha_1)_{m+n}}{(1)_n(1)_m(\beta-B+1)_m(\beta-C+1)_n} x^m y^n,$$

$x = (c_3 c_4)/(c_1 c_6)$ and $y = (c_2 c_5)/(c_1 c_6)$.

Example: Triangle

Integration constants from positions of zero's in roots

$$K_r = \frac{1}{\Gamma(\beta)} \prod_{i\neq 0} \Gamma(-\gamma_r^i)$$

Restriction to physical values

$c_1 = c_2 = c_3 = 1$ and $c_4 = s_3,\ c_5 = s_1,\ c_6 = s_2$

$I(\alpha, \beta) =$

$$K_1 s_2^{\beta-A} F_4(\alpha_1, A-\beta; -\beta+\alpha_{13}+1, -\beta+\alpha_{12}+1; s_3/s_2, s_1/s_2)$$
$$+ K_2 s_2^{-\alpha_3} s_3^{\beta-B} F_4(\beta-\alpha_2, \alpha_3; C-\beta+1, -B+\beta+1; s_3/s_2, s_1/s_2)$$
$$+ K_3 s_2^{-\alpha_2} s_1^{\beta-C} F_4(\beta-\alpha_3, \alpha_2; B-\beta+1, -C+\beta+1; s_3/s_2, s_1/s_2)$$
$$+ K_4 s_1^{\beta-C} s_2^{\alpha_1-\beta} s_3^{\beta-B} F_4(2\beta-A, \beta-\alpha_1; -B+\beta+1, -C+\beta+1; s_3/s_2, s_1/s_2)$$

- Mellin-Barnes [Boos, Davydychev, '91] and negative dimension approach [Anastasiou-Glover-Oleari, '00]

Holonomic properties of scattering amplitudes

- GKZ ideal $H_{\mathsf{A}}(\kappa)$ is a holonomic D-ideal

$$H_{\mathsf{A}}(\kappa) = I_{\mathsf{A}} \cup \langle \mathsf{A}\theta - \kappa \rangle$$

- Key property of Feynman integrals: holonomicity
  Kashiwara-Kawai, '77, Bitoun-Bogner-Klausen-Panzer '17

- Basic equation of generalized unitarity [Bern-Dixon-Dunbar-Kosower, '94]

$$\mathcal{A}_n = \sum_i c_i(A^{\text{trees}}) I_i^{\text{basis}} + \text{rational}$$

- Coefficients c_i can be computed from tree-level amplitudes (rational functions of spinor variables)
- More generally c_i are algebraic functions
- Algebraic functions are also holonomic hence amplitudes! Elementary consequence of holonomic D-modules (See Chapter 20 of Coutinho's book)
- Let us start with trees ...

Biadjoint scalars

- Biadjoint scalar amplitudes

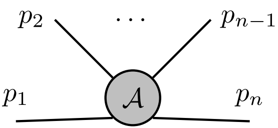
$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi_{a\alpha}\partial^\mu \varphi_{a\alpha} - \frac{\lambda}{3!} f^{abc}\tilde{f}^{\alpha\beta\gamma}\varphi_{a\alpha}\varphi_{b\beta}\varphi_{c\gamma}$$

- Admit a recursive formula Mafra, '16

$$\phi_{w_1,w_2} = \frac{1}{s_{w_1}} \sum_{xy=w_1} \sum_{ab=w_2} \left[\phi_{x,a}\phi_{y,b} - (x \leftrightarrow y)\right], \quad \phi_{w_1,w_2} \equiv 0, \quad \text{if} \quad w_1 \setminus w_2 \neq e$$

with the start of the recursion defined as $\phi_{i,j} = \delta_{ij}$. The n-point amplitude is

$$m_n(w_1 n|w_2 n) = (-1)^{(n-3)} s_{w_1}\varphi_{w_1,w_2}$$



Weyl algebra

- Take $w_1 n = 12\dots n$
- Ring of Mandelstam invariants

$$s_{ijk\dots} := (p_i + p_j + p_k + \dots)^2$$

- Examples

$$m_4 = -\frac{1}{s_{12}} - \frac{1}{s_{23}}, \quad m_5 = \frac{1}{s_{12}s_{123}} + \frac{1}{s_{12}s_{34}} + \frac{1}{s_{123}s_{23}} + \frac{1}{s_{23}s_{234}} + \frac{1}{s_{234}s_{34}}$$

