

Dits: Flattles: Understand the mathematical structure  
 here: use the mathematical structure.

only hyperlogs (generalize to algebraic integrals ...)

grand picture:

$\mathcal{F}$ : amplitude  $\rightarrow$  f-alphabet  
 tree, shuffle, deconcatenation

$\leadsto$  solving the integrals decomposition algorithm by F. Brown.

Classical  $\mathcal{F}$ : Periods  $\xrightarrow{M\mathcal{F}V}$  f-alphabet  $\xrightarrow{\text{primitives}}$   
 $\langle \{t_1, t_2, t_3, \dots\} \rangle_{\mathbb{Q}[2\pi i]}$

$2\pi i$  period of pure Hodge structure on the topology of the punctured sphere.

examples

$1 \mapsto 1$

$\zeta(2) \mapsto \frac{\pi^2}{6}$

$\zeta(3) \mapsto t_3$

$\zeta(4) \mapsto \pi^4/90$

$\zeta(5) \mapsto t_5$

$\zeta(3)\zeta(5) \mapsto t_3 t_5 + t_5 t_3$

$\zeta(3,5) \mapsto -5 t_3 t_5$

$\zeta(5,3) \mapsto 6 t_3 t_5 + t_5 t_3 - \frac{\pi^8}{9450}$

$\zeta(3,5) + \zeta(5,3) \mapsto t_3 t_5 + t_5 t_3 - \frac{\pi^8}{9450} = \zeta(3)\zeta(5) - \frac{\pi^8}{9450}$

• ambiguity / symmetry.

• Implemented up to weight 32  $\rightarrow$  Do you know higher weights?  
 Euler sums wt 21

Here: functions

example multiple polylogs

$$Li_w(z) \sim \sum_{1 \leq k_1 < \dots < k_r} \frac{z^{k_r}}{k_1^{w_1} \dots k_r^{w_r}}$$

w word in letters 0,1.

Decomposition algorithm needs an algebra basis -2-

→  $Li_w(z)$ ,  $w$  Lyndon.

- Special case; every  $Li_w(z)$  is the shuffle product of the algebra basis → no primitives (see algebra)
- conversion is particularly simple and needs no numerical calculations. (except for TTUs)

primitives:  $\log(z) \xrightarrow{4} z_0$        $\log(1-z) \xrightarrow{4} z_1$

$Li_{10}(z) \rightarrow z_1 z_0$	$\log^2(z) \rightarrow 2 z_0 z_0$
$Li_{1010}(z) \xrightarrow{4} z_1 z_0 z_1 z_0 - 2 z_1 z_0^2$	$\uparrow$ S(3) letter from M7V

$Li_w(z) \leftrightarrow \langle z_0, z_1, z_3, z_5, \dots \rangle @$

Not surjective

$z_1 z_0 z_1 z_0$  alone has no meaning as a function.

$Li_{1010}(z) + 2 Li_1(z) \cdot S(3) \xrightarrow{4} z_1 z_0 z_1 z_0 + 2 z_3 z_1$

Where does the  $\frac{1}{3}$  come from?

Monodromy

$M_1 Li_{10}(z) = Li_{10}(z) + 2\pi i \log(z)$
$M_1 Li_{1010}(z) = Li_{1010}(z) + 2\pi i Li_{1010}(z) - 4\pi i S(3) Li_0(z) - 4\pi i S(3)$

with  $M_1 = e^{2\pi i w_1}$   $w_1 \mapsto \log_1 z$

$w_1^{DR} = w_1 \pmod{2\pi i} = \frac{\text{disc}}{2\pi i} \pmod{2\pi i}$

$w_1^{DR} Li_{10}(z) = \log z \xrightarrow{4} z_0$

$w_1^{DR} Li_{1010}(z) = Li_{1010}(z) - 2 S(3) \xrightarrow{4} z_0 z_1 z_0 - 2 z_3$

$w_1^{DR} \xrightarrow{4} \int_{z_1}^z \frac{1}{z}$  (clips off the first letter  $z_1$ )

$z_2 \xrightarrow{4} \frac{1}{z} \int_{z_0}^{DR} + \frac{1}{z-1} \int_{z_1}^{DR}$   
 clips off the rightmost letter

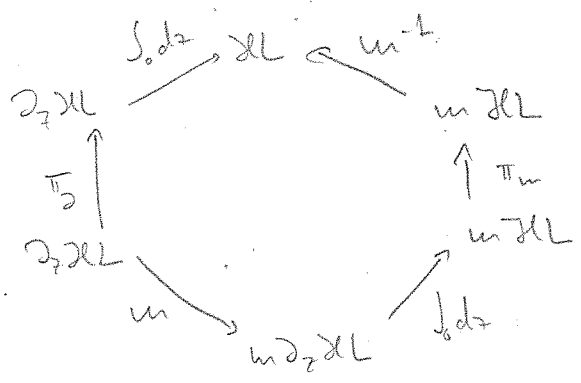
Intrinsic constructions of  $f$ -hypology

