AdS Virasoro-Shapiro from single-valued periods

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Outline

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How to bootstrap a string amplitude?

Veneziano (1968) and Virasoro/Shapiro (1969/1970)
made assumptions on amplitude $A(S, T)$ (from pheno):

1. crossing symmetry
2. IR: only simple poles: on linear Regge trajectories
3. UV: Regge behaviour: $\lim_{|S|\to\infty} A(S, T) \sim S^{\alpha' T+\alpha_0}$
4. superconvergence sum rules relating IR and UV

$\rightarrow$ The birth of string theory

Virasoro-Shapiro amplitude in flat space

Tree-level 4 graviton amplitude in type IIb superstring ($g_s \ll 1$)

$$A(S,T) = -\frac{\Gamma(-S)\,\Gamma(-T)\,\Gamma(-U)}{\Gamma(S+1)\,\Gamma(T+1)\,\Gamma(U+1)}$$

$$S = -\frac{\alpha'}{4}(p_1+p_2)^2\,, \qquad T = -\frac{\alpha'}{4}(p_1+p_3)^2\,, \qquad U = -\frac{\alpha'}{4}(p_1+p_4)^2$$

Low energy expansion ($g_s \ll \alpha' \ll 1$)

$$A(S,T) = \frac{1}{STU} + 2\sum_{a,b=0}^{\infty} \hat{\sigma}_2^a \hat{\sigma}_3^b \alpha^{(0)}_{a,b}$$

$$\hat{\sigma}_2 = \tfrac{1}{2}(S^2+T^2+U^2)\,, \qquad \hat{\sigma}_3 = STU\,, \qquad S+T+U=0$$

AdS/CFT

type IIb string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$

=

$\mathcal{N} = 4$ SYM theory
with $SU(N)$ gauge group

- worldsheet theory unknown
- We want to derive the genus 0, 4pt amplitude (Virasoro-Shapiro of AdS)

- CFT provides rigid structure

Dictionary: $$g_s = \frac{g_{YM}^2}{4\pi} = \frac{1}{4\pi}\frac{\lambda}{N}, \qquad \alpha' = \frac{R_{\mathrm{AdS}}^2}{\sqrt{\lambda}}$$

Tree level string theory at low energy

$$g_s \ll \alpha' \ll 1 \qquad \Leftrightarrow \qquad N \gg \sqrt{\lambda} \gg 1$$

The correlator and its Mellin transform

Consider the four-point function

$$\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle \sim \mathcal{T}(u,v) = \text{ reduced correlator}$$

$\mathcal{O}_2$ = superconformal primary of stress-tensor supermultiplet in $\mathcal{N} = 4$ SYM theory

We will concentrate on

$$M(s_1, s_2) = \text{ Mellin transform of } \mathcal{T}(u,v)$$

Crossing symmetry:

$$M(s_1, s_2) = M(s_2, s_1) = M(s_1, s_3)\,, \qquad s_1 + s_2 + s_3 = 0$$

$M(s_1, s_2)$ has analytic structure similar to a scattering amplitude.

OPE and poles of $M(s_1, s_2)$

Operator product expansion (OPE)

$$\mathcal{O}_2(x)\mathcal{O}_2(0) = \sum_{\mathcal{O}_{\tau,\ell} \text{ primaries}} C^2_{\tau,\ell}\; c_{\tau,\ell}(x, \partial_y)\mathcal{O}_{\tau,\ell}(y)\Big|_{y=0}$$

OPE data

- $\ell =$ spin
- $\tau = \Delta - \ell =$ twist
- $C^2_{\tau,\ell} =$ OPE coefficients

$M(s_1, s_2)$ has only simple poles, given by [Mack;2009], [Penedones,Silva,Zhiboedov;2019]

Poles and residues of $M(s_1, s_2)$

$$M(s_1, s_2) \sim \frac{C^2_{\tau,\ell} Q^{\tau+4,d=4}_{\ell,m}(s_2 - \frac{8}{3})}{s_1 - \tau - 2m + \frac{4}{3}}$$

The low energy expansion

Low energy expansion in flat space

$$A(S,T) = \frac{1}{STU} + 2\sum_{a,b=0}^{\infty} \hat{\sigma}_2^a \hat{\sigma}_3^b \alpha_{a,b}^{(0)}$$

Low energy expansion of Mellin amplitude

$$M(s_1,s_2) = \frac{8}{(s_1-\frac{2}{3})(s_2-\frac{2}{3})(s_3-\frac{2}{3})} + \sum_{a,b=0}^{\infty} \frac{\Gamma(2a+3b+6)}{8^{a+b}\lambda^{\frac{3}{2}+a+\frac{3}{2}b}} \sigma_2^a \sigma_3^b \left(\alpha_{a,b}^{(0)} + \frac{\alpha_{a,b}^{(1)}}{\sqrt{\lambda}} + \frac{\alpha_{a,b}^{(2)}}{\lambda} + \cdots \right)$$

$$\sigma_2^a \sigma_3^b = \text{ contact diagrams }, \quad \sigma_2 = s_1^2 + s_2^2 + s_3^2 , \quad \sigma_3 = s_1 s_2 s_3$$

$$\alpha_{a,b}^{(k)} = \text{ Wilson coefficients}$$

- AdS: expand also in dimensionless parameter $1/\sqrt{\lambda} = \alpha'/R_{\mathrm{AdS}}^2$
- $\alpha_{a,b}^{(0)}$ are the same due to flat space limit [Penedones;2010]

How to bootstrap a string amplitude in AdS?