- kinematic invariants $S_n = \{ s_w \mid w \in B_n \}$, where $|S_n| = \frac{1}{2}n(n-3) = N$, so its associated ring is $\mathbb{C}[S_n]$.
- We then define the corresponding set of operators by $\partial_{S_n} := \{ \partial_{s_w} \mid w \in B_n \}$ so the associated Weyl algebra is

$$D_N = \mathbb{C}[S_n] \langle \partial_{S_n} \rangle$$

Differential equations for biajoint scalars

- Annihihilators of amplitudes ($m_n = f/g$)

$$P_i = gf\partial_i + (f\partial_i g - g\partial_i f), \quad i = 1, \dots, N,$$
$$H_n = \left[\sum_{w\in B_n} \theta_{s_w} + (n-3)\right]$$

$$\langle P_1, \dots, P_N, H_n\rangle \subset D_N \, .$$

- Canonical holonomic representation Zeilberger, '90

$$I_n = \langle \mathsf{A}_n\theta_n - \kappa_n\rangle \Rightarrow s_w\theta_{s_w} m_n(S) = \kappa_w, \quad \forall w \in B_n$$

$$\mathsf{A}_n = m_n \mathrm{diag}(s_{12}, s_{23}, \dots), \quad \theta_n = (\theta_{s_{12}}, \theta_{s_{23}}, \dots)^T, \quad \kappa_n = (\iota_n(12), \iota_n(23), \dots)^T,$$

for $2 \leq |w| \leq n-2$ and zero otherwise

- Boundary condition $m_n|_{S_n\to\infty} = 0$.

Summary and Outlook

Summary

- GKZ systems are the most general tools to study hypergeometric functions
- Generalized Feynman integrals are A-hypergeometric,
- SST canonical series provide the tool to evaluate Feynman integrals
- Output of canonical series method equivalent to Mellin-Barnes
- Holonomicity is key to extend the approach to scattering amplitudes

Outlook

- Elephant in the room: efficiency and scaling (restriction of D-modules[Henn-Pratt-Sattelberger-Zoia, '23])
- Relation between GKZ and PDEs from Griffiths-Dwork[LDLC-Vanhove, '24]
- Generalized unitarity gives us hint to extend this approach to general scattering amplitudes

Thanks!

Canonical series algorithm (Saito-Sturmfels-Takayama)

Input: Matrix A, weight vector w, and complex parameters κ. Output: Roots of the fake indicial ideal $\mathrm{fin}_w(H_{\mathsf{A}}(\kappa))$.

1. Compute the toric ideal associated with A

$$I_{\mathsf{A}} = \langle \partial^u - \partial^v : \mathsf{A}u = \mathsf{A}v, \qquad u, v \in \mathbb{N}^n \rangle .$$

2. Let $w \in \mathbb{R}^n$ be a generic weight vector. Compute the initial ideal $\mathrm{in}_w(I_{\mathsf{A}})$ with respect to w and obtain its standard pairs $\mathcal{S}(\mathrm{in}_w(I_A))$.
3. Use the standard pairs to construct the indicial ideal

$$\mathrm{ind}_w(I_{\mathsf{A}}) = \bigcap_{(\partial^a, F) \in \mathcal{S}(\mathrm{in}_w(I_{\mathsf{A}}))} \langle (\theta_j - a_j), j \notin F \rangle \subset \mathbb{C}[\theta_1, \theta_2, \ldots, \theta_n],$$

4. Write the ideal $\langle \mathsf{A}\theta - \kappa^T \rangle \subset \mathbb{C}[\theta_1, \theta_2, \ldots, \theta_n]$.
5. The fake indicial ideal with respect to w is given by

$$\mathrm{fin}_w(H_{\mathsf{A}}(\kappa)) := \mathrm{ind}_w(I_{\mathsf{A}}) + \langle A\theta - \kappa^T \rangle .$$

6. Compute the roots of $\mathrm{fin}_w(H_{\mathsf{A}}(\kappa))$. These are called fake exponents and we denote them by γ.

