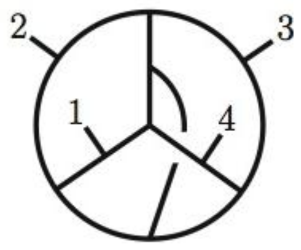
Surprises in UV Behavior of Gravity Theories

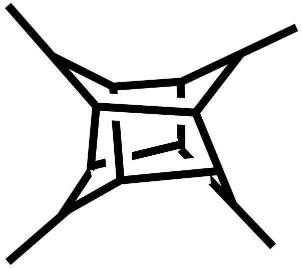
December 6, 2012

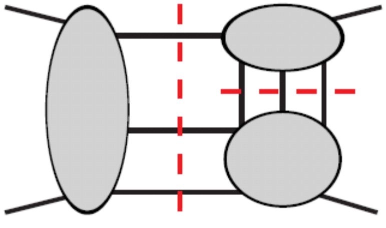
IHES

Zvi Bern, UCLA

Based on papers with John Joseph Carrasco, Scott Davies, Tristan Dennen, Lance Dixon, Yu-tin Huang, Henrik Johansson and Radu Roiban.







Outline

1) A hidden structure in gauge and gravity amplitudes
   - a duality between color and kinematics.
   - gravity as a double copy of gauge theory.

2) Examples of theories where duality and double copy holds.

3) Application: Reexamination of gravity UV divergences
   - Lightning review of supergravity UV properties.
   - Various surprises in UV behavior of supergravity and super-Yang-Mills theory,
   - Prospects for future.

Purpose of Talk

1) Attempt to point out spots where we could use help from our mathematician colleagues.
2) Show some amazing connections between two fundamental theories: gravity and gauge theory.
3) Give you a flavor of how these ideas are applied to questions of interest in physics.

Duality Between Color and Kinematics

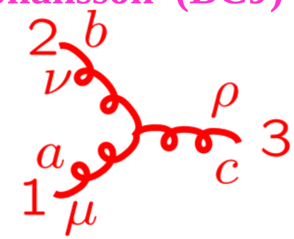
ZB, Carrasco, Johansson (BCJ)

coupling constant

color factor

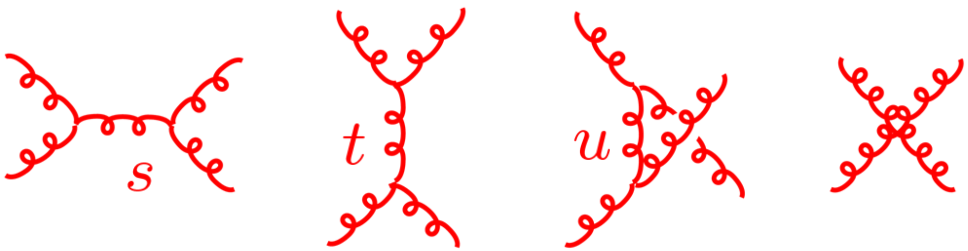
momentum dependent kinematic factor

$$-g f^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = i f^{abc} T^c$

Jacobi Identity $f^{a_1 a_2 b} f^{b a_4 a_3} + f^{a_4 a_2 b} f^{b a_3 a_1} + f^{a_4 a_1 b} f^{b a_2 a_3} = 0$



Use $1 = s/s = t/t = u/u$ to assign 4-point diagram to others.

$$\mathcal{A}_4^{\text{tree}} = g^2 \left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right)$$

$s = (k_1 + k_2)^2$

$t = (k_1 + k_4)^2$

$u = (k_1 + k_3)^2$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

Numerator factors satisfy similar identity: $n_u = n_s - n_t$

Color and kinematics satisfy the same identity

Duality Between Color and Kinematics

Consider five-point tree amplitude:

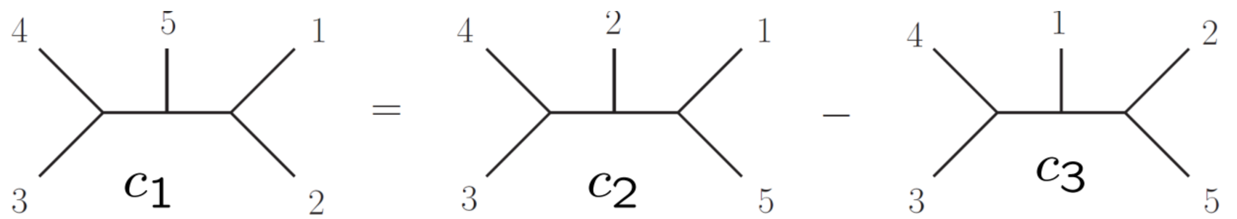
(BCJ)

color factor

kinematic numerator factor

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i\, n_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

Feynman propagators



$$c_1 \equiv f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2}, \quad c_2 \equiv f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5}, \quad c_3 \equiv f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5\, k_2 \cdot \varepsilon_1\, \varepsilon_2 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$\boxed{c_1 - c_2 + c_3 = 0 \Leftrightarrow n_1 - n_2 + n_3 = 0}$$

Lie algebra

Claim: At n-points we can always find a rearrangement so color and kinematics satisfy the same algebraic constraint equations.

Nontrivial constraints on amplitudes in field theory and string theory

BCJ, Bjerrum-Bohr, Feng, Damgaard, Vanhove; Stieberger; Mafra, Stieberger, Schlotterer; Tye and Zhang; Feng, Huang, Jia; Chen, Du, Feng; Du, Feng, Fu; Naculich, Nastase, Schnitzer

See talk from Bjerrum-Bohr

BCJ

Gravity and Gauge Theory

kinematic numerator

color factor

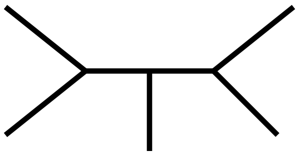
gauge theory: $$\frac{1}{g^{n-2}}\mathcal{A}_n^{\rm tree}(1,2,3,\ldots,n)=\sum_i \frac{n_i\, c_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

sum over diagrams with only 3 vertices

$$c_i \sim f^{a_1 a_2 b_1} f^{b_1 b_2 a_5} f^{b_2 a_4 a_5}$$

Assume we have:

$$c_1+c_2+c_3=0 \quad \Leftrightarrow \quad n_1+n_2+n_3=0$$



Then: $c_i \Rightarrow \tilde{n}_i$ **kinematic numerator of second gauge theory**

Proof: ZB, Dennen, Huang, Kiermaier

gravity: $$\boxed{-i\left(\frac{2}{\kappa}\right)^{(n-2)}\mathcal{M}_n^{\rm tree}(1,2,\ldots,n)=\sum_i \frac{n_i\, \tilde{n}_i}{\prod_{\alpha_i} p^2_{\alpha_i}}}$$

Gravity numerators are a double copy of gauge-theory ones.

This works for ordinary Einstein gravity and susy versions.

Cries out for a unified description of the sort given by string theory!

Gravity From Gauge Theory

BCJ

$$-i\left(\frac{2}{\kappa}\right)^{(n-2)}\mathcal{M}_n^{\rm tree}(1,2,\ldots,n)=\sum_i\frac{n_i\,\tilde{n}_i}{\prod_{\alpha_i}p^2_{\alpha_i}}$$

	n		$\tilde{n}$
***N* = 8 sugra:**	**(*N* = 4 sYM)**	$\times$	**(*N* = 4 sYM)**
***N* = 4 sugra:**	**(*N* = 4 sYM)**	$\times$	**(*N* = 0 sYM)**
***N* = 0 sugra:**	**(*N* = 0 sYM)**	$\times$	**(*N* = 0 sYM)**

