Resurgence and generalised Eisenstein series in string theory

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Appearances of modular symmetry in string theory

Introduction

Invariance under $SL(2,\mathbb{Z})$ plays a central theme in string theory and two-dimensional conformal field. There are (at least) 2 different ways how the modular group appears in this context:

- In perturbation theory at genus-1 the amplitudes are modular invariant functions, since $SL(2,\mathbb{Z})$ is the mapping-class of the torus. This leads to the introduction of Modular Graph Functions (MGFs).
- In Type IIB string theory there is a non-perturbative duality. The action of this duality on the axio-dilaton τ is the standard action of the modular group on the upper half-plane.

In this talk, we focus on a space of functions called generalised Eisenstein series, which naturally appear in both contexts described above.

4 graviton scattering in Type II theories

In string theory, we are interested in calculating the scattering amplitude of massless string excitations. Particularly, we look at the amplitude of 4 gravitons. A prescription for doing this in perturbation theory is by integrating over the conformal structures of genus-h Riemann surfaces. This results in an asymptotic series

$$\mathcal{A}(\epsilon_i, k_i) \sim \kappa_{10}^2 \mathcal{R}^4 \sum_{h=0}^{\infty} g_s^{2h-2} \mathcal{A}^{(h)}(s_{ij}) \tag{1}$$

in the string coupling g_s. Here κ_{10} is related to Newton's gravitational constant in 10 dimensions, $\mathcal{R}^4$ is a contraction of 4 linearised Riemann tensors and $s_{ij} = -\frac{\alpha'}{4} k_i \cdot k_j$ are dimensionless Mandelstam invariants. The form of the amplitude is largely fixed by supersymmetry.

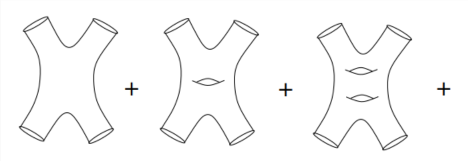

Figure 1: Some of the worldsheet topologies that contribute to the 4-point amplitude.

Amplitude on the torus

The genus-0 amplitude $\mathcal{A}^{(0)}(s_{ij})$ can be evaluated exactly, but on the torus, a direct calculation is no longer possible. Instead, we may express the amplitude as

$$\mathcal{A}^{(1)}(s_{ij}) = \frac{\pi}{16}\int_{\mathcal{F}} \frac{|d\tau|^2}{\mathrm{Im}(\tau)^2}\mathcal{B}^{(1)}(s_{ij}|\tau), \tag{2}$$

where $\mathcal{F}$ is the fundamental domain of $SL(2,\mathbb{Z})$ and

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\Sigma^4}\prod_{i=1}^{4}\frac{d^2 z_i}{\mathrm{Im}(\tau)}\exp\left\{\sum_{i<j} s_{ij}G(z_i - z_j|\tau)\right\}, \tag{3}$$

with $\Sigma = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ the torus and $G(z|\tau)$ the Green's function on it. When equation (3) is expanded in s_{ij} it tells us about the low energy behavior of the amplitude and allows for the calculation of corrections to supergravity.

Modular Graph Functions

In order to analyse the properties of (3), a new class of modular functions associated with directed graphs Γ was introduced [D'Hoker, Green, Gurdogan, Vanhove 2015]

$$C_\Gamma[A](\tau) = \left(\frac{\tau_2}{\pi}\right)^w \sum_{p_1,\ldots,p_R \in \Lambda'} \frac{1}{|p_1|^{2a_1} \ldots |p_R|^{2a_R}} \prod_{v=1}^{V} \delta\Big(\sum_{s=1}^{R} \Gamma_{vs} p_s\Big), \qquad (4)$$

where V is the number of vertices, R is the number of edges, Γ_{vs} is the connectivity matrix, $\Lambda' = (\mathbb{Z} + \tau\mathbb{Z}) \backslash \{0\}$ a lattice that is summed over, $A = (a_1, \ldots, a_R)$ a collection of weights associated to edges and $w = \sum_{i=1}^{R} a_i$ the total weight. The weight provides a grading on the space of MGFs and controls the kind of interaction the graph contributes to at low energy.

MGFs as low energy corrections

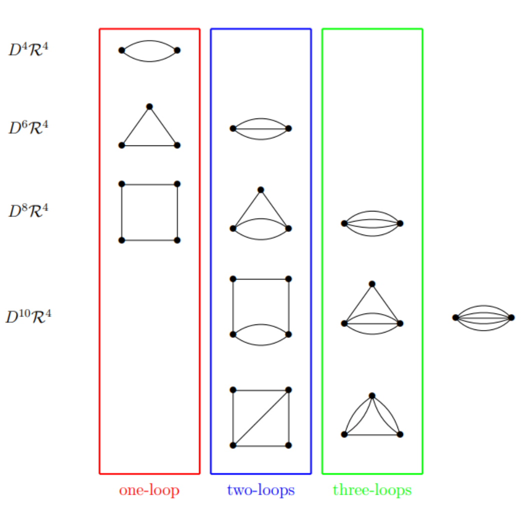

Figure 2: Organisation of MGFs by loop order and weight, the weight gives a grading into which type of low-energy interaction the graph contributes to (from [D'Hoker, Kaidi 2022]).

