f-functions (f-hyperlogs) 13.1.21 IHES / ZOOM

Nils Matthes: Understand the mathematical structure

him: use the mathematical structure.

<u>only</u> hyperlogs (generalize to algebraic integrals ...)

grand picture:

4: amplitude $\longrightarrow$ f-alphabet

free, shuffle, deconcatenation

$\rightsquigarrow$ solving the integrals

decomposition algorithm by F. Brown.

Classical 4: Periods $\mapsto$ f-alphabet primitives

MZV $\mapsto$ $\langle\{f_3, f_5, f_7, \ldots\}^*\rangle_{\mathbb{Q}[2\pi i]}$

$2\pi i$ period of pure Hodge structure on the topology of the punctured sphere.

<u>examples</u>

$$1 \mapsto 1$$
$$\zeta(2) \mapsto \frac{\pi^2}{6}$$
$$\zeta(3) \mapsto f_3$$
$$\zeta(4) \mapsto \pi^4/90$$
$$\zeta(5) \mapsto f_5$$
$$\zeta(3)\zeta(5) \mapsto f_3 f_5 + f_5 f_3$$
$$\zeta(3,5) \mapsto -5 f_3 f_5$$
$$\zeta(5,3) \mapsto 6 f_3 f_5 + f_5 f_3 - \frac{\pi^8}{9450}$$
$$\zeta(3,5)+\zeta(5,3) \mapsto f_3 f_5 + f_5 f_3 - \frac{\pi^8}{9450} = \zeta(3)\zeta(5) - \frac{\pi^8}{9450}.$$

- ambiguity / symmetry.
- Implemented up to weight 32 → Jos Vermaseren higher weights?
  Euler sums wt 21

<u>Here: functions</u>

example multiple polylogs $\mathrm{Li}_w(z) \sim \sum_{1 \le k_1 < \ldots < k_r} \frac{z^{k_r}}{k_1^{n_1} \cdots k_r^{n_r}}$

w word in letters 0,1.

Decomposition algorithm needs an algebra basis

→ $\mathrm{Li}_w(z)$, w Lyndon.

- Special case: every $\mathrm{Li}_w(z)$ is the shuffle product of the algebra basis → no primitives (free algebra)

  → conversion is particularly simple and needs no numerical calculations (except for MZVs)

primitives: $\log(z) \xrightarrow{\phi} z_0$ $\qquad \log(1-z) \xrightarrow{\phi} z_1$

$$\boxed{\begin{aligned} \mathrm{Li}_{10}(z) &\longrightarrow z_1 z_0 \\ \mathrm{Li}_{1010}(z) &\xrightarrow{\phi} z_1 z_0 z_1 z_0 - 2 z_1 f_3 \end{aligned}} \qquad \log^2(z) \to 2\, z_0 z_0 .$$

$\zeta(3)$ letter from MZV

$$\mathrm{Li}_w(z) \hookrightarrow \langle z_0, z_1, f_3, f_5, \dots \rangle_{\mathbb{Q}} .$$

not surjective

$z_1 z_0 z_1 z_0$ alone has no meaning as a function.

$$\mathrm{Li}_{1010}(z) + 2\,\mathrm{Li}_1(z)\cdot\zeta(3) \xrightarrow{\phi} z_1 z_0 z_1 z_0 + 2 f_3 z_1$$

Where does the f_3 come from?

monodromy

$$\boxed{\begin{aligned} \mathcal{M}_1 \mathrm{Li}_{10}(z) &= \mathrm{Li}_{10}(z) + 2\pi i \log(z) \\ \mathcal{M}_1 \mathrm{Li}_{1010}(z) &= \mathrm{Li}_{1010}(z) + 2\pi i\, \mathrm{Li}_{010}(z) - 4\pi i\, \zeta(2)\, \mathrm{Li}_0(z) - 4\pi i\, \zeta(3) \end{aligned}}$$

write $\mathcal{M}_1 = e^{2\pi i\, m_1}$ $\qquad m_1 \sim \mathrm{res}_1 \partial_z$

$$m_1^{dR} = m_1 \bmod 2\pi i = \frac{\mathrm{disc}_1}{2\pi i} \bmod 2\pi i .$$

$$m_1^{dR} \overset{z_1 z_0}{\mathrm{Li}_{10}(z)} = \log z \xrightarrow{\phi} z_0$$

$$m_1^{dR} \overset{z_1 z_0 z_1 z_0 - 2 z_1 f_3}{\mathrm{Li}_{1010}(z)} = \mathrm{Li}_{010}(z) - 2\zeta(3) \xrightarrow{\phi} z_0 z_1 z_0 - 2 f_3$$

$$m_1^{dR} \xrightarrow{\phi} \delta^{B}_{z_1} \qquad \text{(clips off the first letter } z_1\text{)}$$

$$\partial_z \xrightarrow{\phi} \frac{1}{z}\, \delta^{dR}_{z_0} + \frac{1}{z-1}\, \delta^{dR}_{z_1}$$

clips off the rightmost letter

Intrinsic construction of f-hyperlogs

generalize $m_1 \to m = \sum_a a \otimes m_a$

π_∂ kills residues

π_m kills evaluations

m^{-1} inverts m.