Constraints on $M(s_1, s_2)$ (from CFT):

1. superconformal and crossing symmetry
2. IR: only simple poles: positions and residues $\leftrightarrow$ CFT data
3. UV: bound on chaos: $\lim_{|s_1|\to\infty} |M(s_1, s_2)| \lesssim |s_1|^{-2}, \; \mathrm{Re}(s_2) < \frac{2}{3}$
   [Maldacena,Shenker,Stanford;2015]
4. low energy expansion

Dispersive sum rules

$\Rightarrow$
$$\alpha^{(k)}_{a,b} = \sum_{\tau,\ell} f\,(\text{OPE data of stringy operators})$$

stringy operators: single-trace, $\Delta \propto \lambda^{\frac{1}{4}}$

Flat space Wilson coefficients

Extract $\alpha^{(0)}_{a,b}$ from the flat space amplitude:

$$\hat{\sigma}_2 = \tfrac{1}{2}(S^2+T^2+U^2)$$
$$\hat{\sigma}_3 = STU$$

$$\frac{1}{STU} + 2\sum_{a,b=0}^{\infty} \hat{\sigma}_2^a \hat{\sigma}_3^b \alpha^{(0)}_{a,b} = -\frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}$$
$$= \frac{1}{STU}\exp\left(2\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{2n+1}\left(S^{2n+1}+T^{2n+1}+U^{2n+1}\right)\right)$$

Example:

$$\alpha^{(0)}_{a,0} = \zeta(3+2a)\,, \qquad \alpha^{(0)}_{a,1} = \sum_{\substack{i_1,i_2=0\\ i_1+i_2=a}}^{a} \zeta(3+2i_1)\zeta(3+2i_2)\,, \qquad \ldots$$

$\alpha^{(0)}_{a,b}$ is in the ring of single-valued MZVs of weight $3+2a+3b$ and depth $\leq b+1$.

One can solve the dispersive sum rule for leading OPE data (in $1/\sqrt{\lambda}$ expansion).

AdS Wilson coefficients

The next sum rule

$$\alpha^{(1)}_{a,b} = \sum_{\tau,\ell} f\,(\text{OPE data})$$

has unknown data on both sides. We find a unique solution by assuming:

- $\alpha^{(1)}_{a,b}$ is in the ring of single-valued MZVs of weight $4+2a+3b$ and depth $\leq b+2$
- an ansatz for $f\,(\text{OPE data})$ in terms of Euler-Zagier sums

Solution agrees with integrability and passes various consistency checks!

Summing the low energy expansion

$$-\frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)} = \frac{1}{STU} + 2\sum_{a,b=0}^{\infty} \hat{\sigma}_2^a \hat{\sigma}_3^b \alpha_{a,b}^{(0)}$$

Sum for flat space [Zagier,Zerbini;2019]

$$2\sum_{a,b=0}^{\infty} \hat{\sigma}_2^a \hat{\sigma}_3^b \alpha_{a,b}^{(0)} = \sum_{\delta=1}^{\infty} \frac{1}{\delta^3} \frac{y+2}{1-x-y} \binom{z+\delta-1}{\delta-1}^2 \qquad \text{(poles at } S, T, U = \delta)$$

Sum for $1/\sqrt{\lambda}$ correction

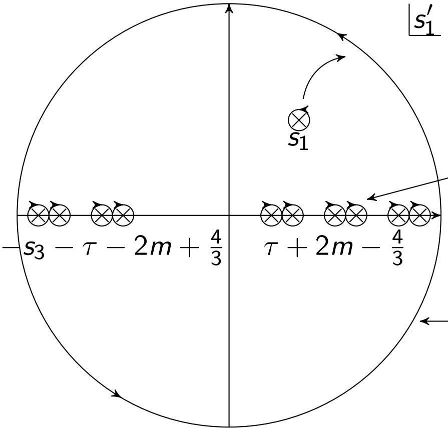
$$2\sum_{a,b=0}^{\infty} \hat{\sigma}_2^a \hat{\sigma}_3^b \alpha_{a,b}^{(1)} = \sum_{\delta=1}^{\infty} \sum_{n=0}^{\delta-1} \frac{1}{\delta^4} \mathcal{D}_n(\delta) \frac{y+2}{1-x-y} \binom{z+\delta-\frac{n}{2}-1}{\delta-n-1}^2$$

$$x = \hat{\sigma}_2/\delta^2\,, \quad y = \hat{\sigma}_3/\delta^3\,, \quad z = \delta\left(\sqrt{1-4y}-1\right)/2$$

$$\mathcal{D}_n(\delta) = \text{ degree 3 differential operator in } x, y, z$$

Outline

Dispersion relation



$$M(s_1, s_2) = \oint_{s_1} \frac{ds_1'}{2\pi i} \frac{M(s_1', -s_1' - s_3)}{(s_1' - s_1)}$$

poles $\sim \dfrac{C^2_{\tau,\ell} Q^{\tau+4,d=4}_{\ell,m}(s_2 - \frac{8}{3})}{s_1' - \tau - 2m + \frac{4}{3}}$

arc vanishes due to bound on chaos

Result: $$M(s_1, s_2) = \sum_{\tau,\ell} C^2_{\tau,\ell} \omega_{\tau,\ell}(s_1, s_2)$$