Two loop MGFs

At two loops every connected MGF can be written as

$$C_{a,b,c}(\tau) = \left(\frac{\tau_2}{\pi}\right)^w \sum_{p_1,p_2,p_3\in\Lambda'} \frac{\delta(p_1+p_2+p_3)}{|p_1|^{2a}|p_2|^{2b}|p_3|^{2c}} \tag{5}$$

with $w = a+b+c$ the weight of the MGF. It is possible to show that all two-loop MGFs are contained in a space of generalised Eisenstein series $\mathcal{E}\left(\lambda; m, k\right)$ [D'Hoker, Green, Vanhove 2015], defined by the differential equation

$$(\Delta - \lambda(\lambda-1))\mathcal{E}\left(\lambda; m, k\middle|\tau\right) = \mathcal{E}_m(\tau)\mathcal{E}_k(\tau) \tag{6}$$

with $\Delta = 4\tau_2^2\partial_\tau\partial_{\bar\tau}$, and the non-holomorphic Eisenstein given by

$$\mathcal{E}_s(\tau = \tau_1 + i\tau_2) = \sum_{(m,n)\neq(0,0)} \frac{(\tau_2/\pi)^s}{|m+n\tau|^{2s}}. \tag{7}$$

The weights $m, k \in \mathbb{N}_{\geq 2}$ and the corresponding total weight is given by $w = k+m$. At fixed values of k, m the eigen-parameter is constrained to lie in the bounded spectrum

$$\lambda \in \mathrm{Spec_1(k,m)} = \{|\mathrm{k}-\mathrm{m}|+2, |\mathrm{k}-\mathrm{m}|+4, ..., \mathrm{k}+\mathrm{m}-2\} \tag{8}$$

This space is larger than 2-loop MGFs, also including cuspidal objects.

Examples

Weight	MGFs	Eisensteins	Relations
3	$C_{1,1,1}$	–	$C_{1,1,1} = \mathcal{E}_3 + \zeta_3$
4	$C_{2,1,1}$	$\mathcal{E}(2;2,2)$	$C_{2,1,1} = -\mathcal{E}(2;2,2) + \frac{9}{10}\mathcal{E}_4$
5	$C_{2,2,1}$, $C_{3,1,1}$	$\mathcal{E}(2;2,3)$	$C_{2,2,1} = \frac{2}{5}\mathcal{E}_5 + \frac{\zeta_5}{30}$, $C_{3,1,1} = -4\mathcal{E}(3;2,3) + \frac{43}{35}\mathcal{E}_5 - \frac{\zeta_5}{60}$
6	$C_{2,2,2}$, $C_{3,2,1}$, $C_{4,1,1}$	$\mathcal{E}(4;2,4)$, $\mathcal{E}(4;3,3)$, $\mathcal{E}(2;3,3)$	$C_{2,2,2} = -\frac{12}{5}\mathcal{E}(2;3,3) + \frac{72}{5}\mathcal{E}(4;3,3) - \frac{9}{7}\mathcal{E}_6$, $C_{3,2,1} = -\frac{2}{5}\mathcal{E}(2;3,3) - \frac{18}{5}\mathcal{E}(4;3,3) + \frac{11}{4}\mathcal{E}_6$, $C_{4,1,1} = \frac{2}{5}\mathcal{E}(2;3,3) - \frac{2}{5}\mathcal{E}(4;3,3) - 6\mathcal{E}(4;2,4) + \frac{167}{126}\mathcal{E}_6$
7	$C_{3,2,2}$, $C_{3,3,1}$, $C_{4,2,1}$, $C_{5,1,1}$	$\mathcal{E}(5;3,4)$, $\mathcal{E}(3;3,4)$, $\mathcal{E}(5;2,5)$	$C_{3,2,2} = -\frac{24}{7}\mathcal{E}(3;3,4) + \frac{108}{7}\mathcal{E}(5;3,4) - \frac{23}{21}\mathcal{E}_7 + \frac{\zeta_7}{630}$, $C_{3,3,1} = \frac{24}{7}\mathcal{E}(3;3,4) - \frac{108}{7}\mathcal{E}(5;3,4) + \frac{32}{21}\mathcal{E}_7 + \frac{\zeta_7}{420}$, $C_{4,2,1} = -\frac{24}{7}\mathcal{E}(3;3,4) - \frac{18}{7}\mathcal{E}(5;3,4) + \frac{16}{21}\mathcal{E}_7 - \frac{\zeta_7}{630}$, $C_{5,1,1} = -\frac{12}{7}\mathcal{E}(3;3,4) - \frac{12}{7}\mathcal{E}(5;3,4) - 8\mathcal{E}(5;2,5) + \frac{661}{462}\mathcal{E}_7 + \frac{\zeta_7}{2520}$, $C_{3,3,1} + C_{3,2,2} = \frac{3}{7}\mathcal{E}_7 + \frac{\zeta_7}{252}$
8	$C_{3,3,2}$, $C_{4,2,2}$, $C_{4,3,1}$, $C_{5,2,1}$, $C_{6,1,1}$	$\mathcal{E}(6;4,4)$, $\mathcal{E}(6;3,5)$, $\mathcal{E}(6;2,6)$, $\mathcal{E}(4;4,4)$, $\mathcal{E}(4;3,5)$, $\mathcal{E}(2;4,4)$	First weight for which the space of Eisenstein series is larger than that of MGFs

At low energies, string theory reduces to a theory of supergravity. It is described by a Lagrangian involving all of the massless fields; in particular, gravitons are to the first approximation described by general relativity. But there are systematic corrections encoded in additional terms [Green, Gutperle 1997; Green, Vanhove 2005]

$$\mathcal{L}_{\mathrm{eff}} = (\alpha')^{-4} g_s^{-2} R + \mathcal{E}_{\frac{3}{2}}(\tau)(\alpha')^{-1} g_s^{-\frac{1}{2}} \mathcal{R}^4 + \mathcal{E}_{\frac{5}{2}}(\tau)\alpha' g_s^{\frac{1}{2}} D^4 \mathcal{R}^4 - \pi^3 \mathcal{E}\left(4; \frac{3}{2}, \frac{3}{2} \middle| \tau\right) (\alpha')^2 g_s D^6 \mathcal{R}^4 + \dots, \quad (9)$$

In this case, we see the appearance of generalised Eisenstein series $\mathcal{E}(\lambda; s_1, s_2)$ with half-integer weights s_1, s_2. It would be interesting to see if the spectrum extends further, but only the first four corrections in (9) are protected by supersymmetry.