Standard Pairs

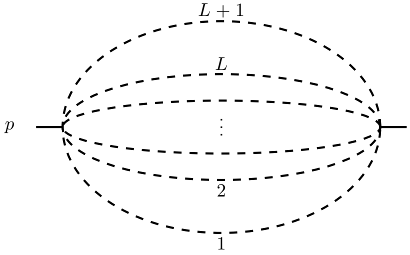
Let $R = \mathbb{K}[\partial_1, \ldots, \partial_n]$ and let I be a monomial ideal in R. Furthermore, let ∂^α be a monomial and $F \subseteq \{1, \ldots, n\}$, where $\alpha \in \mathbb{N}^n$. A standard pair of a monomial ideal I is a pair (∂^α, F) satisfying three conditions:

1. $\alpha_i = 0$ for all $i \in F$,
2. for all choices of integers $\beta_j \geq 0$, the monomial $\partial^\alpha \prod_{j \in F} \partial_j^{\beta_j} \notin I$,
3. for all $l \notin F$, there exist $\beta_j \geq 0$ such that $\partial^\alpha \partial_l^{\beta_l} \prod_{j \in F} \partial_j^{\beta_j} \in I$.

Let us denoted by $\mathcal{S}(I)$ the set of all standard pairs of I. The decomposition of I into irreducible monomial ideals can be obtained from the identity.

$$I = \bigcap_{(\partial^\alpha, F) \in \mathcal{S}(I)} \left\langle \partial_i^{\alpha_i + 1} : i \in F \right\rangle .$$

Example: Cantaloupe or dealing with deformation



$$g(z_1, \ldots, z_{L+1}) = \sum_{i=1}^{L+1} \prod_{j \neq i}^{L} z_j + s \prod_{i=1}^{L+1} z_i,$$

where $s = -p^2$. The integral to be computed reads

$$I(\alpha) = \int_{\mathbb{R}_+^{L+1}} \mathrm{d}\eta_{L+1} \frac{z_1^{\alpha_1} \cdots z_{L+1}^{\alpha_{L+1}}}{g(z)^{\beta}}.$$

In order to perform such deformation systematically, let us introduce some notation. Let 1_i denote a sequence of 1's of length i and similarly for 0_j. We have the relation $i + j = L + 1$. Furthermore, let

$$v := (1_{L-1}, 0_2).$$

At each loop, we set a deformation monomial

$$r(z) = c_1 z^v,$$

hence we have

$$g_r(c, z) = c_1 z^v + \sum_{i=1}^{L+1} c_{L+3-i} \prod_{j \neq i}^{L} z_j + c_{L+3} \prod_{i=1}^{L+1} z_i,$$

where $c_{L+3} = s$. Let us give an example. For $L = 3$, $v = (1, 1, 0, 0)$ and $r(z) = c_1 z_1 z_2$, then we have the deformed toric polynomial

$$g_r(c, z) = c_1 z_1 z_2 + c_2 z_1 z_2 z_3 + c_3 z_1 z_2 z_4 + c_4 z_1 z_3 z_4 + c_5 z_2 z_3 z_4 + c_6 z_1 z_2 z_3 z_4.$$

$$
\mathsf{A} = \begin{pmatrix}
1 & 1 & \dots & 1 & 1 \\
 & \mathsf{1}_{L+1} & & 0 & \mathsf{1}_1 \\
 & \mathsf{1}_L & & 0 & \mathsf{1}_2 \\
 & & \vdots & & \\
 & \mathsf{1}_3 & 0 & & \mathsf{1}_{L-1} \\
0 & \mathsf{1} & 0 & & \mathsf{1}_L \\
0 & 0 & \mathsf{1} & & \mathsf{1}_L
\end{pmatrix}.
$$

$$
I_{g_r}(\kappa) = \int_\Omega \mathrm{d}\eta_{L+1} \frac{z_1^{\alpha_1} \cdots z_{L+1}^{\alpha_{L+1}}}{g_r(c,z)^\beta},
$$