Dispersive sum rules for Wilson coefficients

Imposing

$$\text{low energy expansion} \; = M(s_1, s_2) = \sum_{\tau,\ell} C^2_{\tau,\ell}\omega_{\tau,\ell}(s_1, s_2)$$

relates Wilson coefficients and OPE data:

$$\alpha^{(k)}_{a,b} = \sum_{\tau,\ell} f\,(\text{OPE data of stringy operators})$$

The stringy operators must have $\tau \sim \lambda^{\frac{1}{4}}$ to produce string states of mass $m \sim 1/\sqrt{\alpha}$ in the flat space limit.

Stringy OPE data

$r =$ quantum numbers

OPE coefficients

$$\mathcal{C}^2(r;\lambda) = \underbrace{\frac{\pi^3}{4^{6+\ell+\tau(r;\lambda)}} \frac{\tau(r;\lambda)^6}{\sin^2(\frac{\pi\tau(r;\lambda)}{2})} \frac{1}{\ell+1}}_{\text{cancels factor in } \lim_{\tau\to\infty}\omega_{\tau,\ell}(s_1,s_2)} f(r;\lambda)$$

Expansion in $1/\lambda^{\frac{1}{4}}$

$$\tau(r;\lambda) = \tau_0(r)\lambda^{\frac{1}{4}} + \tau_1(r) + \tau_2(r)\lambda^{-\frac{1}{4}} + \ldots$$
$$f(r;\lambda) = f_0(r) + f_1(r)\lambda^{-\frac{1}{4}} + f_2(r)\lambda^{-\frac{1}{2}} + \ldots$$

Quantisation of the twists

First dispersive sum rule:

$$\zeta(3+2a) \overset{\text{flat space}}{=} \alpha^{(0)}_{a,0} = \sum_r \left(\frac{4}{\tau_0^2(r)}\right)^{3+2a} f_0(r)$$

Expand at large a:

$$\frac{1}{1^{3+2a}} + \frac{1}{2^{3+2a}} + \frac{1}{3^{3+2a}} + \ldots = \left(\frac{4}{\tau_0^2(r_1)}\right)^{3+2a} \sum_{r_1} f_0(r_1) + \left(\frac{4}{\tau_0^2(r_2)}\right)^{3+2a} \sum_{r_2} f_0(r_2) + \ldots$$

$$\Rightarrow \quad \tau_0(r) = 2\sqrt{\delta}\,, \quad \delta \in \mathbb{N}$$

Agrees with [Gubser,Klebanov,Polyakov;1998]!

Quantum numbers

In terms of spin ℓ and Regge trajectory n:

$$\tau_0(r) = 2\sqrt{\delta} = 2\sqrt{\ell/2 + n + 1}$$

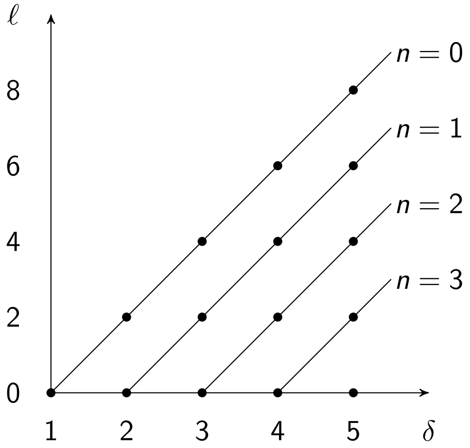
$$\ell = 0, 2, \ldots, 2(\delta - 1)$$

Our quantum numbers:

$$r = (\delta, \ell, \hat{r})$$

$\hat{r}$ not accessible from $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle$ alone:

$$\sum_{\hat{r}} \cdots = \langle \ldots \rangle$$



Outline

Sum rule for $\alpha^{(0)}_{a,b}$

First dispersive sum rule

$$\alpha^{(0)}_{a,b} = \sum_{\delta=1}^{\infty} \sum_{m=0}^{b} \frac{c^{(0)}_{a,b,m}}{\delta^{3+2a+3b}} F^{(0)}_m(\delta)\,, \qquad F^{(0)}_m(\delta) = \sum_{\ell=0,2,\ldots}^{2(\delta-1)} (\ell-m+1)_m(\ell+2)_m \langle f_0(\delta,\ell)\rangle$$