A second spectrum for half-integral Eisenstein series

We use the fact there is a duality between Type IIB string theory in an $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ super Yang-Mills theory in 4-dimensions. Integrated correlators [Binder, Chester, Pufu, Wang 2019] give information about the class of functions that appear in relation to higher corrections $D^{2k}\mathcal{R}^4$ for $k > 3$. These are generalised Eisenstein series $\mathcal{E}\left(\lambda; s_1, s_2 \middle| \tau\right)$ with $s_1, s_2 \in \mathbb{N} + \frac{1}{2}$ and

$$\lambda \in \mathrm{Spec}_2(s_1, s_2) := \{s_1 + s_2 + 1, s_1 + s_2 + 3, ...\} \tag{10}$$

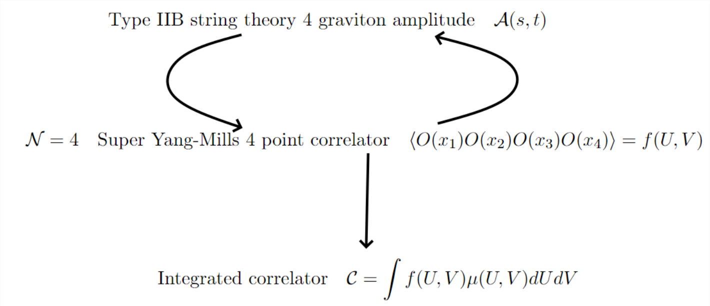

Figure 3: In the process of integrating, information about the observable is lost.

Resurgence techniques

Solving Laplace equation by using Poincaré series

We now use resurgence to analyse generalised Eisenstein series with integer weights. This is done by embedding them in an ambient space of functions that have asymptotic tails at the cusp. To proceed, remember we want to solve the equation

$$(\Delta - \lambda(\lambda-1))\mathcal{E}\left(\lambda; m, k \middle| \tau\right) = \mathcal{E}_m(\tau)\mathcal{E}_k(\tau) \tag{11}$$

with $m, k \in \mathbb{N}_{\geq 2}$, $k \geq m$ and $\lambda \in \mathrm{Spec}_1(\mathrm{k}, \mathrm{m})$. Write the answer using Poincaré series

$$\mathcal{E}\left(\lambda; m, k \middle| \tau\right) = \sum_{\gamma \in \mathrm{B}(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} e(\lambda; m, k | \gamma \cdot \tau) \tag{12}$$

of some periodic seed functions $e(\lambda; m, k|\tau)$ over the quotient of the modular group by its Borel subgroup

$$\mathrm{B}(\mathbb{Z}) := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| \, n \in \mathbb{Z} \right\} \subset \mathrm{SL}(2, \mathbb{Z}). \tag{13}$$

The reason for doing this is the generic reduction in complexity when comparing a modular function and its seed. For example, the seed function of $\mathcal{E}_s(\tau)$ is proportional to τ_2^s.

Solving a simplified equation for the seed

After folding one of the Eisenstein series, in view of the integer character of the weights, we are left with a simpler equation for the seed function

$$(\Delta - \lambda(\lambda-1))e(\lambda; m, k|\tau) = \frac{(-1)^{k+1}B_{2k}}{(2k)!}(4\pi\tau_2)^k \mathcal{E}_m(\tau), \tag{14}$$

where B_{2k} are Bernoulli numbers. By Fourier expanding the Eisenstein series and imposing appropriate boundary conditions, this is solved by $e(\lambda; m, k|\tau) = \sum_{n\in\mathbb{Z}} c_n(\tau_2)e^{2\pi i n\tau_1}$ with $c_0(\tau_2)$ a polynomial and

$$c_n(\tau_2) = (-1)^k \frac{2B_{2k}}{(2k)!\Gamma(m)}\sigma_{1-2m}(|n|)|n|^{m-k-1} \sum_{\ell=k-m+1}^{k-1} g_{m,k,\ell,\lambda}(4\pi|n|\tau_2)^\ell e^{-2\pi|n|\tau_2} \tag{15}$$

for $n \neq 0$ where $g_{m,k,l,\lambda}$ are rational numbers. This formula motivates us to introduce a larger space of functions with asymptotic expansions as $y \to \infty$.