generalization  $m_1 \rightarrow m = \sum_a a \otimes m_a$

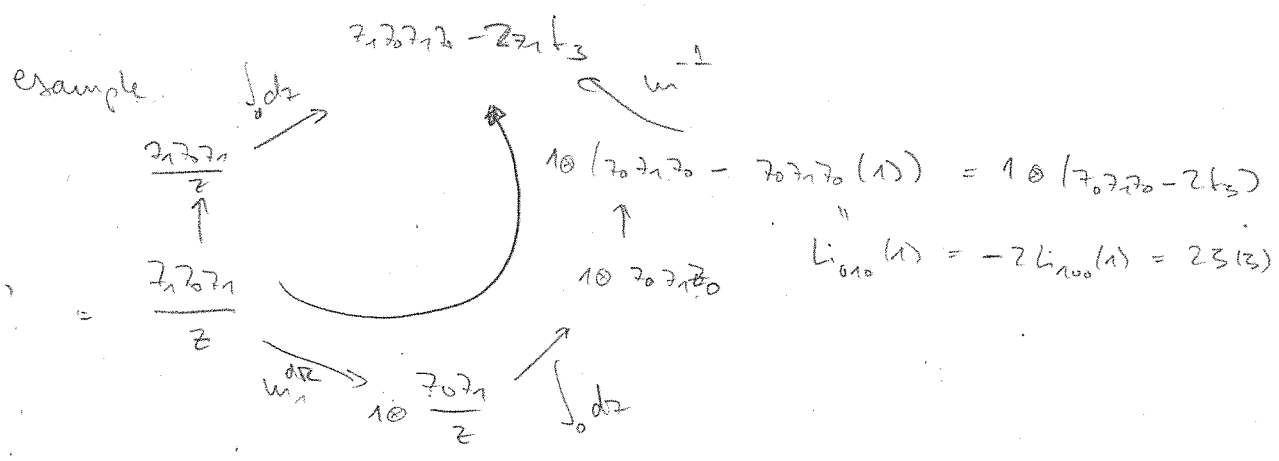
$\Pi_0$  kills residues

$\Pi_m$  kills evaluations

$m^{-1}$  inverts  $m$ .



Commutative (Thm).



unshuffle  $f$ 's from the left  $\rightarrow$  gives  $\int dz$



Benefits of  $f$ -hypology

- \* simpler formula for monodromy
- \* simpler evaluation with the decomposition algorithm

example:  $eval_{z=1/2}(z_1 z_2 z_3 - 2z_1 z_3) = \overbrace{\left[ \frac{1}{1+1+1+1} - \frac{\pi^2}{12} \frac{1}{1+1} + \frac{1}{3} \frac{1}{3+1} + \frac{1}{3} \frac{1}{3+1} + \frac{\pi^4}{1440} \right]}^{eval_{z=1/2}(z_1 z_2 z_3) / z^2}$

$eval_{z=1/2} z_1 \left[ \frac{dR}{dz} \frac{eval(z_0 z_1 z_2 - 2z_3)}{z=1/2} \right] / z^2 \uparrow$  numerical

\* single-valued map

$$SU: W \rightarrow \sum_{w=uv} \bar{u} \cup v$$

$\bar{u}$  is  $u$  in reversed order and complex conjugated.

$$SV Li_{10} = z_1 z_0 + \bar{z}_1 \cup z_0 + \bar{z}_0 \bar{z}_1$$

$$= (z_0 + \bar{z}_0) \bar{z}_1 + (z_1 + \bar{z}_1) z_0 \quad \text{trivial in } \mathbb{R}^2$$

$$4iD(\gamma) = \underset{\substack{\uparrow \\ \text{Black - loop}}}{SV(L_{10} - L_{01})} = (z_0 + \bar{z}_0)(\bar{z}_1 - z_1) + (z_1 + \bar{z}_1)(z_0 - \bar{z}_0)$$

$$SV Li_{1000} = \text{Terms with 4 } z\text{'s} - 2SV(z_1 z_2)$$

$$= \dots - 2z_1 z_2 - 2\bar{z}_1 \cup z_2 - 2z_2 \bar{z}_1$$

$$= \dots - 2(z_1 + \bar{z}_1)z_2 - 4z_2 \bar{z}_1$$

$\hookrightarrow -4S(z) Li_1(\bar{z})$  term in  $SV Li_{1000}(\gamma)$

GSVHS

$$\int_{SV} \frac{\log(z\bar{z})}{z - \bar{z}^{-1}} dz$$

$\uparrow$   
let  $z = e^{i\theta} = e^{i \log(1-\bar{z})}$

$$\int_{SV} \frac{4iD(\gamma)}{z - \bar{z}} dz$$

$\uparrow$   
let  $z = e^{i\theta} = e^{i \log(1-\frac{z}{2})}$

$\hookrightarrow f$ -alphabet  $\downarrow 4$

$$(z_0 + \bar{z}_0) z_{\frac{z}{2} - 1}$$

$\downarrow 4$

$$(z_0 + \bar{z}_0)(\bar{z}_1 z_{\frac{z}{2}} - z_1 z_{\frac{z}{2}} + z_1 \bar{z}_1 - z_1 \bar{z}_0) + (z_1 + \bar{z}_1)(z_0 z_{\frac{z}{2}} - \bar{z}_0 z_{\frac{z}{2}} + \bar{z}_0 \bar{z}_1 - \bar{z}_0 \bar{z}_0)$$

Single-valued integration:

• Intrinsic construction by 2 commutative hexagons.

$\rightarrow$  Can one use  $f$ -hypercyls for a high performance implementation with FORTRAN (Duc) or C++?

$\rightarrow$  Do  $f$ -hypercyls generalize to  $f$ -functions?  
How?