$\alpha^{(0)}_{a,b}$ is known from flat space in terms of multiple zeta values (MZVs):

$$\zeta(s_1,\ldots,s_d) = \sum_{n_1>\ldots>n_d>0} \frac{1}{n_1^{s_1}\cdots n_d^{s_d}} = \sum_{\delta=1}^{\infty} \frac{Z_{s_2,s_3,\ldots}(\delta-1)}{\delta^{s_1}}$$

Euler-Zagier sums:

$$Z_{s_1,\ldots,s_d}(N) = \sum_{\substack{n_1,\ldots,n_d \\ N\geq n_1>\ldots>n_d>0}} \frac{1}{n_1^{s_1}\cdots n_d^{s_d}}$$

Sum rules implies:

$$F^{(0)}_m(\delta) = \sum_{d=\lfloor \frac{m+1}{2}\rfloor}^{m} \sum_{\substack{s_1,\ldots,s_d\in\{1,2\} \\ s_1+\ldots+s_d=m}} 2^{\sum_i \delta_{s_i,1}} \delta^m Z_{s_1,\ldots,s_d}(\delta-1)$$

Resulting $\langle f_0(\delta,\ell)\rangle$ agrees with [Costa,Goncalves,Penedones;2012].

Outline

Sum rule for $\alpha^{(1)}_{a,b}$

Second dispersive sum rule

$$\alpha^{(1)}_{a,b} = \sum_{\delta=1}^{\infty} \sum_{m=0}^{b} \frac{1}{\delta^{4+2a+3b}} \left(c^{(0)}_{a,b,m} \left(F^{(2)}_m(\delta) - (3+2a+3b)\, T^{(2)}_m(\delta) \right) + c^{(2)}_{a,b,m} F^{(0)}_m(\delta) \right)$$

$$T^{(2)}_m(\delta) = \sum_{\ell=0,2,\dots}^{2(\delta-1)} \sqrt{\delta}(\ell - m + 1)_m (\ell+2)_m \langle f_0(\delta,\ell) \tau_2(\delta,\ell) \rangle$$

$$F^{(2)}_m(\delta) = \sum_{\ell=0,2,\dots}^{2(\delta-1)} (\ell - m + 1)_m (\ell+2)_m \left(\delta \langle f_2(\delta,\ell) \rangle - \frac{39}{4} \ell \langle f_0(\delta,\ell) \rangle \right)$$

$$c^{(2)}_{a,b,m} = c^{(0)}_{a,b,m} \times \text{ degree 3 polynomial in } a, b, m$$

Now: Unknown data on both sides of equation.

Constraints at negative a

The dispersive sum rules hold for

$$b = 0, 1, 2, \ldots , \qquad a = -b, -b + 1, \ldots$$

Low energy expansion requires

$$\alpha^{(k)}_{a,b} = 0 \,, \qquad \text{for } a = -b, \ldots, -1$$

One can check that this holds for $\alpha^{(0)}_{a,b}$.

Constrains $\alpha^{(1)}_{a,b}$, but not enough to fix it.

We will demand that $\alpha^{(1)}_{a,b}$ for $a \geq 0$ is in the ring of single-valued MZVs.

Single-valued multiple zeta values

weight $= s_1 + \ldots + s_d$, depth $= d$

Multiple polylogarithms

$$\mathrm{Li}_{s_1,\ldots,s_d}(z) = \sum_{n_1>\ldots>n_d>0} \frac{z^{n_1}}{n_1^{s_1}\cdots n_d^{s_d}}$$

Multiple zeta values (MZVs)

$$\zeta(s_1,\ldots,s_d) = \mathrm{Li}_{s_1,\ldots,s_d}(1) = \sum_{n_1>\ldots>n_d>0} \frac{1}{n_1^{s_1}\cdots n_d^{s_d}}$$

Francis Brown constructed single-valued multiple polylogarithms $\mathrm{Li}^{\mathrm{sv}}_{s_1,\ldots,s_d}(z)$

Single-valued MZVs

$$\zeta^{\mathrm{sv}}(s_1,\ldots,s_d) = \mathrm{Li}^{\mathrm{sv}}_{s_1,\ldots,s_d}(1)$$

$$\zeta^{\mathrm{sv}}(2k) = 0\,, \qquad \zeta^{\mathrm{sv}}(2k+1) = 2\zeta(2k+1)\,, \qquad k \in \mathbb{N}$$

Vector space of MZVs

MZVs of given weight form a finite-dimensional vector space over the rational numbers. We expand in a basis to take into account relations between MZVs of the same weight.

weight	2	3	4	5	6	7	8
generators	$\zeta(2)$	$\zeta(3)$		$\zeta(5)$		$\zeta(7)$	$\zeta(5,3)$
basis	$\zeta(2)$	$\zeta(3)$	$\zeta(2)^2$	$\zeta(5)$	$\zeta(3)^2$	$\zeta(7)$	$\zeta(5,3)$
				$\zeta(3)\zeta(2)$	$\zeta(2)^3$	$\zeta(5)\zeta(2)$	$\zeta(5)\zeta(3)$
						$\zeta(3)\zeta(2)^2$	$\zeta(3)^2\zeta(2)$
							$\zeta(2)^4$