A deformation of the problem

To generate an infinite asymptotic tail, we define a new modular function [Dorigoni, Kleinschmidt 2019]

$$\Phi(a,b,r|\tau) = \sum_{\gamma\in \mathrm{B}(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \Big[\sum_{m\neq 0} \sigma_a(|m|)|m|^b \tau_2^r e^{-2\pi|m|\tau_2 + 2\pi i m\tau_1} \Big]_\gamma , \qquad (16)$$

where $\sigma_a(m) = \sum_{d|m} d^a$ is a divisor-sum and $[\ldots]_\gamma$ indicates standard action of γ on everything in the brackets. We find that for generic values of (a,b,r) the asymptotic series as $y := \pi\tau_2 \to \infty$ of the Fourier zero-mode of $\Phi(a,b,r|\tau) = \sum_{n\in\mathbb{Z}} a_n(y) e^{2\pi i n\tau_1}$ is given by

$$a_0(y) \sim \alpha_1 y^{2+b-r} + \alpha_2 y^{2+a+b-r} + I_{asy}(a,b,r;y), \qquad (17)$$

where α_1, α_2 are constants and the asymptotic tail is

$$I_{asy}(a,b,r;y) = \frac{(4y)^{2+a+b-r}\pi^{2r-a-2b-2}}{2^{a+2b}\Gamma(r)\zeta(2r-a-2b-1)} \sum_{n>0} \sigma_a(n)\sigma_{a+2b+2-2r}(n)$$
$$\sum_{m\geq 0} \frac{\Gamma(m+a+b+1)}{(4ny)^{m+a+b+1}} \frac{\Gamma(2r+m-1)\Gamma(1+b+m)}{\Gamma(m+r)\Gamma(m+1)} \qquad (18)$$
$$\times \Big[(-1)^m \cos\Big(\frac{a\pi}{2}\Big) - \cos\Big(\frac{(a+2b)\pi}{2}\Big) \Big].$$

A crash course in resurgence

In physics and mathematics, one often encounters divergent series which are *asymptotic* to an answer, but don't provide an unambiguous definition. Resurgence is a framework of how to make sense of such series. Let $I(y) = \sum_{n=0}^{\infty} a_n y^{-n-1}$ be a formal series of Gevrey order-1 ($|a_n| < AB^n n!$ for some A, B), then define the Borel transform of this series as

$$\mathcal{B}[I](t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, \tag{19}$$

which converges to a holomorphic function in an open disk. To make contact with the original series, we define a directional Laplace transform that brings us back to the original variable

$$\mathcal{S}_\theta[I](y) = \int_0^{e^{i\theta}\infty} e^{-yt} \mathcal{B}[I](t) dt. \tag{20}$$

We have introduced an angle parameter θ, since the Borel transform has singularities in the t plane and not every direction of integration is valid. The newly constructed function $\mathcal{S}_\theta[I](y)$ has asymptotic series $I(y)$ as $y \to \infty$.

Recovering non-perturbative information

Because of the presence of singularities, this procedure defines multiple possible resummations. To (partially) remedy this shortcoming, we instead work with *transseries*. Consider a very basic example with a single singularity at $\omega \in \mathbb{R}_{>0}$. If the singularity in the vicinity of ω is a simple pole or a logarithmic branch cut, the function is said to be a simple resurgent function. Then define a new formal series

$$\tilde{I}(\sigma, y) = I(y) + \sigma e^{-\omega y} I_\omega(y), \tag{21}$$

where $I_\omega(y)$ is a new asymptotic series and σ is an arbitrary parameter encoding the modified expression. As one moves across the Stokes ray $\theta = 0$ the discontinuity in the Laplace transform is canceled by a shift in σ. More precisely, in the case of a simple resurgent function, the discontinuity takes the form of an exponentially suppressed function

$$\lim_{\epsilon \to 0^+} \left(\mathcal{S}_\epsilon - \mathcal{S}_{-\epsilon}\right)[I](y) = -ie^{-\omega y} \mathcal{S}_0[I_\omega](y). \tag{22}$$

Then a shift $\mathrm{Im}(\sigma) \to \mathrm{Im}(\sigma) + 1$, as you move from below to above the real axis, would cancel the discontinuity.

Median resummation

In a physics context, we are usually interested in observables that are real and the generalised Eisenstein series are also real analytic. A natural way how to resum the asymptotic series associated with such an observable is median resummation

$$\mathcal{S}_{med}[I](y) = \lim_{\epsilon\to 0^+} \mathcal{S}_\epsilon[I](y) + \frac{i}{2} e^{-\omega y} \mathcal{S}_0[I_\omega](y)$$
$$= \lim_{\epsilon\to 0^-} \mathcal{S}_\epsilon[I](y) - \frac{i}{2} e^{-\omega y} \mathcal{S}_0[I_\omega](y) \qquad (23)$$

Figure 4: In order to resum a divergent series, we need to consider the discontinuity along a Stokes ray.

Resumming of $I_{asy}(a, b, r; y)$

Since $I_{asy}(a, b, r; y)$ is a factorially divergent series, we compute the Borel transform in the variable $4ny$, which is given by [Dorigoni, Kleinschmidt, RT 2022]

$$B(t) = t^{a+b}\left[{}_2F_1(2r-1, 1+b; r|-t)\cos\left(\frac{a\pi}{2}\right) - {}_2F_1(2r-1, 1+b; r|t)\cos\left(\frac{(a+2b)\pi}{2}\right)\right], \tag{24}$$

so that the directional Laplace transform is calculated by

$$\mathcal{S}_\theta\left[I_{asy}(a, b, r; y)\right] = \frac{(4y)^{2+a+b-r}\pi^{2r-a-2b-2}}{2^{a+2b}\Gamma(r)\zeta(2r-a-2b-1)}\frac{\Gamma(2r-1)\Gamma(1+b)}{\Gamma(r)}$$
$$\sum_{n>0}\sigma_a(n)\sigma_{a+2b+2-2r}(n)\int_0^{e^{i\theta}\infty} e^{-4nyt}B(t)\mathrm{d}t\,. \tag{25}$$

Observe this has branch point singularities at $t = 1, -1$, but we only pick up the non-perturbative terms from the singularity at $t = 1$, since we expect these contributions to be exponentially suppressed and are interested in $y > 0$.

