Feynman Integrals and Intersection Theory

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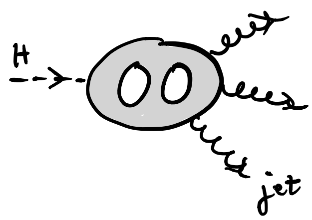
based on work with Mastrolia (Padova); Frellesvig (Copenhagen), Gasparotto, Laporta, Mandal, Mattiazzi; Pokraka (McGill)

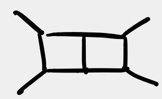
Disclaimer: I'm a physicist!

Motivation

Scattering amplitudes used to make theoretical predictions for particle colliders are linear combinations of many Feynman integrals



$$= \sum_{i=1}^{10^5} \text{coefficient}_i$$



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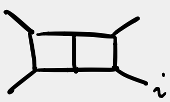
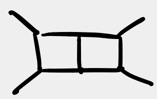
naively $O(10^5)$ scalar Feynman diagrams to evaluate!

But it turns out these are hugely redundant, and only $O(10^3)$ are actually linearly-independent.

- How to predict the size of the basis?
- What makes a good basis?
- How to expand an arbitrary integral in such a basis?

Once a basis is established, the "master integrals" still need to be computed. This requires differential equations (Gauss-Manin system).

∂  $_i = \sum_{j=1}^{\chi} \Omega_{ij}$  $_j$

kinematic differential

- → How to compute the connection matrix Ω?
- → Can we predict the positions and types of singularities?

- → These are one of the main bottlenecks in making predictions in particle colliders
- → Mathematical insights could be very impactful!

Q: Can we compute coefficients of the basis expansion and the Gauss-Manin connection directly?

A: Yes, they are intersection numbers of twisted cohomology classes!

[with Mastrolia '18]

much easier to compute that the full Feynman integrals

The goal of this talk is to review what's known about these objects and how to compute them.

Outline

1. Quick introduction to twisted co/homology

   [cf. talk by Dupont]

2. Intersection numbers of twisted cohomology groups

3. Three strategies for computing them in practice

Setup

We're interested in Feynman integrals with E edges, L loops, n external legs, in D space-time dimensions

polynomial in α's

$$\int \frac{d^E \alpha}{GL(1)} \frac{N(\alpha)}{\mathcal{U}^{D/2-\gamma} \mathcal{F}^{\gamma}}$$

$\gamma = E - LD/2 \in \mathbb{Z}$
degree of divergence

projective measure, fix $\alpha_E = 1$

Here the Symanzik polynomials are given by

$$\mathcal{U} := \sum_{\substack{\text{spanning} \\ \text{trees } T}} \prod_{e \notin T} \alpha_e$$

degree L

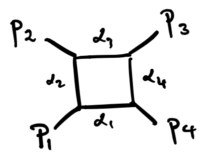
$$\mathcal{F} := \sum_{\substack{\text{spanning} \\ \text{2-forests} \\ F = T_L \sqcup T_R}} p_F^2 \prod_{e \notin F} \alpha_e - \left(\sum_{e=1}^{E} m_e^2 \alpha_e \right) \mathcal{U}$$

degree $L+1$

total external momentum

mass of the edge e

Example:



$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \,,$$

etc.

$$\mathcal{F} = (p_1+p_2)^2 \alpha_2 \alpha_4 + (p_2+p_3)^2 \alpha_2 \alpha_4 + \sum_{i=1}^{4} p_i^2 \alpha_i \alpha_{i+1}$$

$$- \Big(\sum_{e=1}^{4} m_e^2 \alpha_e \Big) \mathcal{U} \,.$$

However, such integrals generically **don't converge**:

we'll use analytic regularization [Speer '60s]

$$\int_{\mathbb{R}_+^{E-1}} d^{E-1}\alpha \; \frac{N \cdot \prod_{e=1}^{E} \alpha_e^{\delta_e}}{\mathcal{U}^{\frac{D}{2}-\gamma-\delta} \; \mathcal{F}^{\gamma+\delta}}$$

