Motives and period integrals

in QFT and string theory

Modular graph forms (MGFs) from equivariant iterated Eisenstein integrals

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based on 2209.06772 with D. Dorigoni, M. Doroudiani, J. Drewitt, M. Hidding, A. Kleinschmidt, N. Matthes, B. Verbeek and F. Brown 1407.5167, 1707.01230, 1708.03354

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Outline

I. String theory context of MGFs

II. MGFs from iterated integrals: first look

III. Genus-zero inspiration

IV. MGFs from equivariant iterated Eisenstein integrals

V. Further directions

I. String theory context of MGFs

I. 1 Four closed strings on a torus

Four-point closed-string amplitude at one loop (gravitons in type IIA/B)

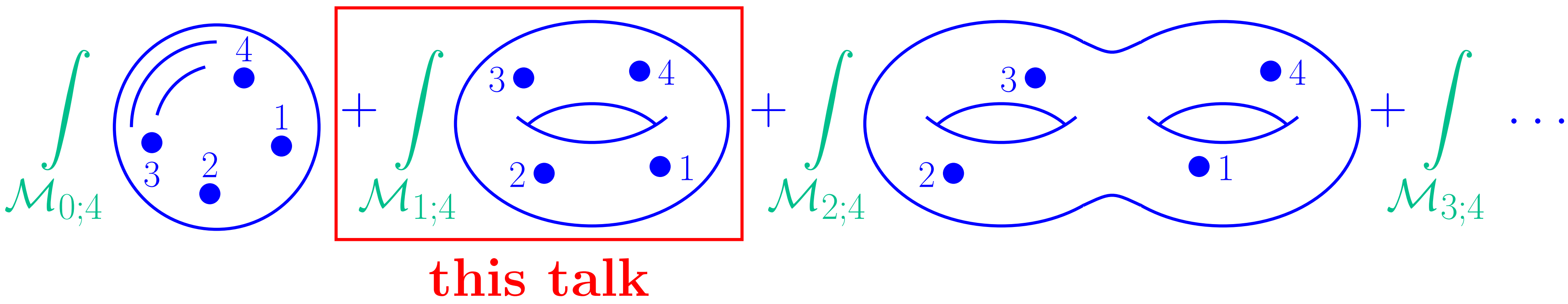
$$\mathcal{M}^{\text{1-loop}}_{\text{IIA/B}}(1,2,3,4) = \left| s_{12} s_{23} A^{\text{tree}}_{\text{YM}}(1,2,3,4) \right|^2 \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^2} \, J(s_{ij},\tau)$$

$$J(s_{ij},\tau) = \left(\prod_{j=1}^{4} \int_{T^2} \frac{\mathrm{d}^2 z_j}{\mathrm{Im}\,\tau} \right) \exp\left(\sum_{i<j}^{4} s_{ij}\, \mathcal{G}(z_i - z_j, \tau) \right)$$

modular invariant

[Brink, Green, Schwarz 1982]

"1-loop correction" in the topological expansion of closed-string amplitudes



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- fund. domain $\mathcal{F}$ of modular group $\mathrm{SL}_2(\mathbb{Z})$ and torus $T^2 = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$
- torus Green function $\mathcal{G}(z,\tau)$ in 2× Fourier expansion w.r.t. $z = u\tau + v$

$$\mathcal{G}(z,\tau) = \frac{\operatorname{Im}\tau}{\pi} \sum_{p\in\Lambda'} \frac{e^{2\pi i\langle p,z\rangle}}{|p|^2}\,, \qquad \langle p,z\rangle = nu - mv$$

lattice $\Lambda' = (\mathbb{Z}+\tau\mathbb{Z})\setminus\{0\}$ for torus momentum $p = m\tau + n$ @ $m,n\in\mathbb{Z}$

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- torus Green function $\mathcal{G}(z,\tau)$ in 2× Fourier expansion w.r.t. $z = u\tau + v$
- coeff's in α'-expansion of $J(s_{ij},\tau)$ dubbed modular graph forms (MGFs)

[Green, Vanhove 9910056; Green, Russo, Vanhove 0801.0322]

[D'Hoker, Gürdogan, Green, Vanhove 1512.06779]

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- fund. domain $\mathcal{F}$ of modular group $\mathrm{SL}_2(\mathbb{Z})$ and torus $T^2 = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$
- note: α'-expanding $J(s_{ij},\tau)$ at fixed τ misses discontinuities in $\alpha' s_{ij}$,

  $\exists$ separate expansion method for "non-analytic part" involving $\log(\alpha' s_{ij})$

[e.g. Green, Russo, Vanhove 0801.0322; D'Hoker, Green 1906.01652]

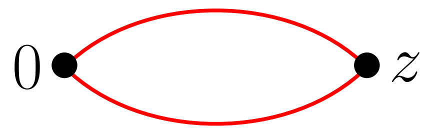
[Edison, Guillen, Johansson, OS, Teng 2107.08009; Eberhardt, Mizera 2208.12233]

I. 2 Simplest examples of MGFs and low-energy interactions

MGFs $\ni$ integrate polynomials in $\underbrace{\mathcal{G}(z_{ij}{=}z_i{-}z_j,\tau)}_{\text{edge } z_i\to z_j}$ over $\underbrace{z_1,z_2,\ldots\in T^2}_{\text{vertices}}$

- by the absence of zero-modes in $p\in\Lambda'=(\mathbb{Z}{+}\tau\mathbb{Z})\setminus\{0\}$, 1-particle reducible graphs vanish $\int \mathrm{d}^2z\,\mathcal{G}(z,\tau)=0$, so simplest nonzero MGF is

$$\int\frac{\mathrm{d}^2z}{\operatorname{Im}\tau}\mathcal{G}(z,\tau)^2=\left(\frac{\operatorname{Im}\tau}{\pi}\right)^2\sum_{p\in\Lambda'}\frac{1}{|p|^4}$$



recall : $\mathcal{G}(z,\tau)=\dfrac{\operatorname{Im}\tau}{\pi}\displaystyle\sum_{p\in\Lambda'}\frac{e^{2\pi i\langle p,z\rangle}}{|p|^2}$ without zero mode

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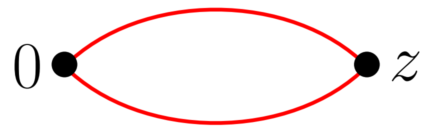
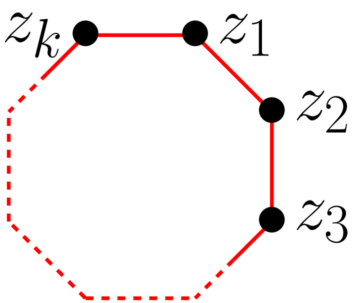


- more generally, 1-loop graphs on $T^2\Rightarrow$ non-holo. Eisenstein series E_k



$$\int\left(\prod_{j=1}^k\frac{\mathrm{d}^2z_j}{\mathrm{Im}\,\tau}\right)\mathcal{G}(z_{12},\tau)\mathcal{G}(z_{23},\tau)\ldots\mathcal{G}(z_{k1},\tau)$$
$$=\left(\frac{\mathrm{Im}\,\tau}{\pi}\right)^k\sum_{p\in\Lambda'}\frac{1}{|p|^{2k}}=\mathrm{E}_k(\tau)$$

I. 2 Simplest examples of MGFs and low-energy interactions

Up to and including 3rd subleading order in α',

$$J(s_{ij},\tau) \ = \ \Bigg(\prod_{j=1}^{4} \int_{T^2} \frac{\mathrm{d}^2 z_j}{\mathrm{Im}\,\tau} \Bigg) \ \exp\Bigg(\sum_{i<j}^{4} s_{ij}\, \mathcal{G}(z_i{-}z_j,\tau) \Bigg)$$

$$= \ 1 + (s_{12}^2{+}s_{13}^2{+}s_{23}^2)\, \mathrm{E}_2(\tau) + s_{12}s_{13}s_{23}\big(5\,\mathrm{E}_3(\tau) + \zeta_3\big) + \mathcal{O}(\alpha'^4)$$

all $\mathrm{E}_k(\tau)$ integrate to zero by Laplace equation $\Delta_\tau \mathrm{E}_k(\tau) = k(k{-}1)\mathrm{E}_k(\tau)$

$$\mathcal{M}^{\text{1-loop}}_{\text{IIA/B}}(1,2,3,4) \ \to \ \frac{\pi}{3} \left| s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4)\right|^2 \big(1 + s_{12}s_{13}s_{23}\zeta_3 + \mathcal{O}(\alpha'^4)\big)$$

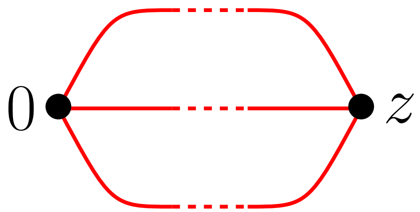
- determines 1-loop contributions to low-energy interactions $\mathcal{R}^4$ and $D^6\mathcal{R}^4$