The discontinuity across $\arg(t) = 0$ captures exponentially suppressed terms in y. Since the discontinuity of the hypergeometric function is well known, we employ median resummation to find that the exact asymptotics of the Fourier zero-mode is given by

$$a_0(y) \sim \alpha_1 y^{2+b-r} + \alpha_2 y^{2+a+b-r} + I_{asy}(a,b,r;y) + \mathrm{NP}(a,b,r;y), \tag{26}$$

where

$$\mathrm{NP}(a,b,r;y) = -\frac{(4y)^{2+a+b-r}\pi^{2r-a-2b-1}}{2^{a+2b}\Gamma(r)\zeta(2r-a-2b-1)} \sum_{n>0} \sigma_a(n)\sigma_{a+2b+2-2r}(n) e^{-4ny} \times \int_0^\infty e^{-4nyt}(t+1)^{a+b} t^{-r-b} {}_2\tilde{F}_1(1-r, r-b-1; 1-r-b| -t)\mathrm{d}t\,. \tag{27}$$

In this equation ${}_2\tilde{F}_1(a,b;c|z) = {}_2F_1(a,b;c|z)/\Gamma(c)$ denotes the regularised hypergeometric function. When the parameters (a,b,r) are set to special values, both the perturbative tail $I_{asy}(a,b,r)$, as well as the non-perturbative terms $\mathrm{NP}(a,b,r)$ truncate to Laurent polynomials. The resurgent structure has disappeared, nevertheless leaving behind the exact answer.

Let $f_0(\lambda; m, k|y)$ be the Fourier zero-mode of $\mathcal{E}\left(\lambda; m, k\middle|\tau\right)$. Then

$$f_0(2;2,2|y) = \frac{y^4}{20250} - \frac{y\zeta_3}{45} - \frac{5\zeta_5}{12y} + \frac{\zeta_3^2}{4y^2} + \sum_{n=1}^{\infty} \frac{e^{-4ny}\sigma_{-3}(n)^2}{2y^2}\,, \tag{28}$$

$$f_0(3;2,3|y) = \frac{y^5}{297675} - \frac{y^2\zeta_3}{1890} - \frac{\zeta_5}{360} - \frac{7\zeta_7}{64y^2} + \frac{\zeta_3\zeta_5}{8y^3} + \sum_{n=1}^{\infty} e^{-4ny}\sigma_{-5}(n)\sigma_{-3}(n)\Big[\frac{1}{4y^3} + \frac{n}{4y^2}\Big]\,, \tag{29}$$

$$f_0(5;3,4|y) = \frac{y^7}{49116375} - \frac{y^2\zeta_5}{113400} - \frac{\zeta_7}{15120} - \frac{77\zeta_{11}}{4608y^4} + \frac{3\zeta_5\zeta_7}{64y^5} + \sum_{n=1}^{\infty} e^{-4ny}\sigma_{-7}(n)\sigma_{-5}(n)\Big[\frac{3}{32y^5} + \frac{37n}{192y^4} + \frac{7n^2}{48y^3} + \frac{n^3}{24y^2}\Big]\,. \tag{30}$$

Small y limit

Lemma. *If* $\mathrm{F}(\tau)$ *is an invariant function on the upper half-plane such that at the cusp* $y \to \infty$ *it satisfies the growth condition* $\mathrm{F}(\tau) = O(y^s)$ *with* $s > 1$*, then each of its Fourier modes* $\mathrm{F}_n(y) = \int_0^1 \mathrm{F}(\tau_1 + iy/\pi) e^{-2\pi i n \tau_1} \mathrm{d}\tau_1$ *satisfies the bound* $\mathrm{F}_n(y) = O(y^{1-s})$ *in the limit* $y \to 0$ *[Green, Miller, Vanhove 2015].*

It's not obvious that the examples given before satisfy this bound, hence we need to analyse the small y behaviour of the non-perturbative terms. To do this introduce a function

$$D_{a,b;c}(y) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^c} e^{-ny}, \tag{31}$$

which can be rewritten in a way that allows for the evaluation of its asymptotics by using a Mellin transform

$$D_{a,b;c}(y) = \frac{1}{2\pi i} \int_{t_1 - i\infty}^{t_1 + i\infty} \frac{\Gamma(t)\zeta(t+c)\zeta(t+c-a)\zeta(t+c-b)\zeta(t+c-a-b)}{\zeta(2t+2c-a-b)} y^{-t} \mathrm{d}t, \tag{32}$$

for an arbitrary t_1 to the right of all the singularities.

Cancellations in the small y limit

We find that for all the Eisenstein series with integer coefficients, there is perfect cancellation between the non-perturbative terms and the Laurent polynomial in the limit $y \to 0$. For example, in the case $\lambda = 3, m = 2, k = 3$ we have

$$\mathrm{NP}(3;2,3|y) \sim \frac{11\zeta_9}{128y^4} - \mathrm{P}(3;2,3|y) - \frac{\zeta_3^2}{42y} - \frac{\zeta_7 y^3}{3240\zeta_5} + \frac{\zeta_3\zeta_5 y^4}{23625\zeta_7} + \sum_{\rho_n} \beta_n y^{\frac{3}{4}+i\frac{\rho_n}{2}}, \tag{33}$$

where NP denotes the non-perturbative part, P the Laurent polynomial and the sum is over all the non-trivial zeros of the zeta function $\frac{1}{2} + i\rho_n$ (with β_n just constants).