Here $\delta_e \in \mathbb{R} \setminus \mathbb{Z}$ are called **analytic regulators**

and $\delta := \sum_{e=1}^{E} \delta_e$

→ **Immensely simplifies** the co/homology structure,

but obscures generalized unitarity

→ In practice we actually use other representations (Baikov, parametric), but this will not matter for this talk

Introduce a "potential function"

$$W := \left(\gamma+\delta-\tfrac{D}{2}\right)\log \mathcal{U} - (\gamma+\delta)\log F + \sum_{e=1}^{E} \delta_e \log \alpha_e$$

on the space of Schwinger parameters

$$M := (\mathbb{C}^*)^{E-1} \setminus \{ \mathcal{U} F = 0 \}.$$

Monodromies of W define a rank-one local system $\mathcal{L}_W$.
This allows us to talk about locally-finite homology with coefficients in $\mathcal{L}_W$:

$$H_k^{dW} := H_k^{lf}(M, \mathcal{L}_W).$$

and similarly

$$H^k_{dW} := H^k(M, \nabla_{dW})$$

$\nabla_{dW} := d + dW\wedge$

A theorem of Aomoto, applied to this case, states that they are concentrated in the middle dimension $k = E-1$, so the only non-trivial ones are

$$H^{dW}_{E-1} \,, \quad H^{E-1}_{dW}$$ [Aomoto '75]

"twisted co/homology"

For our purposes we need $[\mathbb{R}^{E-1}_{+} \otimes e^{W}] \in H^{dW}_{E-1}$

and $[\varphi] \in H^{E-1}_{dW}$.

Their pairing is the Feynman integral in analytic regularization

$$\left(\mathbb{R}^{E-1}_{+} \otimes e^{W} , \varphi \right) = \int_{\mathbb{R}^{E-1}_{+}} e^{W} \varphi$$

$$= \int_{\mathbb{R}^{E-1}_{+}} \underbrace{\frac{\prod_{e=1}^{E} \alpha_e^{\delta_e}}{\mathcal{U}^{\frac{D}{2}-\gamma-\delta} \, F^{\gamma+\delta}}}_{e^{W}} \cdot \underbrace{N \, d^{E-1}\alpha}_{\varphi}$$

e^W: universal for a family of Feynman integrals (multi-valued)

φ: depends on a specific numerator (single-valued)

→ Much more rigorous and motivic: Brown, Dupont, Fresán, Tapušković

Since the integration contour stays fixed, the twisted cohomology H^{E-1}_{dW} becomes the "vector space of Feynman integrals" spanned by $[\varphi]$ (up to the kernel of integration).

Since $\mathcal{L}_W$ is flat, the Euler characteristic of M equals:

$$\chi(M) = \sum_k (-1)^k \dim H^k(M, \nabla_{dW}) = (-1)^{E-1} \dim H^{E-1}_{dW}.$$

Not always practical to compute. On the other hand, we can construct a Morse homology based on the function $\mathrm{Re}(W)$. Introduce the critical set:

$$\mathrm{Crit}(W) := \{ (\alpha_1, \alpha_2, \dots, \alpha_{E-1}) \in M \mid dW = 0 \}.$$

→ Assume all critical points are isolated and non-degenerate

→ Since W is holomorphic, $\mathrm{Crit}(W) = \mathrm{Crit}(\mathrm{Re}(W))$ by Cauchy-Riemann eqs. and all critical points have the middle Morse index (the same number of upwards & downwards dir.)

This gives a more practical way of computing the Euler characteristic:

$$\chi(M) = (-1)^{E-1} \# \mathrm{Crit}(W)$$
$$= (-1)^{E-1} \dim H^{E-1}_{dW} ,$$

master integrals

given the above assumptions

[with Mastrolia et al. '18-20]

[see also Lee-Pomeransky '13

Bitoun-Bogner-Klausen-Panzer '17

from different perspectives]

As we'll see, the critical points contain much more information beyond just counting the number of master integrals...