- type IIB superstrings: key input for exact, S-duality invariant couplings

[Green, Gutperle hep-th/9701093, Green, Vanhove hep-th/0510027]

I. 3 Simplest relations among MGFs

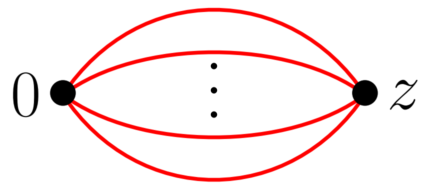
Beyond 1-loop graphs on the torus, get nested lattice sums, e.g.



$$\leftrightarrow \quad C_{a,b,c}(\tau) = \left(\frac{\operatorname{Im}\tau}{\pi}\right)^{a+b+c} \sum_{p_1,p_2,p_3\in\Lambda'} \frac{\delta(p_1+p_2+p_3)}{|p_1|^{2a}|p_2|^{2b}|p_3|^{2c}}$$

Higher-loop graphs often simplify to lower loop and MZVs, e.g.

$$\int \frac{\mathrm{d}^2 z}{\operatorname{Im}\tau}\mathcal{G}(z,\tau)^3 = C_{1,1,1}(\tau) = \mathrm{E}_3(\tau) + \zeta_3$$



$$\int \frac{\mathrm{d}^2 z}{\operatorname{Im}\tau}\mathcal{G}(z,\tau)^4 = 24C_{2,1,1}(\tau) - 18\mathrm{E}_4(\tau) + 3\mathrm{E}_2(\tau)^2$$

$$\int \frac{\mathrm{d}^2 z}{\operatorname{Im}\tau}\mathcal{G}(z,\tau)^5 = 60C_{3,1,1}(\tau) + 10\mathrm{E}_2(\tau)C_{1,1,1}(\tau) - 48\mathrm{E}_5(\tau) + 16\zeta_5$$

[Zagier '08; D'Hoker, Green, Vanhove '15; D'Hoker, Green '16; D'Hoker, Kaidi '16]

Problem: How to anticipate such relations?

What is the set of independent MGFs (over $\mathbb{Q}[\mathrm{MZV}]$)?

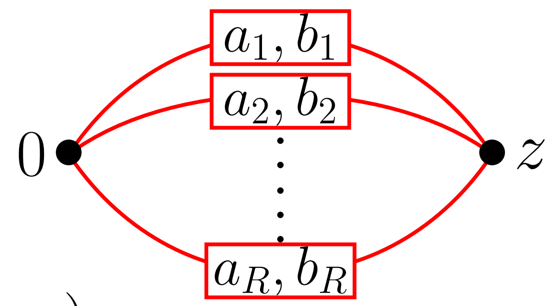
I. 4 MGFs with non-zero modular weight

Generalize $\int_{T^2}$ of Green fcts $\mathcal{G}(z,\tau)$ to MGFs with different exponents

$a_i, b_i \in \mathbb{Z}$ of (anti-)holomorphic momenta $p_i = m_i\tau + n$ & $\bar{p}_i = m_i\bar{\tau} + n_i$

$$\mathcal{C}^+\begin{bmatrix} a_1 & a_2 & \ldots & a_R \\ b_1 & b_2 & \ldots & b_R \end{bmatrix} = \frac{(\operatorname{Im}\tau)^{a_1+\ldots+a_R}}{\pi^{b_1+\ldots+b_R}} \sum_{p_1,p_2,\ldots,p_R\in\Lambda'} \frac{\delta(p_1+p_2+\ldots+p_R)}{p_1^{a_1}\bar{p}_1^{b_1}\ldots p_R^{a_R}\bar{p}_R^{b_R}}$$

- corresponds to dihedral graph with $R{-}1$ loops
- closed under Maaß operators $\nabla_\tau = 2i(\operatorname{Im}\tau)^2\partial_\tau$
- above normalization $\Rightarrow$ modular weight $(0, \sum_{i=1}^R b_i - a_i)$
- naturally appear in type IIA/B amplitudes beyond four points



[Richards 0807.2421; Green, Mafra, OS 1307.3534]

... as well as $(n \geq 4)$-point amplitudes of heterotic strings

[Gerken, Kleinschmidt, OS 1811.02548]

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After repeated action of Maaß operators $\nabla_\tau = 2i(\mathrm{Im}\,\tau)^2\partial_\tau$, can factor out

holomorphic Eisenstein series (using "holomorphic subgraph reduction")

$$\mathrm{G}_k(\tau) = \sum_{p\in\Lambda'} \frac{1}{p^k}\,, \qquad k \geq 4 \text{ even}$$

[D'Hoker, Green 1603.00839; Gerken, Kaidi 1809.05122, Gerken 2007.05476]

for instance $\quad (\pi\nabla_\tau)^k \mathrm{E}_k = \dfrac{(2k{-}1)!}{(k{-}1)!}(\mathrm{Im}\,\tau)^{2k}\,\mathrm{G}_{2k}$ $\quad$ and

$$(\pi\nabla_\tau)^3\mathcal{C}^+\left[\begin{smallmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \end{smallmatrix}\right] = \tfrac{9}{10}(\pi\nabla_\tau^3)\mathrm{E}_4 - 6\,(\mathrm{Im}\,\tau)^4\,\mathrm{G}_4\,(\pi\nabla_\tau)\mathrm{E}_2$$

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$\Longrightarrow$ **MGFs expressible via iterated Eisenstein integrals!**

II. MGFs from iterated integrals

II. 1 Towards equivariant iterated Eisenstein integrals

Recall holo Eisenstein series $\mathrm{G}_k(\tau) = \sum_{p\in\Lambda'} p^{-k}$ of mod. weight $(k,0)$

$\longrightarrow$ consider primitives of $\mathrm{G}_k(\tau)$ or $\tau^j \mathrm{G}_k(\tau)$ w.r.t. τ

$$\int_\tau^{i\infty} \mathrm{d}\tau_1\, (\tau_1)^j \mathrm{G}_k(\tau_1)$$

- endpoint divergence $\int_\tau^{i\infty}$ of zero mode $\mathrm{G}_k = 2\zeta_k + \mathcal{O}(q)$

  regularized via "tangential base point" (where $q = e^{2\pi i\tau}$)

$$\Longrightarrow \quad \int_\tau^{i\infty} \mathrm{d}\tau_1\, \mathrm{G}_k(\tau) = -2\zeta_k\tau + \mathcal{O}(q)$$

[Brown 1407.5167]

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Recall holo Eisenstein series $\mathrm{G}_k(\tau) = \sum_{p\in\Lambda'} p^{-k}$ of mod. weight $(k,0)$

$\longrightarrow$ primitives of $\mathrm{G}_k(\tau)$ or $\tau^j \mathrm{G}_k(\tau)$ w.r.t. τ are not modular forms,

$$\int_\tau^{i\infty} \mathrm{d}\tau_1 \, (\tau_1)^j \mathrm{G}_k(\tau_1) \overset{\tau \to -\frac{1}{\tau}}{\longrightarrow} (-1)^j \left(\int_\tau^{i\infty} - \int_0^{i\infty} \right) \mathrm{d}\tau_1 \, (\tau_1)^{k-j-2} \mathrm{G}_k(\tau_1)$$

- endpoint divergence $\int_\tau^{i\infty}$ of zero mode $\mathrm{G}_k = 2\zeta_k + \mathcal{O}(q)$

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$$\Longrightarrow \quad \int_\tau^{i\infty} \mathrm{d}\tau_1 \, \mathrm{G}_k(\tau) = -2\zeta_k \tau + \mathcal{O}(q)$$

- additive constants from $\int_0^{i\infty}$ known as multiple modular values (MMVs)

  $\longrightarrow$ often $\mathbb{Q}[(i\pi)^{\pm 1}]$-multiples of MZVs or L-values of cusp forms

[Brown 1407.5167; Brown 1904.00179]

II. 1 Towards equivariant iterated Eisenstein integrals

Construct equivariant forms via commutative, real indeterminates X, Y

$$\underline{\mathrm{G}}_k[X,Y;\tau_1] \;=\; \mathrm{d}\tau_1\,(X-\tau_1 Y)^{k-2}\,\mathrm{G}_k(\tau_1)$$

Under modular transformation with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$,

$$\underline{\mathrm{G}}_k\left[aX+bY,\, cX+dY;\, \frac{a\tau_1+b}{c\tau_1+d}\right] \;=\; \underline{\mathrm{G}}_k[X,Y;\tau_1]$$

Since τ_1 is integration variable of $\int_\tau \underline{\mathrm{G}}_k[X,Y;\tau_1]$, re-expand in powers