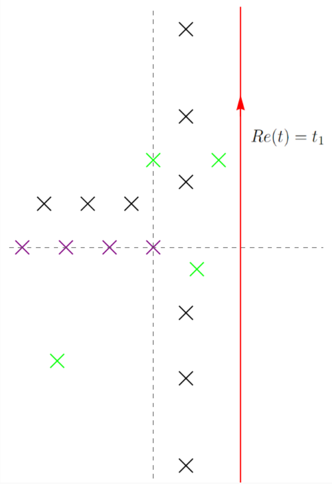

Figure 5: Singularity structure of (32). Poles in purple are from Γ function, in green from ζ functions in the numerator, and in black from ζ function in the denominator.

A curious observation is the presence of the non-trivial zeros of the zeta function in the limit $y \to 0$ of the generalised Eisenstein series. These have a clear manifestation numerically, yet are somewhat odd in the context of corrections to supergravity.

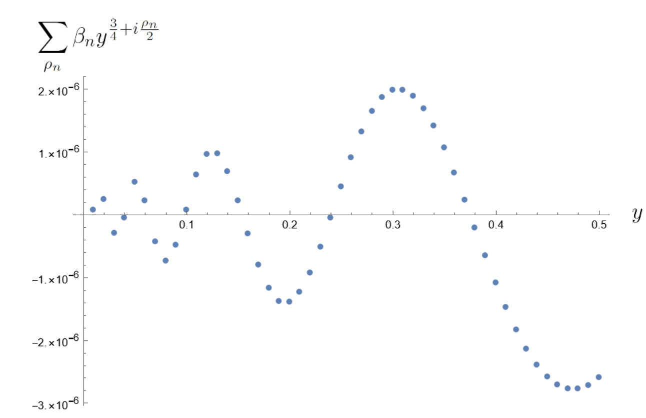

Figure 6: A graph showing the behaviour associated with $y \to 0$ limit of the Fourier zero-mode of $\mathcal{E}(3; 3, 2)$ coming from the non-trivial zeros.

More general comments about NP terms

It is possible to analyse generalised Eisenstein series directly from spectral theory. This gives an explicit formula for the non-perturbative terms as $\tau_2 \to \infty$

$$\mathrm{NP}(\lambda; s_1, s_2|\tau_2) = \sum_{n=1}^{\infty} \frac{4\sigma_{1-2s_1}(n)\sigma_{1-2s_2}(n) n^{s_1+s_2-1}\tau_2}{\Gamma(s_1)\Gamma(s_2)} \times \int_{\mathrm{Re}(t)=\gamma} \frac{\Gamma(\frac{t+s_1+s_2-1}{2})\Gamma(\frac{t+s_1-s_2}{2})\Gamma(\frac{t+s_2-s_1}{2})\Gamma(\frac{t+1-s_1-s_2}{2})}{(t-\lambda)(t+\lambda-1)\Gamma(t)} (\pi n \tau_2)^{-t} \frac{\mathrm{d}t}{2\pi i} \tag{34}$$

With γ to the right of all singularities. The integral can be evaluated using a saddle point in order to obtain another asymptotic series

$$\mathrm{NP}(\lambda; s_1, s_2|\tau_2) \sim \sum_{n=1}^{\infty} \frac{\sigma_{1-2s_1}(n)\sigma_{1-2s_2}(n) n^{s_1+s_2-2}}{\Gamma(s_1)\Gamma(s_2)} e^{-4\pi n \tau_2} \Big(\frac{8}{(4\pi n \tau_2)^2} + \frac{8[s_1(s_1-1)+s_2(s_2-1)-4]}{(4\pi n\tau_2)^3} + O(\tau_2^{-4}) \Big) . \tag{35}$$

This series truncated for the integer-weight generalised Eisensteins.

Exact strong coupling limit

The spectral representation also gives access to the exact asymptotics of the strong coupling $\tau_2 \to 0$ limit

$$f_0(\lambda; s_1, s_2|\tau_2) \sim \#\tau_2^{s_1+s_2-1} + \#\tau_2^{s_1-s_2} + \#\tau_2^{s_2-s_1} + \#\tau_2^{1-s_1-s_2} + \#\tau_2^{\lambda}$$
$$+ \sum_{\rho_n} \frac{2\xi(t+s_1+s_2-1)\xi(t+s_1-s_2)\xi(t+s_2-s_1)\xi(t+1-s_1-s_2)}{(1-t-\lambda)(t-\lambda)\Gamma(s_1)\Gamma(s_2)\pi^{\frac{1}{2}-t}\Gamma(t-\frac{1}{2})\zeta'(2t-1)}\tau_2^t\Big|_{t=\frac{3}{4}+i\frac{\rho_n}{2}}$$
$$+ \widetilde{\mathrm{NP}}(\lambda; s_1, s_2|\tau_2), \tag{36}$$

where $\frac{1}{2} + i\rho_n$ are the non-trivial zeros of the Riemann zeta as before and the new strong coupling non-perturbative terms also have an asymptotic expansion

$$\widetilde{\mathrm{NP}}(\lambda; s_1, s_2|\tau_2)) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sigma_{1-2s_1}(m)\sigma_{1-2s_2}(m)m^{s_1+s_2-\frac{3}{2}}\varphi^{-1}(n)}{\Gamma(s_1)\Gamma(s_2)\,n}\sqrt{4\tau_2}\,e^{-\frac{4\pi mn^2}{\tau_2}}$$
$$\times\left(8\Big(\frac{\tau_2}{4\pi mn^2}\Big)^2 + 8\Big[s_1(s_1-1)+s_2(s_2-1)-\frac{11}{4}\Big]\Big(\frac{\tau_2}{4\pi mn^2}\Big)^3 + O(\tau_2^4)\right), \tag{37}$$

with $\varphi^{-1}(n) = \sum_{d|n} d\mu(d)$ the Dirichlet inverse of the Euler totient function.