We can introduce other isomorphic co/homology groups

cohomology intersection #

$$\begin{tikzcd}
H^{E-1}_{d\bar{W}} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & {[\varphi_+] \in H^{E-1}_{dW}} \arrow[rr, leftrightarrow, "\langle \varphi_+ | \varphi_- \rangle"] \arrow[d, leftrightarrow, "\int_{\Gamma_+} e^{W} \varphi_+"'] & & H^{E-1}_{-dW} \ni [\varphi_-] \arrow[d, leftrightarrow, "\int_{\Gamma_-} e^{-W} \varphi_-"] \\
H^{d\bar{W}}_{E-1} \arrow[r, leftrightarrow] & {[\Gamma_+ \otimes e^{W}] \in H^{dW}_{E-1}} \arrow[rr, leftrightarrow, "\langle \Gamma_- | \Gamma_+ \rangle"'] & & H^{-dW}_{E-1} \ni [\Gamma_- \otimes e^{-W}]
\end{tikzcd}$$

antiholomorphic version

(important for $M_{0,n}$)

homology intersection #

just use linear algebra!

[cf. talks by Fresán, Zhou, ...]

The main novelty will be the intersection pairing of twisted cohomology groups:

$$H^{E-1}_{+dW} \times H^{E-1}_{-dW} \longrightarrow \mathbb{C}$$

[Cho, Matsumoto '95]

$$\langle \varphi_+ | \varphi_- \rangle := \left(\frac{1}{2\pi i}\right)^{E-1} \int_M \varphi_- \wedge \iota(\varphi_+)$$

compact support $\iota : H^{E-1}_{dW} \to H^{E-1,\,\mathrm{comp}}_{dW}$

Before diving into the details, let's how it's useful:

$$\underbrace{\int_{\mathbb{R}_+^{E-1}} e^{W} \varphi_+}_{\text{an arbitrary integral in the family}} = \sum_{i=1}^{|\chi(M)|} \underbrace{\langle \varphi_+ | \varphi^\vee_{-,i} \rangle}_{\text{coefficients}} \underbrace{\int_{\mathbb{R}_+^{E-1}} e^{W} \varphi_{+,i}}_{\text{master integrals}}$$

where $\langle \underset{\text{basis}}{\varphi_{+,i}} | \underset{\text{dual basis}}{\varphi^\vee_{-,j}} \rangle = \delta_{ij}$

In general, it's a difficult problem to choose a "nice" basis $[\varphi_{+,i}] \in H^{E-1}_{dW}$, let alone $[\varphi^\vee_{-,i}] \in H^{E-1}_{-dW}$ [..., Henn, ...]

At one-loop, progress in [Caron-Huot, Pokraka '21]

In practice, to find the dual basis we choose any basis $[\Phi^\vee_{-,j}] \in H^{E-1}_{-dw}$. Then

invertible matrix

$$\langle \varphi_{+,i} \,|\, \Phi^\vee_{-,j} \rangle =: G_{ij}.$$

Then $\varphi^\vee_{-,i} = \sum_{k=1}^{|\chi(M)|} \Phi^\vee_{-,k} \, G^{-1}_{kj}$, because

$$\langle \varphi_{+,i} \,|\, \varphi^\vee_{-,j} \rangle = \sum_{k=1}^{|\chi(M)|} \underbrace{\langle \varphi_{+,i} \,|\, \Phi^\vee_{-,k} \rangle}_{G_{ik}} G^{-1}_{kj} = \delta_{ij}.$$