$(X-\tau Y)^j$ & $(X-\bar\tau Y)^{k-j-2} \Rightarrow$ coeff. forms ω_+ of mod. weight $(0,k-2j-2)$

$$\underline{\mathrm{G}}_k[X,Y;\tau_1] \;=\; \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{(2\pi i)^{k-1}\,(X-\tau Y)^j (X-\bar\tau Y)^{k-j-2}}{(-4\pi \mathrm{Im}\,\tau)^j}\, \omega_+\!\left[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau,\tau_1\right]$$

$$\omega_+\!\left[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau,\tau_1\right] \;=\; \frac{\mathrm{d}\tau_1}{2\pi i}\left(\frac{\tau-\tau_1}{4\pi \mathrm{Im}\,\tau}\right)^{k-j-2} (\bar\tau-\tau_1)^j\,\mathrm{G}_k(\tau_1)$$

II. 1 Towards equivariant iterated Eisenstein integrals

Starting from modular coefficient forms ...

$$\omega_+\left[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau,\tau_1\right] = \frac{\mathrm{d}\tau_1}{2\pi i}\left(\frac{\tau-\tau_1}{4\pi \mathrm{Im}\,\tau}\right)^{k-j-2}(\bar\tau-\tau_1)^j\,\mathrm{G}_k(\tau_1)$$

$$\omega_-\left[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau,\tau_1\right] = -\frac{\mathrm{d}\bar\tau_1}{2\pi i}\left(\frac{\tau-\bar\tau_1}{4\pi \mathrm{Im}\,\tau}\right)^{k-j-2}(\bar\tau-\bar\tau_1)^j\,\overline{\mathrm{G}_k(\tau_1)}$$

... construct (homotopy invariant) iterated Eisenstein integrals & cc

$$\beta_+\left[\begin{smallmatrix} j_1 & j_2 & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{smallmatrix};\tau\right] = \int_\tau^{i\infty}\omega_+\left[\begin{smallmatrix} j_\ell \\ k_\ell \end{smallmatrix};\tau,\tau_\ell\right]\int_{\tau_\ell}^{i\infty}\ldots\int_{\tau_2}^{i\infty}\omega_+\left[\begin{smallmatrix} j_1 \\ k_1 \end{smallmatrix};\tau,\tau_1\right]$$

$$\beta_-\left[\begin{smallmatrix} j_1 & j_2 & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{smallmatrix};\tau\right] = \int_{\bar\tau}^{-i\infty}\omega_-\left[\begin{smallmatrix} j_\ell \\ k_\ell \end{smallmatrix};\tau,\tau_\ell\right]\int_{\bar\tau_\ell}^{-i\infty}\ldots\int_{\bar\tau_2}^{-i\infty}\omega_-\left[\begin{smallmatrix} j_1 \\ k_1 \end{smallmatrix};\tau,\tau_1\right]$$

Fail to be modular forms by MMVs & shorter $\beta_\pm$[less than ℓ]

$$\beta_\pm\left[\begin{smallmatrix} j_1 & \cdots & j_\ell \\ k_1 & \cdots & k_\ell \end{smallmatrix};\frac{a\tau+b}{c\tau+d}\right] = \left(\prod_{i=1}^{\ell}(c\bar\tau+d)^{k_i-2-2j_i}\right)\beta_\pm\left[\begin{smallmatrix} j_1 & \cdots & j_\ell \\ k_1 & \cdots & k_\ell \end{smallmatrix};\tau\right] \text{ mod MMVs}$$

II. 2 Non-holomorphic Eisenstein series vs. iterated integrals

With the above iterated Eisenstein integrals $\beta_\pm$ and rational $c_w = \frac{(2w-1)!}{(w-1)!^2}$,

realize non-holomorphic Eisenstein series $\mathrm{E}_w = (\frac{\mathrm{Im}\,\tau}{\pi})^w \sum_{p\in\Lambda'} |p|^{-2w}$ as

$$\begin{aligned}
\mathrm{E}_w(\tau) &= -c_w\left(\beta_+\left[\begin{smallmatrix} w-1 \\ 2w \end{smallmatrix};\tau\right] + \beta_-\left[\begin{smallmatrix} w-1 \\ 2w \end{smallmatrix};\tau\right] - \frac{2\zeta_{2w-1}}{(2w-1)(4\pi \mathrm{Im}\,\tau)^{w-1}}\right) \\
&= \frac{c_w}{(4\pi \mathrm{Im}\,\tau)^{w-1}}\left(\frac{2\zeta_{2w-1}}{(2w-1)}\right. \\
&\qquad \left. - \,\mathrm{Im}\left[\int_\tau^{i\infty} \frac{\mathrm{d}\tau_1}{\pi}(\tau-\tau_1)^{w-1}(\bar\tau-\tau_1)^{w-1}\mathrm{G}_{2w}(\tau_1)\right]\right)
\end{aligned}$$

- $\frac{\zeta_{2k-1}}{(\mathrm{Im}\,\tau)^{w-1}}$ cancels non-modular behaviour of $\beta_\pm\left[\begin{smallmatrix} w-1 \\ 2w \end{smallmatrix}\right]$ ($\longrightarrow$ MMVs)
- similar expressions for $\tau, \bar\tau$-derivatives of E_w (Maaß operator ∇_τ)

$$\frac{\mathrm{Im}\,\tau^a}{\pi^b}\sum_{p\in\Lambda'}\frac{1}{p^a\bar p^b} = -\frac{(2i)^{b-a}(a+b-1)!}{(a-1)!(b-1)!}\left(\beta_+\left[\begin{smallmatrix} a-1 \\ a+b \end{smallmatrix}\right] + \beta_-\left[\begin{smallmatrix} a-1 \\ a+b \end{smallmatrix}\right] - \frac{2\zeta_{a+b-1}}{(a+b-1)(4\pi \mathrm{Im}\,\tau)^{b-1}}\right)$$

II. 3 Equivariant iterated integrals

At higher depth, can find completions to modular forms $\beta^{\rm eqv}\left[\begin{smallmatrix} j_1 & \dots & j_\ell \\ k_1 & \dots & k_\ell \end{smallmatrix}\right]$ of wt's $(0, \sum_{i=1}^{\ell}(k_i-2-2j_i))$, or *equivariant iterated Eisenstein integrals*

$$\beta^{\rm eqv}\left[\begin{matrix} j_1 & \dots & j_\ell \\ k_1 & \dots & k_\ell \end{matrix}\right] = \sum_{i=0}^{\ell} \beta_-\left[\begin{matrix} j_i & \dots & j_1 \\ k_i & \dots & k_1 \end{matrix}\right] \beta_+\left[\begin{matrix} j_{i+1} & \dots & j_\ell \\ k_{i+1} & \dots & k_\ell \end{matrix}\right] + \left(\begin{matrix} \text{MZVs and} \\ \text{shorter } \beta_\pm \end{matrix}\right)$$

[Brown 1407.5167, 1707.01230, 1708.03354]

Already saw depth $\ell=1$ example: $\mathrm{E}_{w=k/2}$ and their $\tau,\bar\tau$-derivatives

$$\beta^{\rm eqv}\left[\begin{matrix} j \\ k \end{matrix}\right] = \beta_+\left[\begin{matrix} j \\ k \end{matrix}\right] + \beta_-\left[\begin{matrix} j \\ k \end{matrix}\right] - \frac{2\zeta_{k-1}}{(k-1)(4\pi \mathrm{Im}\,\tau)^{k-2-j}}$$

General MGFs are $\mathbb{Q}[\text{svMZV}]$ combinations of $\beta^{\rm eqv}$ and thereby modular $\mathbb{Q}\left[\text{MZV}, \tau, \bar\tau, \frac{1}{\mathrm{Im}\,\tau}\right]$ combinations of $\beta_\pm\left[\begin{matrix} j_1 & \dots & j_\ell \\ k_1 & \dots & k_\ell \end{matrix}\right]$

[conjectured in Brown 1707.01230, 1708.03354 & worked out in: DDDHKMSV 2209.06772]

II. 4 Higher-depth examples

General MGFs are $\mathbb{Q}$[svMZV] combinations of $\beta^{\rm eqv}$ and thereby

modular $\mathbb{Q}\Big[\mathrm{MZV}, \tau, \bar\tau, \frac{1}{\mathrm{Im}\,\tau}\Big]$ combinations of $\beta_\pm\begin{bmatrix} j_1 & \dots & j_\ell \\ k_1 & \dots & k_\ell \end{bmatrix}$

[conjectured in Brown 1707.01230, 1708.03354
& worked out in: DDDHKMSV 2209.06772]