Example: $\zeta(3,2,1) = 3\zeta(3)^2 - \frac{29}{30}\zeta(2)^3$

Single-valued MZVs form a much smaller vector space. All generators have odd weight.

weight	3	5	7	9	11	13
generators	$\zeta^{sv}(3)$	$\zeta^{sv}(5)$	$\zeta^{sv}(7)$	$\zeta^{sv}(9)$	$\zeta^{sv}(11)$	$\zeta^{sv}(13)$
					$\zeta^{sv}(5,3,3)$	$\zeta^{sv}(7,3,3)$
						$\zeta^{sv}(5,5,3)$

Example: $\zeta^{sv}(5,3,3) = 2\zeta(5,3,3) - \frac{8}{7}\zeta(5)\zeta(2)^3 + \frac{12}{5}\zeta(7)\zeta(2)^2 + 90\zeta(9)\zeta(2) - 5\zeta(3)^2\zeta(5)$

The ansatz

We make the assumptions:

- $\alpha^{(1)}_{a,b}$ is in the ring of single-valued MZVs and has uniform weight $4 + 2a + 3b$.
- $T^{(2)}_m(\delta)$: Euler-Zagier sums (and svMZVs) of weight $\leq m + 2$ and depth $\leq m + 1$
- $F^{(2)}_m(\delta)$: Euler-Zagier sums (and svMZVs) of weight $\leq m + 3$ and depth $\leq m + 1$

Fixing $\alpha^{(1)}_{a,0}$

Ansatz for unknown terms in $\alpha^{(1)}_{a,0}$ (8 parameters):

$$T_0^{(2)}(\delta) = d_0 + d_1 \delta Z_1(\delta-1) + d_2 \delta^2 Z_2(\delta-1)$$
$$F_0^{(2)}(\delta) = c_0 + c_1 \delta Z_1(\delta-1) + c_2 \delta^2 Z_2(\delta-1) + c_3 \delta^3 Z_3(\delta-1) + \tilde{c}\delta^3 \zeta(3)$$

Demand convergence:

$$\alpha^{(1)}_{0,0} = \sum_{\delta=1}^{\infty} \left(\frac{\tilde{c}\zeta(3) + c_3 Z_3(\delta-1)}{\delta} + O(\delta^{-2}) \right) \qquad \Rightarrow \qquad \tilde{c} = -c_3$$

Demand single-valuedness:

$$\alpha^{(1)}_{0,0} = \frac{1}{10}(4c_0 + c_1 + 3c_2 - 5c_3 - 12d_0 - 3d_1 - 9d_2 - 44)\,\zeta(2)^2\,,$$
$$\alpha^{(1)}_{1,0} = (\ldots)\,\zeta(2)^3 + (\ldots)\,\zeta(3)^2\,, \qquad \alpha^{(1)}_{2,0} = (\ldots)\,\zeta(2)^4 + (\ldots)\,\zeta(3)\zeta(5) + (\ldots)\,\zeta(5,3)\,,$$
$$\alpha^{(1)}_{3,0} = (\ldots)\,\zeta(2)^5 + (\ldots)\,\zeta(3)\zeta(7) + (\ldots)\,\zeta(5)^2 + (\ldots)\,\zeta(7,3)\,.$$

Use localisation result: $\alpha^{(1)}_{1,0} = -2\zeta(3)^2$ [Chester,Pufu;2020] All parameters fixed!

Result for $\alpha^{(1)}_{a,0}$

The result is

$$\alpha^{(1)}_{a,0} = 2\zeta(2a+1)\zeta(3) - 2\zeta(2a+1,3) - (2a+1)\zeta(2a+2,2) + \frac{1}{3}\left(6a^2+23a+3\right)\zeta(2a+3,1) - \frac{1}{3}\left(4a^3+16a^2+13a+6\right)\zeta(2a+4)$$

Can be rewritten as

$$\alpha^{(1)}_{a,0} = -\left(a^2+\frac{35}{6}a+\frac{1}{2}\right)\sum_{\substack{i_1,i_2=0\\ i_1+i_2=a-1}}^{a-1} \zeta(3+2i_1)\zeta(3+2i_2) - 2\sum_{\substack{i_1,i_2=0\\ i_1+i_2=a-1}}^{a-1} i_1 i_2 \zeta(3+2i_1)\zeta(3+2i_2)$$

This shows it is in the ring of single-valued MZVs for all values of a.