A unified framework

We are interested in describing generalised Eisenstein series with integer, as well as half-integer weights, but the class of functions studied in the previous section is not appropriate for the latter. To understand how to proceed, remember the Fourier series

$$\mathcal{E}_s(\tau) = \frac{2\zeta(2s)}{\pi^s}\tau_2^s + \frac{2\xi(2s-1)}{\Gamma(s)}\tau_2^{1-s} + \frac{4}{\Gamma(s)}\sum_{m\neq 0}|m|^{s-\frac{1}{2}}\sigma_{1-2s}(|m|)\tau_2^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|m|\tau_2)e^{2\pi i m\tau_1} \tag{38}$$

In order to view both spectra in a unified framework, we need to extend the space of functions to an even larger one, hence we look at [Dorigoni, RT 2023]

$$\Upsilon(a,b,r,s|\tau) = \sum_{\gamma\in \mathrm{B}(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})}\sum_{m\neq 0}\left[\sigma_a(|m|)|m|^{b-\frac{1}{2}}\tau_2^{r+\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|m|\tau_2)e^{2\pi i m\tau_1}\right]_\gamma . \tag{39}$$

The series converges absolutely if

$$\min\left(\mathrm{Re}(r+1-s), \mathrm{Re}(r+s), \mathrm{Re}(r-b), \mathrm{Re}(r-a-b)\right) > 1\,. \tag{40}$$

Algebraic and differential identities

Some simple relations follow instantly from the definition of the seed functions

$$\Upsilon(a,b,r,s) = \Upsilon(a,b,r,1-s) \quad (41)$$

$$\Upsilon(a,b,r,s) = \Upsilon(-a,b+a,r,s)\,. \quad (42)$$

By using standard properties of the Bessel function, one can also derive a recursion relation for the modular functions

$$\Upsilon(a,b,r,s+1) - \Upsilon(a,b,r,s-1) = \frac{2s-1}{2\pi}\Upsilon(a,b-1,r-1,s)\,. \quad (43)$$

Additionally, we also have an action for the Laplace operator with a fixed eigenvalue

$$\big[\Delta - (r+s)(r+s-1)\big]\Upsilon(a,b,r,s) = -4\pi r\Upsilon(a,b+1,r+1,s+1)\;. \quad (44)$$

An application of these identities will allow us to construct a tower of solutions to inhomogeneous the Laplace equation.

Some functions that lie in this space

- Since for $r = 0$ the function $\Upsilon(a, b, 0, s)$ is annihilated by $\Delta - s(s-1)$ and it has polynomial growth at the cusp, it must be proportional to $\mathcal{E}_s(\tau)$. In fact, one can show

$$\Upsilon(a, b, 0, s|\tau) = \frac{\pi \tan(\pi s)\Gamma(s)\zeta(1-b-s)\zeta(1-a-b-s)\zeta(s-b)\zeta(s-a-b)}{(2s-1)\pi^{s-1}\zeta(1-a-2b)\zeta(2-2s)\zeta(2s)} \mathcal{E}_s(\tau). \tag{45}$$

- Products of two Eisenstein series are also in this space of functions, since Poincaré expanding one and Fourier expanding the other gives precisely a function of this kind

$$\mathcal{E}_{s_1}(\tau)\mathcal{E}_{s_2}(\tau) = \frac{8\xi(2s_1)}{\Gamma(s_1)\Gamma(s_2)}\Upsilon(1-2s_2, s_2, s_1, s_2|\tau) \tag{46}$$
$$+ \frac{2\Gamma(s_1+s_2)\xi(2s_1)\xi(2s_2)}{\Gamma(s_1)\Gamma(s_2)\xi(2(s_1+s_2))}\mathcal{E}_{s_1+s_2}(\tau) + \frac{2\Gamma(s_1+1-s_2)\xi(2s_1)\xi(2s_2-1)}{\Gamma(s_1)\Gamma(s_2)\xi(2(s_1+1-s_2))}\mathcal{E}_{s_1+1-s_2}(\tau)$$

- In view of the differential and algebraic identities noted before, many generalised Eisenstein series must also be present. We will see that the spectra we are interested in fall precisely in this category.

A formula for the Fourier zero-mode

In the case of absolute convergence, the Fourier zero-mode can be derived in terms of a contour integral

$$\Upsilon_0(a,b,r,s|\tau_2) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} U(a,b,r,s|t)\tau_2^t dt, \tag{47}$$

where the integrand is given by

$$\begin{aligned} U(a,b,r,s|t) := & \frac{\pi^{\frac{1}{2}-r}\Gamma\left(\frac{r-s-t}{2}\right)\Gamma\left(\frac{r+s-t}{2}\right)\Gamma\left(\frac{t+r-s-1}{2}\right)\Gamma\left(\frac{t+r+s-1}{2}\right)}{2\Gamma(r-\frac{1}{2})\xi(2-2t)} \\ & \times \frac{\zeta(r-b-t)\zeta(r-a-b-t)\zeta(t+r-b-1)\zeta(t+r-a-b-1)}{\zeta(2r-a-2b-1)}. \end{aligned} \tag{48}$$

This can be shown either by starting with the general Formula for the Fourier modes of a function from its seed, or by computing the spectral overlap of this function with Eisenstein series.