At depth two, for instance (recall $C_{a,b,c} \sim \sum_{p_i \in \Lambda'} \frac{\delta(p_1+p_2+p_3)}{|p_1|^{2a}|p_2|^{2b}|p_3|^{2c}}$)

$$C_{2,1,1} \ = \ -18\beta^{\rm eqv}\big[\begin{smallmatrix} 2 & 0 \\ 4 & 4 \end{smallmatrix}\big] - 126\beta^{\rm eqv}\big[\begin{smallmatrix} 3 \\ 8 \end{smallmatrix}\big]$$

where (see later for general construction of MZV admixtures)

$$\beta^{\rm eqv}\big[\begin{smallmatrix} 2 & 0 \\ 4 & 4 \end{smallmatrix}\big] \ = \ \beta_+\big[\begin{smallmatrix} 2 & 0 \\ 4 & 4 \end{smallmatrix}\big] + \beta_+\big[\begin{smallmatrix} 0 \\ 4 \end{smallmatrix}\big]\,\beta_-\big[\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\big] + \beta_-\big[\begin{smallmatrix} 0 & 2 \\ 4 & 4 \end{smallmatrix}\big]$$
$$-\frac{2\zeta_3}{3}\Big(\beta_+\big[\begin{smallmatrix} 0 \\ 4 \end{smallmatrix}\big] + \beta_-\big[\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\big]\Big) - \frac{\tau\bar\tau\pi\zeta_3}{1080\,\mathrm{Im}\,\tau} - \frac{5\zeta_5}{216\pi\,\mathrm{Im}\,\tau} + \frac{\zeta_3^2}{72\pi^2(\mathrm{Im}\,\tau)^2}$$

Highest-depth terms match those of $\beta^{\rm sv}$ of **[Gerken, Kleinschmidt, OS 2004.05156]**.

II. 4 Higher-depth examples

General MGFs are $\mathbb{Q}[\text{svMZV}]$ combinations of $\beta^{\rm eqv}$ and thereby modular $\mathbb{Q}\big[\text{MZV}, \tau, \bar\tau, \frac{1}{\text{Im}\,\tau}\big]$ combinations of $\beta_\pm\begin{bmatrix} j_1 & \dots & j_\ell \\ k_1 & \dots & k_\ell \end{bmatrix}$

[conjectured in Brown 1707.01230, 1708.03354
& worked out in: DDDHKMSV 2209.06772]

At depth two, for instance (recall $C_{a,b,c} \sim \sum_{p_i \in \Lambda'} \frac{\delta(p_1+p_2+p_3)}{|p_1|^{2a}|p_2|^{2b}|p_3|^{2c}}$)

$$C_{2,1,1} \ = \ -18\beta^{\rm eqv}\big[\begin{smallmatrix} 2 & 0 \\ 4 & 4 \end{smallmatrix}\big] - 126\beta^{\rm eqv}\big[\begin{smallmatrix} 3 \\ 8 \end{smallmatrix}\big]$$

$$C_{3,1,1} \ = \ -120\beta^{\rm eqv}\big[\begin{smallmatrix} 2 & 1 \\ 4 & 6 \end{smallmatrix}\big] - 120\beta^{\rm eqv}\big[\begin{smallmatrix} 3 & 0 \\ 6 & 4 \end{smallmatrix}\big] - 774\beta^{\rm eqv}\big[\begin{smallmatrix} 4 \\ 10 \end{smallmatrix}\big] - \frac{\zeta_5}{60}$$

$$C_{2,2,1} \ = \ -252\beta^{\rm eqv}\big[\begin{smallmatrix} 4 \\ 10 \end{smallmatrix}\big] + \frac{\zeta_5}{30}$$

Exposes MGF relations such as $C_{2,2,1} = \frac{2}{5}\,\text{E}_5 + \frac{\zeta_5}{30}$: the $\beta^{\rm eqv}$ representation is canonical since $\beta_\pm$ with different entries are lin. independent.

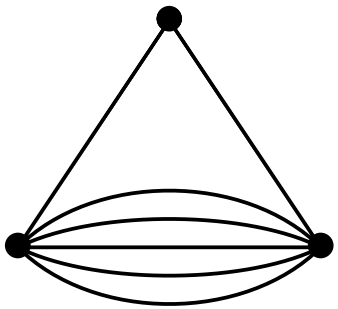
[Matthes 1708.04561]

II. 4 Higher-depth examples

Triple integrals may involve svMZVs of depth 3 in the modular completion

$$\beta^{\rm eqv}\left[\begin{smallmatrix}1&2&1\\4&4&6\end{smallmatrix}\right] = \beta_+\left[\begin{smallmatrix}1&2&1\\4&4&6\end{smallmatrix}\right] + \ldots - \frac{48\zeta^{\rm sv}_{3,5,3}+960\zeta_3^2\zeta_5+221\zeta_{11}}{5529600(\pi {\rm Im}\,\tau)^4}$$

With similar terms in other $\beta^{\rm eqv}$, reproduces triple zeta **[Zerbini 1512.05689]**



$$\longleftrightarrow \quad \mathcal{C}^+\left[\begin{smallmatrix}2&1&1&1&1&1\\2&1&1&1&1&1\end{smallmatrix}\right]\Big|_{\tau\to i\infty} = \ldots - \frac{9\zeta^{\rm sv}_{3,5,3}}{8(\pi{\rm Im}\,\tau)^4} + \ldots$$

via (with convention-dep. $c_{446}\in\mathbb{Q}$ to be fixed) **[DDDHKMSV 2209.06772]**

$$\begin{aligned}
\mathcal{C}^+\left[\begin{smallmatrix}2&1&1&1&1&1\\2&1&1&1&1&1\end{smallmatrix}\right] &= \left(\frac{47}{72}+1814400\,c_{446}\right)\zeta_7 - 3\zeta_5\beta^{\rm eqv}\left[\begin{smallmatrix}1\\4\end{smallmatrix}\right] - 1260\zeta_3\beta^{\rm eqv}\left[\begin{smallmatrix}3\\8\end{smallmatrix}\right] - 180\zeta_3\beta^{\rm eqv}\left[\begin{smallmatrix}2&0\\4&4\end{smallmatrix}\right]\\
&+360(150\beta^{\rm eqv}\left[\begin{smallmatrix}1&2&1\\4&4&6\end{smallmatrix}\right]+150\beta^{\rm eqv}\left[\begin{smallmatrix}3&0&1\\6&4&4\end{smallmatrix}\right]-90\beta^{\rm eqv}\left[\begin{smallmatrix}1&1&2\\4&4&6\end{smallmatrix}\right]-90\beta^{\rm eqv}\left[\begin{smallmatrix}2&1&1\\6&4&4\end{smallmatrix}\right]-90\beta^{\rm eqv}\left[\begin{smallmatrix}1&2&1\\4&6&4\end{smallmatrix}\right] \quad (3.18)\\
&+150\beta^{\rm eqv}\left[\begin{smallmatrix}1&3&0\\4&6&4\end{smallmatrix}\right]+150\beta^{\rm eqv}\left[\begin{smallmatrix}2&1&1\\4&6&4\end{smallmatrix}\right]+195\beta^{\rm eqv}\left[\begin{smallmatrix}2&0&2\\4&4&6\end{smallmatrix}\right]+195\beta^{\rm eqv}\left[\begin{smallmatrix}2&2&0\\6&4&4\end{smallmatrix}\right]+15\beta^{\rm eqv}\left[\begin{smallmatrix}2&2&0\\4&6&4\end{smallmatrix}\right]\\
&-330\beta^{\rm eqv}\left[\begin{smallmatrix}2&1&1\\4&4&6\end{smallmatrix}\right]-330\beta^{\rm eqv}\left[\begin{smallmatrix}3&1&0\\6&4&4\end{smallmatrix}\right]+480\beta^{\rm eqv}\left[\begin{smallmatrix}2&2&0\\4&4&6\end{smallmatrix}\right]+480\beta^{\rm eqv}\left[\begin{smallmatrix}4&0&0\\6&4&4\end{smallmatrix}\right]+315\beta^{\rm eqv}\left[\begin{smallmatrix}2&3\\6&8\end{smallmatrix}\right]\\
&+315\beta^{\rm eqv}\left[\begin{smallmatrix}3&2\\8&6\end{smallmatrix}\right]-1190\beta^{\rm eqv}\left[\begin{smallmatrix}3&2\\6&8\end{smallmatrix}\right]-1190\beta^{\rm eqv}\left[\begin{smallmatrix}4&1\\8&6\end{smallmatrix}\right]+2800\beta^{\rm eqv}\left[\begin{smallmatrix}4&1\\6&8\end{smallmatrix}\right]+2800\beta^{\rm eqv}\left[\begin{smallmatrix}5&0\\8&6\end{smallmatrix}\right]\\
&+243\beta^{\rm eqv}\left[\begin{smallmatrix}4&1\\10&4\end{smallmatrix}\right]+243\beta^{\rm eqv}\left[\begin{smallmatrix}1&4\\4&10\end{smallmatrix}\right]+432\beta^{\rm eqv}\left[\begin{smallmatrix}5&0\\10&4\end{smallmatrix}\right]+432\beta^{\rm eqv}\left[\begin{smallmatrix}2&3\\4&10\end{smallmatrix}\right]+3640\beta^{\rm eqv}\left[\begin{smallmatrix}6\\14\end{smallmatrix}\right]),
\end{aligned}$$