More $\alpha^{(1)}_{a,b}$ solutions

In the same way we find unique solutions for $\alpha^{(1)}_{a,b}$ with $b = 1, \ldots, 6$

Can not be written in terms of single zeta values, example:

$$\alpha^{(1)}_{3,1} = -\frac{209279}{300}\zeta(13) - 166\zeta(3)^2\zeta(7) - 174\zeta(3)\zeta(5)^2 + \frac{2}{25}\zeta^{\text{sv}}(5,5,3)$$

Examples for $T^{(2)}_m(\delta)$: (Z's are evaluated at $\delta - 1$)

$$\begin{aligned}
T^{(2)}_0(\delta) &= \delta^2 Z_2 + \tfrac{1}{4}\delta Z_1 + 2 \\
T^{(2)}_1(\delta) &= \delta^3\left(Z_3 + 2Z_{1,2} + 3Z_{2,1}\right) + \delta^2\left(\tfrac{7}{4}Z_2 + Z_{1,1}\right) + \tfrac{9}{2}\delta Z_1 \\
T^{(2)}_2(\delta) &= \delta^4\left(2Z_{1,3} + 3Z_{3,1} + 3Z_{2,2} + 4Z_{1,1,2} + 6Z_{1,2,1} + 8Z_{2,1,1}\right) \\
&\quad + \delta^3\left(2Z_3 + \tfrac{15}{4}Z_{1,2} + \tfrac{23}{4}Z_{2,1} + 3Z_{1,1,1}\right) + \delta^2\left(2Z_2 + 9Z_{1,1}\right)
\end{aligned}$$

Similar for $F^{(2)}_m(\delta)$

All $\alpha^{(1)}_{a,b}$ solutions

Based on the results for $b = 0, \ldots, 6$:

$$T^{(2)}_m(\delta) = \sum_{w=m}^{m+2} \sum_{d=\lfloor \frac{m+1}{2} \rfloor}^{m+1} \sum_{\substack{s_1,\ldots,s_d \in \{1,2,3\} \\ s_1+\ldots+s_d=w}} t^m_{s_1,\ldots,s_d} \delta^w Z_{s_1,\ldots,s_d}(\delta - 1)$$

$$F^{(2)}_m(\delta) = 2\delta^3 \zeta(3) F^{(0)}_m(\delta) + \sum_{w=m}^{m+3} \sum_{d=\lfloor \frac{m+1}{2} \rfloor}^{m+1} \sum_{\substack{s_1,\ldots,s_d \in \{1,2,3,4\} \\ s_1+\ldots+s_d=w}} f^m_{s_1,\ldots,s_d} \delta^w Z_{s_1,\ldots,s_d}(\delta - 1)$$

Found general formula for coefficients $t^m_{s_1,\ldots,s_d}$ and $f^m_{s_1,\ldots,s_d}$ $\quad\rightarrow\quad$ solution for all $\alpha^{(1)}_{a,b}$!

OPE data

$$r_n(\delta) = \frac{4^{2-2\delta}\delta^{2\delta-2n-1}(2\delta-2n-1)}{\Gamma(\delta)\Gamma\left(\delta-\left\lfloor\frac{n}{2}\right\rfloor\right)}$$

Leading Regge trajectory $\ell = 2(\delta-1)$ (non-degenerate)

$$f_0(\delta, 2(\delta-1)) = \frac{r_0(\delta)}{\delta}, \qquad \tau_2(\delta, 2(\delta-1)) = \frac{3\delta^2-\delta+2}{2\sqrt{\delta}}$$

$$f_2(\delta, 2(\delta-1)) = -\frac{r_0(\delta)}{96\delta^2}\left(112\delta^3 - 1872\delta^2 + 344\delta + 201\right) + 2\delta^2\zeta(3)f_0(\delta, 2(\delta-1))$$

$\tau_2(\delta, 2(\delta-1))$ agrees with integrability result [Gromov,Serban,Shenderovich,Volin;2011]!

Next Regge trajectory

$$\langle f_0\tau_2\rangle(\delta, 2(\delta-2)) = \frac{r_1(\delta)}{18\sqrt{\delta}}\left(18\delta^4 + 25\delta^3 - 57\delta^2 + 50\delta - 72\right) \qquad \text{etc.}$$

We determine $\langle f_0\tau_2\rangle$ and $\langle f_2\rangle$ for many Regge trajectories.

Further checks

1. Single-valuedness (and zeros for $a < 0$) of $\alpha^{(1)}_{a,b}$:
   - imposed for $b = 0, \ldots, 6$ and $a = 0, 1, 2, 3$
   - checked for $b \leq 6$, $4 + 2a + 3b \leq 28$ and $b \leq 12$, $4 + 2a + 3b \leq 25$ (using HyperlogProcedures by Oliver Schnetz)

2. Equations for OPE data are overconstrained, with solution, e.g.

$$T^{(2)}_m(\delta)|_{\text{definition}} = T^{(2)}_m(\delta)|_{\text{solution}}\,, \qquad \delta \text{ fixed}$$

   - δ unknowns: $\ell = 0, 2, \ldots, 2(\delta - 1)$
   - $2\delta - 1$ equations: $m = 0, 1, \ldots, 2(\delta - 1)$

Outline

The flat space transform

Flat space transform achieves Borel summation of low energy expansion.