Similarly, differential equations can be computed with

$$d\!\!l = \sum_I ds_I \frac{\partial}{\partial s_I}$$

external variables (Mandelstam invariants)

$$d\!\!l \int_{\mathbb{R}_+^{E-1}} e^W \varphi_{+,i} = \int_{\mathbb{R}_+^{E-1}} e^W \left(d\!\!l + d\!\!l W \right) \varphi_{+,i}$$

$$= \sum_{j=1}^{|\chi(M)|} \underbrace{\left\langle (d\!\!l + d\!\!l W)\varphi_{+,i} \,\middle|\, \varphi^{\vee}_{-,j} \right\rangle}_{\Omega_{ij}} \int_{\mathbb{R}_+^{E-1}} e^W \varphi_{+,j}$$

Rank-$|\chi(M)|$ Gauss-Manin system on the space $\{s_I\}$ of external variables

This motivates systematic study of the intersection pairings $\langle \varphi_+ | \varphi_- \rangle$. So far we've used three basic strategies

- → Blow-up
- → Fibration

} localize on the max-codim. components of ∂M

- → Critical point

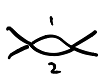
} localize on $\mathrm{Crit}(W)$

[other approaches by Matsubara-Heo, Weinzierl, ...]

Let's see this for $\dim_{\mathbb{C}} M = 1$.

Example:

set $\alpha_2 = 1$
$D = 4$

$s \to$ 

$$= \int_{\mathbb{R}_+} \underbrace{\frac{\alpha_1^{\delta_1}}{(\alpha_1+1)^{2-\delta_1-\delta_2}\left[s\,\alpha_1 - (m_1^2\alpha_1 + m_2^2)(\alpha_1+1)\right]^{\delta_1+\delta_2}}}_{e^W} \underbrace{N(\alpha_1)\, d\alpha_1}_{\varphi_+}$$

$$M = \mathbb{CP}^1 - \{ 5 \text{ pts} \}$$

$$|\chi(M)| = 3$$

$$\varphi_\pm \in \Omega^{1,0}(M).$$

We need to compute

$$\langle \varphi_+ | \varphi_- \rangle = \frac{1}{2\pi i} \int_M \varphi_- \wedge \iota(\varphi_+)$$

zero everywhere except

for neighborhoods of ∂M

" holomorphic anomaly "

$$= \sum_{p \in \partial M} \mathrm{Res}_{\alpha_1 = p} \left(\varphi_- \underbrace{\nabla_{dW}^{-1} \varphi_+}_{} \right).$$

$\xi_p \in \Omega^0(M)$

local solution of

$\nabla_{dW} \xi_p = \varphi_+$

near $\alpha_1 = p$

Useful expansion : $W \to \tau W$

(essentially $\delta_e \to \tau \delta_e$)

and take $\tau \to 0$, then

$$\tau \langle \varphi_+ | \varphi_- \rangle = \sum_{p \in \partial M} \frac{\mathrm{Res}_p(\varphi_-) \; \mathrm{Res}_p(\varphi_+)}{\mathrm{Res}_p(dW)} + \mathcal{O}(\tau)$$

exact (no $\mathcal{O}(\tau)$ corrections)

for logarithmic forms $\varphi_\pm$

[Matsumoto '98]

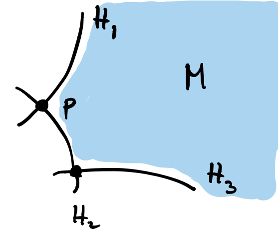
Blow-up strategy

Generalization to arbitrary $\dim_{\mathbb{C}} M = E-1$:

$$\tau^{E-1} \langle \varphi_+ | \varphi_- \rangle = \sum_{p = H_1 \cap \dots \cap H_{E-1}} \frac{\mathrm{Res}_p(\varphi_+)\ \mathrm{Res}(\varphi_-)}{\prod_{a=1}^{E-1} \mathrm{Res}_{H_a}(dW)} + O(\tau)$$