III. Genus-zero inspiration

III. 1 Analogy between genus 0 and 1

Constructing the MZVs in modular completion

$$\beta^{\text{eqv}}\left[\begin{smallmatrix} j_1 & \dots & j_\ell \\ k_1 & \dots & k_\ell \end{smallmatrix};\tau\right] = \sum_{i=0}^{\ell} \beta_-\left[\begin{smallmatrix} j_i & \dots & j_1 \\ k_i & \dots & k_1 \end{smallmatrix};\tau\right] \beta_+\left[\begin{smallmatrix} j_{i+1} & \dots & j_\ell \\ k_{i+1} & \dots & k_\ell \end{smallmatrix};\tau\right] + \left(\begin{matrix}\text{MZVs and} \\ \text{shorter } \beta_\pm\end{matrix}\right)$$

[Brown 1407.5167, 1707.01230, 1708.03354]

partially follows genus-zero analogy in constructing single-valued polylogs

$$G^{\text{sv}}(a_1,\ldots,a_\ell;z) = \sum_{i=0}^{\ell} \overline{G(a_\ell,\ldots,a_{i+1};z)}\, G(a_1,\ldots,a_i;z) + \left(\begin{matrix}\text{svMZVs and} \\ \text{shorter } G,\bar{G}\end{matrix}\right)$$

[Brown 2004]

in one variable $a_i \in \{0,1\}$, with $⧢$-regularization $G(0;z) = \log(z)$ and

$$G(a_1,\ldots,a_\ell;z) = \int_0^z \frac{\mathrm{d}t}{t-a_1} G(a_2,\ldots,a_\ell;t)\,, \qquad G(\emptyset;z) = 1$$

MZVs in G^{sv} & β^{eqv} from change of alphabet in resp. generating series...

III. 2 Rewriting generating series of sv polylogs

Generating series $\mathcal{I}_+$ and $\mathcal{I}_-$ of meromorphic polylogs and their cc

$$\mathcal{I}_\pm(e_0,e_1;z) = \sum_{\ell=0}^{\infty} \sum_{a_1,\ldots,a_\ell\in\{0,1\}} e_{a_1}e_{a_2}\ldots e_{a_\ell}\cdot \begin{cases} G(a_1,\ldots,a_\ell;z) : + \\ \overline{G(a_1,\ldots,a_\ell;z)} : - \end{cases}$$

with non-commutative braid operators e_0, e_1 forming free algebra.

III. 2 Rewriting generating series of sv polylogs

Generating series $\mathcal{I}_+$ and $\mathcal{I}_-$ of meromorphic polylogs and their cc

$$\mathcal{I}_\pm(e_0,e_1;z) = \sum_{\ell=0}^{\infty} \sum_{a_1,\ldots,a_\ell \in \{0,1\}} e_{a_1} e_{a_2} \ldots e_{a_\ell} \cdot \begin{cases} G(a_1,\ldots,a_\ell;z) & : + \\ \overline{G(a_1,\ldots,a_\ell;z)} & : - \end{cases}$$

with non-commutative braid operators e_0, e_1 forming free algebra.

Analogous generating series $\mathcal{I}^{\rm SV}$ of sv polylogs

$$\mathcal{I}^{\rm SV}(e_0,e_1;z) = \sum_{\ell=0}^{\infty} \sum_{a_1,\ldots,a_\ell \in \{0,1\}} e_{a_1} e_{a_2} \ldots e_{a_\ell} G^{\rm SV}(a_1,\ldots,a_\ell;z)$$

$$= \mathcal{I}_+(e_0,e_1;z)\, \mathbb{M}^{\rm SV}\, \widetilde{\mathcal{I}_-}(e_0,e_1;z)\, (\mathbb{M}^{\rm SV})^{-1}$$

from reversal $\widetilde{\mathcal{I}_-}$ on all words $\ldots \widetilde{e_a e_b} \ldots = \ldots e_b e_a \ldots$

... and conjugation by series $\mathbb{M}^{\rm SV}$ in svMZVs.

III. 2 Rewriting generating series of sv polylogs

Target: series $\mathcal{I}^{\rm sv}(e_0,e_1;z)$ in e_0,e_1 with $G^{\rm sv}$ as coeff's,

$$\mathcal{I}^{\rm sv}(e_0,e_1;z) = \mathcal{I}_+(e_0,e_1;z)\,\mathbb{M}^{\rm sv}\,\widetilde{\mathcal{I}_-}(e_0,e_1;z)\,(\mathbb{M}^{\rm sv})^{-1}$$

intermediate steps: derivations $M_3, M_5, M_7, \ldots \longleftrightarrow$ odd zeta values

$$\mathbb{M}^{\rm sv} = 1 + 2\sum_{k\in 2\mathbb{N}+1} \zeta_k\, M_k + 2 \sum_{k_1,k_2\in 2\mathbb{N}+1} \zeta_{k_1}\zeta_{k_2}\, M_{k_1} M_{k_2} + \begin{pmatrix}\text{higher}\\ \text{depth}\end{pmatrix}$$

$$= \sum_{\ell=0} \sum_{k_1,k_2,\ldots,k_\ell\in 2\mathbb{N}+1} {\rm sv}(f_{k_1} f_{k_2}\ldots f_{k_\ell})\, M_{k_1} M_{k_2}\ldots M_{k_\ell}$$

with f-alphabet description of svMZVs in 2nd line.

Remove all M_{k_i} in favor of e_0, e_1 by evaluating nested commutators

$$\mathbb{M}^{\rm sv}\,\widetilde{\mathcal{I}_-}\,(\mathbb{M}^{\rm sv})^{-1} = \widetilde{\mathcal{I}_-} + 2\sum_{k\in 2\mathbb{N}+1} \zeta_k\,[M_k, \widetilde{\mathcal{I}_-}\,] + 2\sum_{k_1,k_2\in 2\mathbb{N}+1} \zeta_{k_1}\zeta_{k_2}\,[M_{k_1},[M_{k_2},\widetilde{\mathcal{I}_-}]] + \begin{pmatrix}\text{higher}\\ \text{depth}\end{pmatrix}$$

III. 2 Rewriting generating series of sv polylogs

Target: series $\mathcal{I}^{\rm sv}(e_0,e_1;z)$ in e_0,e_1 with $G^{\rm sv}$ as coeff's,

$$\mathcal{I}^{\rm sv}(e_0,e_1;z) \;=\; \mathcal{I}_+(e_0,e_1;z)\, \mathbb{M}^{\rm sv}\, \widetilde{\mathcal{I}_-}(e_0,e_1;z)\, (\mathbb{M}^{\rm sv})^{-1}$$

$$\mathbb{M}^{\rm sv}\, \widetilde{\mathcal{I}_-}\, (\mathbb{M}^{\rm sv})^{-1} \;=\; \widetilde{\mathcal{I}_-} + 2 \sum_{k\in 2\mathbb{N}+1} \zeta_k \, [M_k, \widetilde{\mathcal{I}_-}\,] + \begin{pmatrix} \text{higher} \\ \text{depth} \end{pmatrix}$$

where commutators $[M_k, e_a]$ are length-$(k{+}1)$ words in e_0, e_1 from

$$[e_0, M_k] \;=\; 0\,, \qquad [e_1, M_k] \;=\; \Big[\Phi(e_0,e_1)\,\Big|_{\zeta_k}, e_1\Big]$$

with Drinfeld associator $\Phi(e_0,e_1) = \mathcal{I}_+(e_0,e_1;z{=}1)$, e.g.

$$[e_1, M_3] \;=\; [[[e_0,e_1],e_0{+}e_1],e_1]\,, \qquad [e_1, M_k] \;=\; (\text{brackets of } (k{+}1)\ e_a\text{'s})$$

$$\Longrightarrow \quad G^{\rm sv}(0,0,1,1;z) \;=\; \mathcal{I}^{\rm sv}(e_0,e_1;z)\,\Big|_{e_0e_0e_1e_1} \;=\; \ldots + 2\zeta_3 \overline{G(1;z)}$$

III. 2 Rewriting generating series of sv polylogs

Target: series $\mathcal{I}^{\rm sv}(e_0,e_1;z)$ in e_0,e_1 with $G^{\rm sv}$ as coeff's,

$$\mathcal{I}^{\rm sv}(e_0,e_1;z) \;=\; \mathcal{I}_+(e_0,e_1;z)\,\mathbb{M}^{\rm sv}\,\widetilde{\mathcal{I}_-}(e_0,e_1;z)\,(\mathbb{M}^{\rm sv})^{-1}$$

$$\mathbb{M}^{\rm sv}\,\widetilde{\mathcal{I}_-}\,(\mathbb{M}^{\rm sv})^{-1} \;=\; \widetilde{\mathcal{I}_-} + 2\sum_{k\in 2\mathbb{N}+1}\zeta_k\,[M_k,\widetilde{\mathcal{I}_-}\,] + \begin{pmatrix}\text{higher}\\ \text{depth}\end{pmatrix}$$

where commutators $[M_k,e_a]$ are length-$(k{+}1)$ words in e_0,e_1 from

$$[e_0,M_k] \;=\; 0\,, \qquad [e_1,M_k] \;=\; \Big[\Phi(e_0,e_1)\Big|_{\zeta_k},e_1\Big]$$