$$M(s_1, s_2) = \frac{8}{(s_1 - \frac{2}{3})(s_2 - \frac{2}{3})(s_3 - \frac{2}{3})} + \sum_{a,b=0}^{\infty} \frac{\Gamma(2a+3b+6)}{8^{a+b}\lambda^{\frac{3}{2}+a+\frac{3}{2}b}} \sigma_2^a \sigma_3^b \left(\alpha_{a,b}^{(0)} + \frac{\alpha_{a,b}^{(1)}}{\sqrt{\lambda}} \right) + O(1/\lambda)$$

Flat space transform [Penedones;2010]

$$A(S, T) = \mathrm{FS}(M(s_1, s_2)) = 2\lambda^{\frac{3}{2}} c \int_{\kappa - i\infty}^{\kappa + i\infty} \frac{d\alpha}{2\pi i} e^{\alpha} \alpha^{-6} M\left(\frac{2\sqrt{\lambda} S}{\alpha}, \frac{2\sqrt{\lambda} T}{\alpha} \right)$$

Result:

$$A(S, T) = A^{(0)}(S, T) + \frac{1}{\sqrt{\lambda}} A^{(1)}(S, T) + O(1/\lambda)$$

$$= \frac{1}{STU} - \frac{1}{3\sqrt{\lambda}} \frac{\hat{\sigma}_2}{\hat{\sigma}_3^2} + 2 \sum_{a,b=0}^{\infty} \hat{\sigma}_2^a \hat{\sigma}_3^b \left(\alpha_{a,b}^{(0)} + \frac{\alpha_{a,b}^{(1)}}{\sqrt{\lambda}} \right) + O(1/\lambda)$$

Summing $\alpha^{(0)}_{a,b}$

Recall sum rule:

$$\alpha^{(0)}_{a,b} = \sum_{\delta=1}^{\infty} \sum_{m=0}^{b} \frac{c^{(0)}_{a,b,m}}{\delta^{3+2a+3b}} F^{(0)}_m(\delta)$$

We use the generating series [Zagier,Zerbini;2019]:

$$\sum_{a,b=0}^{\infty} c^{(0)}_{a,b,m} x^a y^b = \frac{1}{2} \frac{y+2}{1-x-y} \left(\frac{\sqrt{1-4y}-1}{2} \right)^m, \qquad \sum_{m=0}^{\infty} F^{(0)}_m(\delta) \left(\frac{z}{\delta}\right)^m = \left(\frac{z+\delta-1}{\delta-1} \right)^2$$

Sum over a, b, m

$$2 \sum_{a,b=0}^{\infty} \hat{\sigma}_2^a \hat{\sigma}_3^b \alpha^{(0)}_{a,b} = \sum_{a,b,m=0}^{\infty} \sum_{\delta=1}^{\infty} \frac{2}{\delta^3} x^a y^b c^{(0)}_{a,b,m} F^{(0)}_m(\delta) = \sum_{\delta=1}^{\infty} \frac{1}{\delta^3} \frac{y+2}{1-x-y} \left(\frac{z+\delta-1}{\delta-1} \right)^2$$

$$x = \hat{\sigma}_2/\delta^2,\ y = \hat{\sigma}_3/\delta^3,\ z = \delta \left(\sqrt{1-4y} - 1 \right)/2, \qquad \frac{y+2}{1-x-y} = 2 - \frac{S}{S-\delta} - \frac{T}{T-\delta} - \frac{U}{U-\delta}$$

Summing $\alpha^{(1)}_{a,b}$

Recall sum rule:

$$\alpha^{(1)}_{a,b} = \sum_{\delta=1}^{\infty}\sum_{m=0}^{b} \frac{1}{\delta^{4+2a+3b}}\left(c^{(0)}_{a,b,m}\left(F^{(2)}_m(\delta) - (3+2a+3b)\,T^{(2)}_m(\delta)\right) + c^{(2)}_{a,b,m}F^{(0)}_m(\delta)\right)$$

We find the generating series:

$$\sum_{m=0}^{\infty} F^{(2)}_m(\delta)\left(\frac{z}{\delta}\right)^m = \sum_{n=0}^{\delta-1} h_n(\delta)\binom{z+\delta-\frac{n}{2}-1}{\delta-n-1}^2, \quad \sum_{m=0}^{\infty} T^{(2)}_m(\delta)\left(\frac{z}{\delta}\right)^m = \sum_{n=0}^{\delta-1} g_n(\delta)\binom{z+\delta-\frac{n}{2}-1}{\delta-n-1}^2$$

Sum over a, b, m

$$2\sum_{a,b=0}^{\infty} \hat{\sigma}_2^a\hat{\sigma}_3^b\alpha^{(1)}_{a,b} = \sum_{\delta=1}^{\infty}\sum_{n=0}^{\delta-1}\frac{1}{\delta^4}\mathcal{D}_n(\delta)\frac{y+2}{1-x-y}\binom{z+\delta-\frac{n}{2}-1}{\delta-n-1}^2$$

Using $ax^a = x\partial_x x^a$ etc:

$$\mathcal{D}_n(\delta) = h_n(\delta) - g_n(\delta)\,(3 + 2x\partial_x + 3y\partial_y) + \delta_{n,0}\big(\text{third order differential operator in } x, y, z\big)$$