Poincaré residue

normal crossing after blow-up



For example, specialize to primitive log-divergent diagrams G ($\gamma + \delta = 0$) and take $\varphi_\pm = \bigwedge_{e=1}^{E-1} \frac{d\alpha_e}{\alpha_e}$:

$$\langle \varphi_+ | \varphi_- \rangle = \sum_{\substack{\text{permutations} \\ \text{of } E \text{ edges } \sigma}} \frac{1}{\prod_{k=1}^{E-1} \left(\gamma_{G_{\sigma,k}} + \sum_{e=1}^{k} \delta_e \right)}$$

subdiagram of G with edges $\{\sigma_1, \sigma_2, \dots, \sigma_k\}$

the same as the generalized Hepp bound

[Panzer '19]

Critical-point strategy

Critical-point strategy uses another linear algebra property "twisted period relations":

[Cho, Matsumoto '95]

$$\langle \varphi_+ | \varphi_- \rangle = \sum_{i,j=1}^{|\chi(M)|} \left(\int_{\Gamma_{+,i}} e^{W} \varphi_+ \right) \mathbb{H}^{-1}_{ij} \left(\int_{\Gamma_{-,j}} e^{-W} \varphi_- \right)$$

homology intersection matrix

$\mathbb{H}_{ij} = \langle \Gamma_{-,i} | \Gamma_{+,j} \rangle$.

The idea is to pick bases of Lefschetz thimbles $\mathcal{J}_i$ (steepest descent for e^{W}) and anti-thimbles $\mathcal{K}_i$ (steepest descent for e^{-W}) from the i-th critical point. This gives

$$\langle \mathcal{K}_i | \mathcal{J}_j \rangle = \delta_{ij}$$

and so

$$\langle \varphi_+ | \varphi_- \rangle = \sum_{i=1}^{|\chi(M)|} \int_{\mathcal{J}_i} e^{W} \varphi_+ \int_{\mathcal{K}_i} e^{-W} \varphi_-$$

We can now expand around the critical points ($\tau \to \infty$):

$$\tau^{E-1} \langle \varphi_+ | \varphi_- \rangle = \sum_{p \in \mathrm{Crit}(W)} \frac{\varphi_+ \varphi_- / d^{E-1}\alpha}{\det\left(\partial^2 W / \partial\alpha_i \partial\alpha_j \right)} \Bigg|_p + \mathcal{O}(\tau^{-1})$$

[SM '17]

It turns out this expansion can be made more systematic in the Čech - de Rham formulation.

Intersection numbers are expressed in terms of "higher residue pairings"

[Saito '83]

$$\tau^{E-1} \langle \varphi_+ | \varphi_- \rangle = \sum_{p \in \mathrm{Crit}(W)} \mathrm{Res}_p \left[\frac{\varphi_+ \varphi_- / d^{E-1}\alpha}{\prod_{e=1}^{E-1} \partial W / \partial \alpha_e} \left(1 + \frac{1}{2\tau} \sum_{i=1}^{E-1} \frac{\partial \log(\varphi_+/\varphi_-)/\partial\alpha_i}{\partial W / \partial \alpha_i} + \mathcal{O}(1/\tau^2) \right) \right]$$

Grothendieck residue

[with Pokraka '19]

Fibration strategy

Finally, there's the fibration strategy in which we split M into $E-1$ one-dimensional fibers

$$\begin{tikzcd}
M \arrow[d, "\pi_1"] & M_1 \arrow[l, hook', "i_1"'] \arrow[d, "\pi_2"] & \cdots \arrow[l, hook', "i_2"'] & M_{E-2} \arrow[l, hook'] \arrow[d, "\pi_{E-1}"] & M_{E-1} \arrow[l, hook', "i_{E-1}"'] \\
\Sigma_1 & \Sigma_2 & \cdots & \Sigma_{E-1} &
\end{tikzcd}$$

each fiber is $\simeq \mathbb{CP}^1 - \{\#\text{ pts}\}$.

→ Computations on each Σ_e proceed as before, but now with a higher-rank local system $\mathcal{L}_e$

→ Each $\mathcal{L}_e$ can be determined in terms of intersection numbers on M_e

→ Recursion ends on $M_{E-1} = \{pt\}$.

[SM '19]

Applied to numerous practical examples!

[with Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi '18-20]

Thank you!