Can pull conjugation with $\mathbb{M}^{\rm sv}$ into change of letter $e_1\rightarrow e_1'$ in $\widetilde{\mathcal{I}_-}$,

$$\mathbb{M}^{\rm sv}\,\widetilde{\mathcal{I}_-}(e_0,e_1;z)\,(\mathbb{M}^{\rm sv})^{-1} \;=\; \widetilde{\mathcal{I}_-}(e_0,e_1';z)\,, \qquad e_1' \;=\; \mathbb{M}^{\rm sv}\,e_1\,(\mathbb{M}^{\rm sv})^{-1}$$

Equivalent to change of alphabet $e_1' = \Phi^{\rm sv}(e_0,e_1)^{-1}e_1\Phi^{\rm sv}(e_0,e_1)$ of Brown.

[in progress: proof by Deepak Kamlesh]

III. 3 Towards a change of alphabet for genus one

Next step: Generate MZVs in modular completion of $\beta^{\rm eqv}$ via gen. series

genus zero $\quad \mathcal{I}^{\rm SV}(e_0,e_1;z) \, = \, \mathcal{I}_+(e_0,e_1;z)\, \mathbb{M}^{\rm SV}\, \widetilde{\mathcal{I}_-}(e_0,e_1;z)\, (\mathbb{M}^{\rm SV})^{-1}$

genus one $\quad \mathcal{J}^{\rm eqv}(\epsilon_k;\tau) \, = \, \mathcal{J}_+(\epsilon_k;\tau)\, B^{\rm SV}(\epsilon_k;\tau)\, \mathbb{M}^{\rm SV}\, \widetilde{\mathcal{J}_-}(\epsilon_k;\tau)\, (\mathbb{M}^{\rm SV})^{-1}$

[Brown 1708.03354; DDDHKMSV 2209.06772]

Two kinds of new ingredients at genus one:

- instead of e_0, e_1, get ∞ many letters $\epsilon_k \leftrightarrow {\rm G}_k$ in series $\mathcal{J}_\pm$ at genus 1,

  need to specify $[\mathbb{M}^{\rm SV}, \epsilon_k]$ to obtain change of alphabet $\epsilon_k \rightarrow \epsilon_k'$

  [Pollack undergraduate thesis 2009; Hain, Matsumoto 1512.03975]

- additional series $B^{\rm SV}(\epsilon_k;\tau)$ in ϵ_k with $\mathbb{Q}[{\rm MZV}, \tau, \bar\tau, \frac{1}{{\rm Im}\, \tau}]$ coefficients

  [ancillary files of DDDHKMSV 2209.06772]

IV. MGFs from equiv. iterated Eisenstein integrals

IV. 1 Change of alphabet at genus 1

Generalize $e_0, e_1 \longrightarrow \epsilon_k^{(j)} = \mathrm{ad}^j_{\epsilon_0}(\epsilon_k)$ @ $k \geq 4$ and $0 \leq j \leq k{-}2$

with derivation algebra $\{\epsilon_m,\ m \in 2\mathbb{N}_0\}$ subject to $\mathrm{ad}^{k-1}_{\epsilon_0}(\epsilon_k) = 0$

and a variety of bracket relations $[\epsilon_4, \epsilon_{10}] - 3[\epsilon_6, \epsilon_8] = 0$ & higher order

[Tsunogai 1995, ... , Pollack 2009]

Generating series of meromorphic iterated Eisenstein integrals

$$\mathcal{J}_\pm(\epsilon_k;\tau) = \sum_{\ell=0}^{\infty} \sum_{k_1,\ldots,k_\ell=4}^{\infty} \sum_{j_1=0}^{k_1-2} \cdots \sum_{j_\ell=0}^{k_\ell-2} \Bigg(\prod_{i=1}^{\ell} \frac{(-1)^{j_i}(k_i{-}1)}{(k_i{-}j_i{-}2)!} \Bigg)$$

$$\times\, \epsilon_{k_\ell}^{(k_\ell-j_\ell-2)} \ldots \epsilon_{k_2}^{(k_2-j_2-2)} \epsilon_{k_1}^{(k_1-j_1-2)} \beta_\pm\Big[\begin{smallmatrix} j_1 & j_2 & \ldots & j_\ell \\ k_1 & k_2 & \ldots & k_\ell \end{smallmatrix};\tau\Big]$$

$$= \sum_P \epsilon[P]\, \beta_\pm[P;\tau]$$

with shorthand $\sum_P$ for summing over words P in composite letters $\begin{smallmatrix} j \\ k \end{smallmatrix}$

IV. 1 Change of alphabet at genus 1

For antimeromorphic Eisenstein integrals $\mathcal{J}_-(\epsilon_k;\tau) = \sum_P \epsilon[P]\,\beta_-[P;\tau]$,

generalize genus-zero move $\mathbb{M}^{\rm sv}\,\widetilde{\mathcal{I}_-}\,(\mathbb{M}^{\rm sv})^{-1}$ for polylogs to

$$\mathbb{M}^{\rm sv}(z_i)\,\widetilde{\mathcal{J}_-}(\epsilon_k)\,(\mathbb{M}^{\rm sv}(z_i))^{-1} = \widetilde{\mathcal{J}_-} + 2\sum_{m\in 2\mathbb{N}+1} \zeta_m\,[z_m, \widetilde{\mathcal{J}_-}]$$
$$+ 2 \sum_{m_1,m_2\in 2\mathbb{N}+1} \zeta_{m_1}\zeta_{m_2}\,[z_{m_1},[z_{m_2},\widetilde{\mathcal{J}_-}]] + \begin{pmatrix}\text{higher}\\ \text{depth}\end{pmatrix}$$

with derivations $z_3, z_5, \ldots$ inside $\mathbb{M}^{\rm sv}(z_i)$

$$\mathbb{M}^{\rm sv}(z_i) = 1 + 2\sum_{m\in 2\mathbb{N}+1} \zeta_m\, z_m + 2\sum_{m_1,m_2\in 2\mathbb{N}+1} \zeta_{m_1}\zeta_{m_2}\, z_{m_1} z_{m_2} + \begin{pmatrix}\text{higher}\\ \text{depth}\end{pmatrix}$$
$$= \sum_{\ell\geq 0}\ \sum_{m_1,\ldots,m_\ell\in 2\mathbb{N}+1} {\rm sv}(f_{m_1} f_{m_2}\ldots f_{m_\ell})\, z_{m_1} z_{m_2}\ldots z_{m_\ell}$$

[DDDHKMSV 2209.06772; implicit in Brown 1708.03354]

IV. 1 Change of alphabet at genus 1

For antimeromorphic Eisenstein integrals $\mathcal{J}_-(\epsilon_k;\tau) = \sum_P \epsilon[P]\,\beta_-[P;\tau]$,

generalize genus-zero move $\mathbb{M}^{\rm sv}\,\widetilde{\mathcal{I}_-}\,(\mathbb{M}^{\rm sv})^{-1}$ for polylogs to

$$\mathbb{M}^{\rm sv}(z_i)\,\widetilde{\mathcal{J}_-}(\epsilon_k)\,(\mathbb{M}^{\rm sv}(z_i))^{-1} = \widetilde{\mathcal{J}_-} + 2\sum_{m\in 2\mathbb{N}+1} \zeta_m\,[z_m,\widetilde{\mathcal{J}_-}]$$
$$+ 2\sum_{m_1,m_2\in 2\mathbb{N}+1} \zeta_{m_1}\zeta_{m_2}\,[z_{m_1},[z_{m_2},\widetilde{\mathcal{J}_-}]] + \begin{pmatrix}\text{higher}\\ \text{depth}\end{pmatrix}$$

with derivations $z_3, z_5,\ldots$ inside $\mathbb{M}^{\rm sv}(z_i)$ subject to $[z_m,\epsilon_0]=0$ and

$$[z_m,\epsilon_k] = \text{nested brackets of two and more } \epsilon_{k_i}^{(j_i)}$$