Singularity structure of the integrand

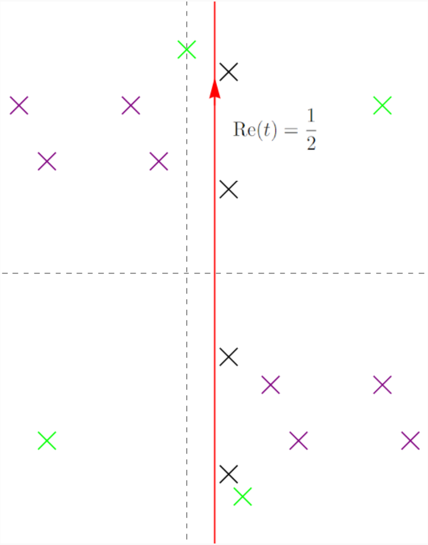

Figure 7: Singularity structure of $U(a, b, r, s|t)$ with poles in purple from Γ functions, poles in green from ζ functions in numerator and poles in black from ζ function in denominator.

Our original interest in this space of functions came from a desire to algorithmically construct Poincaré series expressions of generalised Eisenstein series. We now introduce the machinery to do this. We want to construct the spectrum $\mathrm{Spec}(r+s) = \{r+s-2,\ r+s-4,\ r+s-6,\ ...\}$ of eigenvalues $\lambda_n(r+s) := r+s-2(n+1)$ (with $n = 0, 1, ...$). Start with the highest-weight state

$$\big[\Delta - \lambda_0(r+s)(\lambda_0(r+s)-1)\big]\frac{\Upsilon\,(a, b-1, r-1, s-1)}{4\pi(1-r)} = \Upsilon\,(a, b, r, s)\ , \quad (49)$$

and then look for a general solution within this space of functions to

$$\big[\Delta - \lambda_n(r+s)(\lambda_n(r+s)-1)\big]\mathrm{Y}_n(a, b, r, s) = \Upsilon\,(a, b, r, s)\ . \quad (50)$$

A simple application of identities (44,43) establishes a particular solution

$$\mathrm{Y}_n(a, b, r, s) = \sum_{k=0}^{n}\binom{n}{k}\Big(\prod_{i=0}^{k-1}\frac{2(s+i-n)-1}{2\pi}\Big) \quad (51)$$

$$\times\frac{\Upsilon\,(a, b-k-1, r-k-1, s+k-2n-1)}{4\pi(k+1-r)}\,. \quad (52)$$

Construction of generalised Eisenstein series

The spectra of generalised Eisenstein series mentioned before are both of this type, therefore we simply start with a highest-weight state of the form $\Upsilon\left(1-2s_2, s_2, s_1, s_2\right)$ and can construct the full spectrum relevant to string theory.

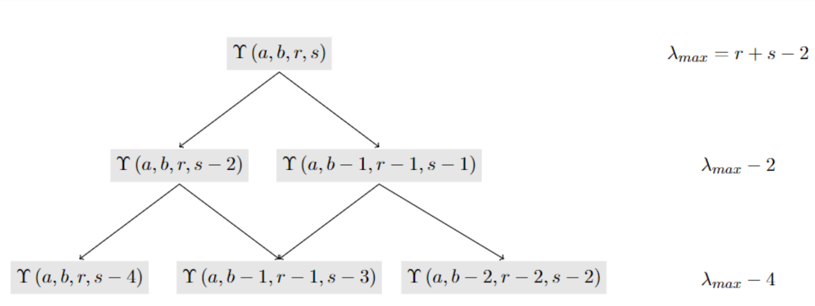

Figure 8: A graphical depiction of the recursion process.

Some examples

Both spectra $\mathrm{Spec}_1(s_1, s_2)$ and $\mathrm{Spec}_2(s_1, s_2)$ may be constructed this way by starting with an appropriate initial representative $\Upsilon(a, b, r, s|\tau)$, corresponding to $\mathcal{E}_{s_1}(\tau)\mathcal{E}_{s_2}(\tau)$. Some examples are

$$\mathcal{E}\left(3; 3, 2\middle|\tau\right) = -\frac{\pi^2}{945}\Upsilon(-3, 1, 2, 1|\tau) + \frac{11}{70}\mathcal{E}_5(\tau) - \frac{\zeta(3)}{42}\mathcal{E}_2(\tau)\,. \tag{53}$$

in the integer case, and

$$\begin{aligned}\mathcal{E}\left(7; \frac{5}{2}, \frac{3}{2}|\tau\right) = &-\frac{16}{15\pi^2}\Upsilon\left(-4, \frac{1}{2}, -\frac{5}{2}, -\frac{7}{2}|\tau\right) + \frac{16}{27\pi}\Upsilon\left(-4, \frac{3}{2}, -\frac{3}{2}, -\frac{9}{2}|\tau\right) \\ &- \frac{4096\pi^{12}}{46\,414\,974\,375\zeta(13)}\mathcal{E}_7(\tau) - \frac{8\pi^4}{10\,935\zeta(5)}\mathcal{E}_3(\tau) - \frac{3\zeta(5)}{2\pi^4}\mathcal{E}_2(\tau)\,.\end{aligned} \tag{54}$$

in the half-integer case.

Conclusions

- We have seen how resurgence techniques can be applied in a modular context as a tool of reconstructing exponentially suppressed contributions at the cusp. Question: How generic is this behaviour? Does modularity somehow enforce resurgent properties?
- In string theory two different spectra of generalised Eisenstein series naturally appear - they are related to both perturbative, as well as non-perturbative effects.
- We provide a natural construction in terms of Poincaré series for these spectra. This is done by studying a space of modular functions which satisfy a tower of differential and algebraic identities. Additionally, the Fourier zero-mode can of these functions can be calculated exactly. Question: What can be said about the non-zero modes? (Complications with either Kloosterman sums or non-holomorphic cusp-forms)