where $\kappa = (-\beta, -\alpha_1, \dots, -\alpha_{L+1})$. Computing the kernel of the above matrix leads to

$$
\mathcal{L} = \mathbb{Z}(1, -1, -1, 0_{L-1}, 1),
$$

where by definition $0_0 := \emptyset$. We choose $w = (1, \mathsf{0}_{L+2})$, thus obtaining

$$
\mathsf{fin}_w(H_{\mathsf{A}}(\kappa)) = \langle \theta_1 \theta_{L+3} \rangle + \langle \mathsf{A}\theta - \kappa^T \rangle .
$$

The roots can be written as

$$\{\gamma_i\} = \{\left(0, \alpha_{L+1} - \beta,\ \dots,\ \alpha_1 - \beta, L\beta - \sum_{i=1}^{L+1} \alpha_i\right),$$

$$\Big(\sum_{i=1}^{L} \alpha_i - L\beta,\ (L-1)\beta - \sum_{i=1}^{L} \alpha_i,\ (L-1)\beta - \sum_{i\neq L}^{L+1} \alpha_i,\ -\beta + \alpha_{L-1},\ \dots,$$

$$-\beta + \alpha_1,\ 0\Big)\},$$

which lead to the canonical series

$$\phi_1 = c^{\gamma_1}\ {}_2F_1\left(\beta - \alpha_{L+1},\ \beta - \alpha_L,\ L\beta - \sum_{i=1}^{L+1} \alpha_i + 1; x\right),$$

$$\phi_2 = c^{\gamma_2}\ {}_2F_1\left(-(L-1)\beta + \sum_{i=1}^{L} \alpha_i,\ -(L-1)\beta + \sum_{i\neq L}^{L+1} \alpha_i; \sum_{i=1}^{L} \alpha_i - L\beta + 1; x\right),$$

where $x = \frac{c_1 c_{L+3}}{c_2 c_3}$. The relevant integration constant reads

$$K_1 = \frac{\Gamma(-L\beta + \sum_{i=1}^{L+1} \alpha_i)}{\Gamma(\beta)} \prod_{i=1}^{L+1} \Gamma(\beta - \alpha_i),$$

Theorem

Let $g_r(c, z)$ be the deformed polynomial in N variables obtained from $g(c, z) = \mathcal{U}(c) + \mathcal{F}(c)$, where $\mathcal{F}(c)$ and $\mathcal{U}(c)$ are obtained by considering the coefficients appearing in the Symanzik polynomials as variables. $g_r(c, z)$ is obtained by introducing a deformation $r(c, z)$ demanding that its matrix satisfies co(A) > 0. Let $A = (a_1 \ a_2 \cdots \ a_n)$ be the configuration matrix associated with $g_r(c, z)$ and consider the polynomial with indeterminate generic coefficients

$$g_r(c, z) = \sum_{i=1}^{n} c_i z^{a_i}, \qquad c_i \in \mathbb{C}_*$$

Let A be its associated $(N + 1) \times n$ matrix

$$\mathsf{A} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$$

Theorem

The Euler-Mellin integral

$$I_{g_r}(\kappa) = \int_\Omega \frac{z^\alpha}{g_r(c,z)^{d/2}} \mathrm{d}\eta_N$$

is a solution of the A-hypergeometric system $H_{\mathsf{A}}(\kappa)$ of degree $\kappa = (-d/2, -\alpha)$. Noncompact cycles Ω can be obtained by taking the coamoeba of $g_r(c,z)$ and choosing representatives θ of connected components $\Theta \in \mathbb{R}^N \backslash \overline{\mathcal{A}'}_{g_r}$. Proof. Show that the above integral satisfies GKZ system. Validity for non-compact cycles demonstrated by Berkesh-Forsgård-Passare.

Remark on cycles

Noncompact cycles for A-hypergeometric functions from coamoebas of $\mathcal{A}'_g$ simply gives $\Omega = \mathbb{R}^N_+$ thanks to positivity of coefficients in $g(z)$.