[Pollack 2009; Hain, Matsumoto 1512.03975]

for instance $[z_3,\epsilon_4] = \frac{1}{504}\big([\epsilon_6^{(2)},\epsilon_4] - 3[\epsilon_6^{(1)},\epsilon_4^{(1)}] + 6[\epsilon_6,\epsilon_4^{(2)}]\big)$ leads to

$$\mathbb{M}^{\rm sv}(z_i)\,\epsilon_4\,(\mathbb{M}^{\rm sv}(z_i))^{-1} = \epsilon_4 + \frac{\zeta_3}{252}\Big([\epsilon_6^{(2)},\epsilon_4] - 3[\epsilon_6^{(1)},\epsilon_4^{(1)}] + 6[\epsilon_6,\epsilon_4^{(2)}]\Big) + \ldots$$

IV. 2 Constructing equivariant iterated Eisenstein integrals

To finish the construction of modular forms $\beta^{\rm eqv}$ from

$$\mathcal{J}^{\rm eqv}(\epsilon_k;\tau) = \mathcal{J}_+(\epsilon_k;\tau)\, B^{\rm sv}(\epsilon_k;\tau)\, \mathbb{M}^{\rm sv}\, \widetilde{\mathcal{J}_-}(\epsilon_k;\tau)\, (\mathbb{M}^{\rm sv})^{-1}$$

$$= \sum_P \epsilon[P]\, \beta^{\rm eqv}[P;\tau]$$

specify the new ingredient $B^{\rm sv}(\epsilon_k;\tau)$ at genus one:

- series in svMZVs $c^{\rm sv}\begin{bmatrix} j_1 & j_2 & \cdots & j_r \\ k_1 & k_2 & \ldots & k_r \end{bmatrix}$ of transc. weight $r+j_1+j_2+\ldots+j_r$
- composed to polynomials $\mathbb{Q}[i\pi\bar\tau, \frac{1}{\pi {\rm Im}\,\tau}]$ "choice of SL_2 frame"

$$B^{\rm sv}(\epsilon_k;\tau) = \sum_P \epsilon[P] b^{\rm sv}[P;\tau]$$

$$b^{\rm sv}\begin{bmatrix} \cdots & j & \cdots \\ \cdots & k & \cdots \end{bmatrix} = \sum_{p=0}^{k-2-j} \sum_{\ell=0}^{j+p} \binom{k-j-2}{p} \binom{j+p}{\ell} \frac{(-2\pi i\bar\tau)^\ell}{(4\pi {\rm Im}\,\tau)^p}\, c^{\rm sv}\begin{bmatrix} \cdots & j-\ell+p & \cdots \\ \cdots & k & \cdots \end{bmatrix}$$

IV. 2 Constructing equivariant iterated Eisenstein integrals

To finish the construction of modular forms $\beta^{\rm eqv}$ from

$$\mathcal{J}^{\rm eqv}(\epsilon_k;\tau) = \mathcal{J}_+(\epsilon_k;\tau)\, B^{\rm sv}(\epsilon_k;\tau)\, \mathbb{M}^{\rm sv}\, \widetilde{\mathcal{J}_-}(\epsilon_k;\tau)\, (\mathbb{M}^{\rm sv})^{-1}$$

specify the new ingredient $B^{\rm sv}(\epsilon_k)$ via composing svMZVs $c^{\rm sv}$

$$c^{\rm sv}\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right] = -\delta_{j,k-2}\frac{2\zeta_{k-1}}{k-1}$$

$$c^{\rm sv}\left[\begin{smallmatrix} 0 & 1 \\ 4 & 4 \end{smallmatrix}\right] = -\frac{\zeta_3}{2160}, \quad c^{\rm sv}\left[\begin{smallmatrix} 1 & 2 \\ 4 & 4 \end{smallmatrix}\right] = \frac{5\zeta_5}{108}, \quad c^{\rm sv}\left[\begin{smallmatrix} 3 & 3 \\ 8 & 6 \end{smallmatrix}\right] = \frac{\zeta_3\zeta_5}{588000}$$

$$c^{\rm sv}\left[\begin{smallmatrix} 2 & 2 & 6 \\ 4 & 4 & 8 \end{smallmatrix}\right] = -\frac{\zeta^{\rm sv}_{3,7,3}}{1764} + \frac{\zeta^{\rm sv}_{5,3,5}}{1470} - \frac{2\zeta_3^2\zeta_7}{63} - \frac{137359\zeta_{13}}{24378480}$$

with conjectural closed formula for "highest components" $j_i = k_i - 2$

$$c^{\rm sv}\left[\begin{smallmatrix} k_1-2 & \ldots & k_\ell-2 \\ k_1 & \ldots & k_\ell \end{smallmatrix}\right] = \left(\prod_{i=1}^{\ell}\frac{1}{1-k_i}\right)\mathrm{sv}(f_{k_1-1}\ldots f_{k_\ell-1}) \text{ mod fewer } f_i$$

[inspired by Saad 2009.09885]

IV. 3 Iterated integrals of holomorphic cusp forms

Some of the iterated Eisenstein integrals in $\mathcal{J}_+(\epsilon_k;\tau)\ldots\widetilde{\mathcal{J}_-}(\epsilon_k;\tau)$ require

primitives of holo' cusp forms $\Delta_k(\tau)=q+\mathcal{O}(q^2)$ in modular completion

[Brown 1407.5167, 1707.01230; Dorigoni, Kleinschmidt, OS, 2109.05018]

$$\beta^{\rm eqv}\big[\begin{smallmatrix}1&4\\6&8\end{smallmatrix}\big] = \Big(\beta_\pm \text{ and MZVs}\Big) + \frac{1}{52920000}\frac{\Lambda(\Delta_{12},12)}{\Lambda(\Delta_{12},10)}\,\beta^{\rm sv}\Big[\begin{smallmatrix}5\\ \Delta^{-}_{12}\end{smallmatrix}\Big]$$

$$\beta^{\rm eqv}\big[\begin{smallmatrix}2&3\\4&10\end{smallmatrix}\big] = \Big(\beta_\pm \text{ and MZVs}\Big) - \frac{1}{122472000}\frac{\Lambda(\Delta_{12},12)}{\Lambda(\Delta_{12},10)}\,\beta^{\rm sv}\Big[\begin{smallmatrix}5\\ \Delta^{-}_{12}\end{smallmatrix}\Big]$$

$$\beta^{\rm sv}\Big[\begin{smallmatrix}5\\ \Delta^{-}_{12}\end{smallmatrix}\Big] = (2\pi i)^{11}\Big\{\int_\tau^{i\infty}\mathrm{d}\tau_1\,(\tau-\tau_1)^5\,(\bar\tau-\tau_1)^5\,\Delta_{12}(\tau_1)+\text{cc}\Big\}$$

- depth-2 coeff's: ratio of critical and non-critical L-values $\frac{\Lambda(\Delta_k,nc)}{\Lambda(\Delta_k,crit)}$
- relations like $[\epsilon_4,\epsilon_{10}]-3[\epsilon_6,\epsilon_8]=0$ project out cusp-form contributions

  to non-holo modular forms in $J^{\rm eqv}\longleftrightarrow$ no $\int_\tau \mathrm{d}\tau_1\,\Delta_k(\tau_1)$ in MGFs

IV. 4 Relation to earlier approaches

Generating series of all MGFs constructed from closed-string integrals

[Gerken, Kleinschmidt, OS 1911.03476 & 2004.05156]

real-analytic iterated Eisenstein integrals $\beta^{\rm sv}$ in refs. related to $\beta^{\rm eqv}$ via

$$\beta^{\rm eqv}\begin{bmatrix} j_1 \\ k_1 \end{bmatrix} = \beta^{\rm sv}\begin{bmatrix} j_1 \\ k_1 \end{bmatrix} + \sum_{p_1=0}^{k_1-j_1-2} \frac{\binom{k_1-j_1-2}{p_1}}{(4\pi {\rm Im}\, \tau)^{p_1}} c^{\rm sv}\begin{bmatrix} j_1+p_1 \\ k_1 \end{bmatrix}$$

$$\beta^{\rm eqv}\begin{bmatrix} j_1 & j_2 \\ k_1 & k_2 \end{bmatrix} = \beta^{\rm sv}\begin{bmatrix} j_1 & j_2 \\ k_1 & k_2 \end{bmatrix} + \sum_{p_1=0}^{k_1-j_1-2} \frac{\binom{k_1-j_1-2}{p_1}}{(4\pi {\rm Im}\, \tau)^{p_1}} c^{\rm sv}\begin{bmatrix} j_1+p_1 \\ k_1 \end{bmatrix} \beta^{\rm sv}\begin{bmatrix} j_2 \\ k_2 \end{bmatrix}$$

$$+ \sum_{p_1=0}^{k_1-j_1-2} \sum_{p_2=0}^{k_2-j_2-2} \frac{\binom{k_1-j_1-2}{p_1}\binom{k_2-j_2-2}{p_2}}{(4\pi {\rm Im}\, \tau)^{p_1+p_2}} c^{\rm sv}\begin{bmatrix} j_1+p_1 & j_2+p_2 \\ k_1 & k_2 \end{bmatrix} + \begin{pmatrix} \text{cusp form} \\ \text{integrals} \end{pmatrix}$$