$\Rightarrow A^{(1)}(S,T)$ has poles up to 4th order at $S, T, U = 1, 2, 3, \ldots$

Summary

- bound on chaos
  $\Rightarrow$ dispersive sum rules
  $\Rightarrow$ operators with $\Delta \sim \lambda^{\frac{1}{4}}$ in planar $\mathcal{N} = 4$ SYM at large λ

- assumption that $\alpha^{(1)}_{a,b}$ are single-valued periods
  $\Rightarrow$ solution of sum rules

- low energy expansion can be Borel summed
  analytic structure generalises the flat space Virasoro-Shapiro amplitude

Future directions

- Next order (WIP): it looks like $\alpha^{(2)}_{a,b}$ can be fixed after we determine $\langle f_0 \tau_2^2 \rangle$.
  This requires solving a mixing problem by studying $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_p \mathcal{O}_p \rangle$.

- $\mathcal{N} = 4$ SYM theory is modular invariant in the gauge coupling.
  Can we identify modular functions that have $\alpha^{(k)}_{a,b}$ as perturbative terms?
  The first few are the same as in flat space: $E(\frac{3}{2}, \tau_s, \bar{\tau}_s), E(\frac{5}{2}, \tau_s, \bar{\tau}_s), \varepsilon(3, \frac{3}{2}, \frac{3}{2}, \tau_s, \bar{\tau}_s)$
  [Green,Gutperle,Kwon,Vanhove;1997-2006], [Chester,Green,Pufu,Wang,Wen;2019,2020]

- What is the worldsheet theory for strings in AdS?
  First step: find single-valued integral representation for $A^{(1)}(S,T)$, generalising

$$(S+T)^2 A^{(0)}(S,T) = -\frac{1}{2\pi i} \int_{\mathbb{P}^1(\mathbb{C})} |z|^{-2S-2} |1-z|^{-2T-2} dz d\bar{z}$$

Thank you!

Questions?

$T_m^{(2)}(\delta)$ formula

$$T_m^{(2)}(\delta) = \sum_{w=m}^{m+2} \sum_{d=\lfloor \frac{m+1}{2} \rfloor}^{m+1} \sum_{\substack{s_1,\ldots,s_d \in \{1,2,3\} \\ s_1+\ldots+s_d=w}} t_{\boldsymbol{s}}^m \delta^w Z_{\boldsymbol{s}}(\delta - 1)$$

$$t_{\boldsymbol{s}}^m = \begin{cases} 2^{n_1^s} \left(Q_w^m(n_1^{\boldsymbol{s}}, n_2^{\boldsymbol{s}}, n_3^{\boldsymbol{s}}) + p_w^m P_{\boldsymbol{s}} \right), & n_3^{\boldsymbol{s}} \in \{0,1\}\,, \\ 0, & n_3^{\boldsymbol{s}} > 1\,, \end{cases}$$

$$\boldsymbol{s} = (s_1, \ldots, s_d)$$

$n_k^{\boldsymbol{s}} = \#$ of k's in $\boldsymbol{s}$

Q_w^m = result for lexicographically ordered $\boldsymbol{s}$

$$Q_{m+2}^m(n_1^{\boldsymbol{s}}, n_2^{\boldsymbol{s}}, 1) = 1\,, \qquad Q_{m+2}^m(n_1^{\boldsymbol{s}}, n_2^{\boldsymbol{s}}, 0) = \frac{n_2^{\boldsymbol{s}}(n_2^{\boldsymbol{s}}+1)}{2}\,,$$

$$Q_{m+1}^m(n_1^{\boldsymbol{s}}, n_2^{\boldsymbol{s}}, 1) = 2\,, \qquad Q_{m+1}^m(n_1^{\boldsymbol{s}}, n_2^{\boldsymbol{s}}, 0) = \frac{n_1^{\boldsymbol{s}} + n_2^{\boldsymbol{s}}(8n_2^{\boldsymbol{s}}+6)}{8}\,,$$

$$Q_m^m(n_1^{\boldsymbol{s}}, n_2^{\boldsymbol{s}}, 1) = 0\,, \qquad Q_m^m(n_1^{\boldsymbol{s}}, n_2^{\boldsymbol{s}}, 0) = 2 - \frac{n_1^{\boldsymbol{s}}(n_1^{\boldsymbol{s}} + 4n_2^{\boldsymbol{s}} - 3)}{8} - \frac{n_2^{\boldsymbol{s}}(n_2^{\boldsymbol{s}} - 1)}{2}\,,$$

$p_w^m P_{\boldsymbol{s}}$ = extra term for unordered $\boldsymbol{s}$

$$p_{m+2}^m = \frac{1}{2}\,, \quad p_{m+1}^m = 1\,, \quad p_m^m = 0\,, \quad P_{\boldsymbol{s}} = \sum_{i=1}^{d} \delta_{s_i,\max(\boldsymbol{s})} \sum_{j=i+1}^{d} s_j(1 - \delta_{s_j,\max(\boldsymbol{s})})$$