$$\begin{tikzcd}
 & \mathcal{HL} & \\
\partial_z\mathcal{HL} \arrow[ur, "\int_0 dz"] & & m\mathcal{HL} \arrow[ul, "m^{-1}"'] \\
\partial_z\mathcal{HL} \arrow[u, "\pi_\partial"] \arrow[dr, "m"'] & & m\mathcal{HL} \arrow[u, "\pi_m"'] \\
 & m\partial_z\mathcal{HL} \arrow[ur, "\int_0 dz"'] &
\end{tikzcd}$$

Commutes (Thm).

example

$$\begin{tikzcd}
 & z_1z_0z_1z_0 - 2z_1f_3 & \\
\frac{z_1z_0z_1}{z} \arrow[ur, "\int_0 dz"] & & 1\otimes(z_0z_1z_0 - z_0z_1z_0(1)) = 1\otimes(z_0z_1z_0 - 2f_3) \arrow[ul, "m^{-1}"'] \\
f: \frac{Li_{101}(z)}{z} = \frac{z_1z_0z_1}{z} \arrow[u] \arrow[uur, bend right] \arrow[dr, "m_1^{dz}"'] & & 1\otimes z_0z_1z_0 \arrow[u] \\
 & 1\otimes\frac{z_0z_1}{z} \arrow[ur, "\int_0 dz"'] &
\end{tikzcd}$$

$z_0z_1z_0(1) = Li_{010}(1) = -2Li_{100}(1) = 2\zeta(3)$

unshuffle f's from the left $\circlearrowright$ gives $\int_0 dz$

Benefits of f-hyperlogs

* simpler formula for monodromy.
* simpler evaluation with the decomposition algorithm

example: $\mathrm{eval}_{z=1/2}(z_1z_0z_1z_0 - 2z_1f_3) = \overbrace{f^2_1f^2_1f^2_1f^2_1 - \frac{\pi^2}{12}f^2_1f^2_1 + \frac{1}{3}f^2_3f^2_1}^{\mathrm{eval}_{z=1/2}(z_1z_0z_1)f^2_1}$

$+\frac{1}{3}f^2_1f^2_3 + \cancel{\frac{1}{3}f^2_3f^2_1} + \frac{\pi^4}{1440}$

$\mathrm{eval}_{z=1/2}\, z_1\left[\delta^{dR}_{f^2_3}\,\mathrm{eval}_{z=1/2}(z_0z_1z_0 - 2f_3)\right]f^2_3$

↑ numerical

* single-valued map

$$sv: w \to \sum_{w=uv} \bar{\tilde{u}} ⧢ v$$

$\bar{\tilde{u}}$ is u in reversed order and complex conjugated.

$$sv\,Li_{10} = z_1 z_0 + \bar{z}_1 ⧢ z_0 + \bar{z}_0 \bar{z}_1$$

$$= (z_0+\bar{z}_0)\bar{z}_1 + (z_1+\bar{z}_1) z_0 \qquad \text{trivial mod}^{dR}$$

$$4iD(z) = sv(L_{10} - L_{01}) = (z_0+\bar{z}_0)(\bar{z}_1 - z_1) + (z_1+\bar{z}_1)(z_0-\bar{z}_0)$$

↑ Bloch-Wigner

$sv\,Li_{1010}$ = Terms with 4 z's $- 2\,sv(z_1 f_3)$

$$= \dots - 2 z_1 f_3 - 2\bar{z}_1 ⧢ f_3 - 2 f_3 \bar{z}_1$$

$$= \dots - 2(z_1+\bar{z}_1) f_3 - 4 f_3 \bar{z}_1$$

↳ $-4\zeta(3)\, Li_1(\bar{z})$ term in $sv\,Li_{1010}(z)$

<u>GSVHs</u> $\quad \int_{sv} \frac{\log(z\bar{z})}{z-\bar{z}^{-1}}\, dz \qquad \int_{sv} \frac{4iD(z)}{z-\bar{z}}\, dz$

↑ letter $z_{\bar{z}^{-1}}$ "≙" $\log(1-z\bar{z})$ $\qquad$ ↑ letter $z_{\bar{z}}$ "≙" $\log(1-\frac{z}{\bar{z}})$

⇝ f-alphabet $\qquad \downarrow 4 \qquad \downarrow 4$

$$(z_0+\bar{z}_0)\, z_{\bar{z}^{-1}}$$

$$(z_0+\bar{z}_0)(\bar{z}_1 z_{\bar{z}} - \bar{z}_1 z_{\bar{z}} + z_1\bar{z}_1 - z_1\bar{z}_0)$$
$$+ (z_1+\bar{z}_1)(z_0 z_{\bar{z}} - \bar{z}_0 z_{\bar{z}} + z_0\bar{z}_1 - z_0\bar{z}_0)$$

single-valued integration:

- intrinsic construction by 2 commutative hexagons.

→ Can one use f-hyperlogs for a high performance implementation with FORM (Jos) or C++?

→ Do f-hyperlogs generalize to f-functions? How?