However, $\beta^{\rm sv}$ beyond depth one not generated by Brown's series

$J^{\rm sv} = J^{\rm eqv}(B^{\rm sv})^{-1}$ of *single-valued iterated Eisenstein integrals*

V. Further directions

- unify B^{sv} & $\mathbb{M}^{\text{sv}}$ in $J^{\text{eqv}} = J_+ B^{\text{sv}} \mathbb{M}^{\text{sv}} \widetilde{J}_-(\mathbb{M}^{\text{sv}})^{-1}$ via zeta elements σ_m

$$\sigma_3 = z_3 - \frac{1}{2}\epsilon_4^{(2)} + \frac{1}{480}[\epsilon_4, \epsilon_4^{(1)}] + \frac{1}{120960}\big(4[\epsilon_4^{(1)}, \epsilon_6] - [\epsilon_4, \epsilon_6^{(1)}]\big) + \frac{1}{7257600}[\epsilon_4, \epsilon_8^{(1)}]$$
$$- \frac{1}{1209600}[\epsilon_4^{(1)}, \epsilon_8] + \frac{1}{383201280}\big(8[\epsilon_4^{(1)}, \epsilon_{10}] - [\epsilon_4, \epsilon_{10}^{(1)}]\big) - \frac{1}{58060800}[\epsilon_4, [\epsilon_4, \epsilon_6]] + \ldots$$
$$\sigma_5 = z_5 - \frac{1}{24}\epsilon_6^{(4)} - \frac{5}{48}[\epsilon_4^{(1)}, \epsilon_4^{(2)}] + \frac{1}{5760}\big([\epsilon_4^{(0)}, \epsilon_6^{(3)}] - [\epsilon_4^{(1)}, \epsilon_6^{(2)}] + [\epsilon_4^{(2)}, \epsilon_6^{(1)}]\big)$$
$$- \frac{1}{145152}\big([\epsilon_6^{(0)}, \epsilon_6^{(3)}] - [\epsilon_6^{(1)}, \epsilon_6^{(2)}]\big) + \frac{1}{6912}\big([\epsilon_4^{(1)}, [\epsilon_4^{(1)}, \epsilon_4^{(0)}]] + 2[\epsilon_4^{(0)}, [\epsilon_4^{(0)}, \epsilon_4^{(2)}]]\big) + \ldots$$

$\longrightarrow$ infer higher depth terms $\text{sv}(f_a f_b)$, $\text{sv}(f_a f_b f_c)$, $\ldots$ in c^{sv} from $\#(f_a)$

V. Further directions

- unify $B^{\rm sv}$ & $\mathbb{M}^{\rm sv}$ in $J^{\rm eqv} = J_+ B^{\rm sv} \mathbb{M}^{\rm sv} \widetilde{J}_- (\mathbb{M}^{\rm sv})^{-1}$ via zeta elements σ_m

- similar generating-function approach to z-dependent *elliptic MGFs* / sv elliptic polylogarithms and their iterated-integral representation

  [D'Hoker, Green, Pioline 1806.02691; D'Hoker, Kleinschmidt, OS 2012.09198]

  [Basu 2010.08331 & 2210.00648; Hidding, OS, Verbeek 2208.11116]

- explore differential eq's of higher-genus MGFs / modular graph tensors and connections with iterated integrals / higher-genus polylogarithms

  [D'Hoker, Green, Pioline 1712.06135; D'Hoker, OS 2010.00924]

Thank you for your attention !

Backup: counting of MGFs

Bases of dihedral & mod. invariant MGFs at $w = \sum_{i=1}^{R} a_i = \sum_{i=1}^{R} b_i$

$$\mathcal{C}^+\left[\begin{smallmatrix} a_1 & a_2 & \dots & a_R \\ b_1 & b_2 & \dots & b_R \end{smallmatrix}\right] = \frac{(\mathrm{Im}\,\tau)^{a_1+\ldots+a_R}}{\pi^{b_1+\ldots+b_R}} \sum_{p_1,p_2,\ldots,p_R\in\Lambda'} \frac{\delta(p_1+p_2+\ldots+p_R)}{p_1^{a_1}\bar{p}_1^{b_1}\ldots p_R^{a_R}\bar{p}_R^{b_R}}$$

w	0	1	2	3	4	5
basis of	1	$\emptyset$	E_2	E_3	$\mathrm{E}_4\,,\ \mathrm{E}_2^2$	$\mathrm{E}_5\,,\ \mathrm{E}_2\mathrm{E}_3\,,\ \mathcal{C}^+\left[\begin{smallmatrix} 3 & 1 & 1 \\ 3 & 1 & 1 \end{smallmatrix}\right]$
mod. inv.					$\mathcal{C}^+\left[\begin{smallmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \end{smallmatrix}\right]$	$\mathrm{Im}\,\mathcal{C}^+\left[\begin{smallmatrix} 0 & 2 & 3 \\ 3 & 0 & 2 \end{smallmatrix}\right]\,,\ \nabla_\tau\mathrm{E}_2\overline{\nabla}_\tau\mathrm{E}_3$
MGF					$\nabla_\tau\mathrm{E}_2\overline{\nabla}_\tau\mathrm{E}_2$	$\mathrm{Im}\,\mathcal{C}^+\left[\begin{smallmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 0 & 3 \end{smallmatrix}\right]\,,\ \nabla_\tau\mathrm{E}_3\overline{\nabla}_\tau\mathrm{E}_2$

Similarly, $\exists$ 19, 43 and 108 mod. inv. MGFs at weights $w = 6, 7, 8$.

[Gerken, Kleinschmidt, OS 2004.05156]

Basis reductions implemented in MATHEMATICA package.

[Gerken 2007.05476]

Backup: counting of MGFs

Bases of dihedral & mod. invariant MGFs at $w = \sum_{i=1}^{R} a_i = \sum_{i=1}^{R} b_i$

w	0	1	2	3	4	5
basis of mod. inv. MGF	1	$\emptyset$	E_2	E_3	$\mathrm{E}_4\,,\ \mathrm{E}_2^2$ $\mathcal{C}^+\big[\begin{smallmatrix} 2&1&1\\2&1&1\end{smallmatrix}\big]$ $\nabla_\tau \mathrm{E}_2 \overline{\nabla}_\tau \mathrm{E}_2$	$\mathrm{E}_5\,,\ \mathrm{E}_2\mathrm{E}_3\,,\ \mathcal{C}^+\big[\begin{smallmatrix} 3&1&1\\3&1&1\end{smallmatrix}\big]$ $\mathrm{Im}\,\mathcal{C}^+\big[\begin{smallmatrix} 0&2&3\\3&0&2\end{smallmatrix}\big]\,,\ \nabla_\tau \mathrm{E}_2 \overline{\nabla}_\tau \mathrm{E}_3$ $\mathrm{Im}\,\mathcal{C}^+\big[\begin{smallmatrix} 0&1&2&2\\1&1&0&3\end{smallmatrix}\big]\,,\ \nabla_\tau \mathrm{E}_3 \overline{\nabla}_\tau \mathrm{E}_2$

w	2	3	4	5
basis of mod. inv. MGF	$\beta^{\rm eqv}\big[\begin{smallmatrix} 1\\4\end{smallmatrix}\big]$	$\beta^{\rm eqv}\big[\begin{smallmatrix} 2\\6\end{smallmatrix}\big]$	$\beta^{\rm eqv}\big[\begin{smallmatrix} 3\\8\end{smallmatrix}\big]\,,\ \beta^{\rm eqv}\big[\begin{smallmatrix} 1&1\\4&4\end{smallmatrix}\big]$ $\beta^{\rm eqv}\big[\begin{smallmatrix} 2&0\\4&4\end{smallmatrix}\big]$ $\beta^{\rm eqv}\big[\begin{smallmatrix} 0&2\\4&4\end{smallmatrix}\big]$	$\beta^{\rm eqv}\big[\begin{smallmatrix} 4\\10\end{smallmatrix}\big]\,,\ \beta^{\rm eqv}\big[\begin{smallmatrix} 2&1\\4&6\end{smallmatrix}\big]\,,\ \beta^{\rm eqv}\big[\begin{smallmatrix} 1&2\\4&6\end{smallmatrix}\big]$ $\beta^{\rm eqv}\big[\begin{smallmatrix} 0&3\\4&6\end{smallmatrix}\big]\,,\ \beta^{\rm eqv}\big[\begin{smallmatrix} 3&0\\6&4\end{smallmatrix}\big]$ $\beta^{\rm eqv}\big[\begin{smallmatrix} 2&1\\6&4\end{smallmatrix}\big]\,,\ \beta^{\rm eqv}\big[\begin{smallmatrix} 1&2\\6&4\end{smallmatrix}\big]$