***N* = 0 sugra: graviton + antisym tensor + dilaton**

Duality for BLG Theory

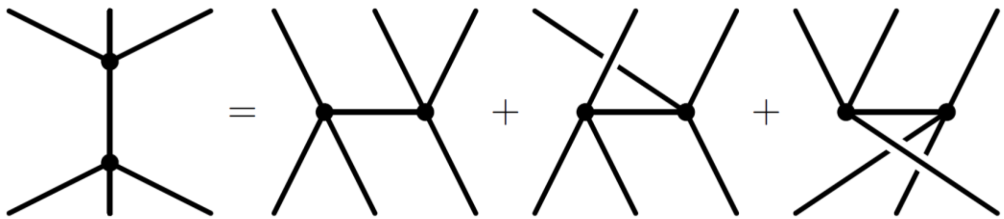
BLG based on a 3 algebra

Bagger, Lambert, Gustavsson (BLG)

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d$$

***D* = 3 Chern-Simons gauge theory**

Four-term color identity:



color factors

kinematic numerators

$$\boxed{c_s = c_t + c_u + c_v \Rightarrow n_s = n_t + n_u + n_v}$$

Such numerators explicitly found at 6 points.

What is the double copy?

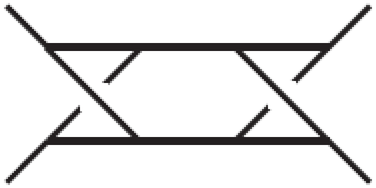
Bargheer, He, and McLoughlin ;
Huang and Johansson

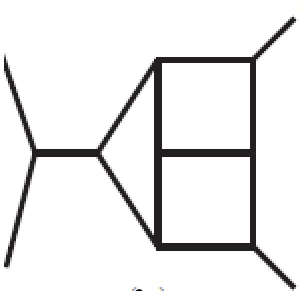
Explicit check at 4 and 6 points shows it is the $E_{8(8)}$ N = 16 supergravity of Marcus and Schwarz. Very non-trivial!

A hidden 3 algebra structure exists in this supergravity.
This story is not limited to ordinary gauge theories.

Loop-Level Conjecture

ZB, Carrasco, Johansson (2010)





$$c_i + c_j + c_k = 0$$
$$n_i + n_j + n_k = 0$$

sum is over diagrams

kinematic numerator

color factor

$$\frac{(-i)^L}{g^{n-2+2L}} \mathcal{A}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

gauge theory

propagators

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} \mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

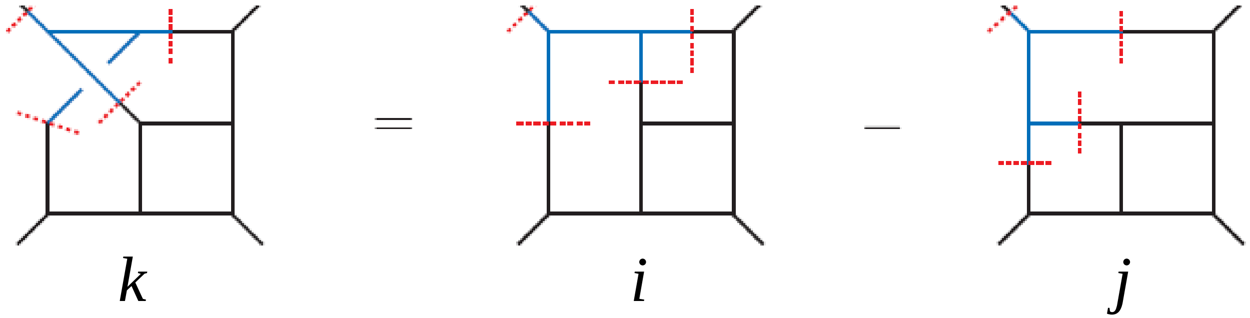
gravity

symmetry factor

Loop-level is identical to tree-level one except for symmetry factors and loop integration.
Double copy works when numerator satisfies duality.

Gravity from Gauge-Theory Amplitudes

BCJ



$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

If you have a set of duality satisfying numerators.
To get:

gauge theory $\longrightarrow$ gravity theory

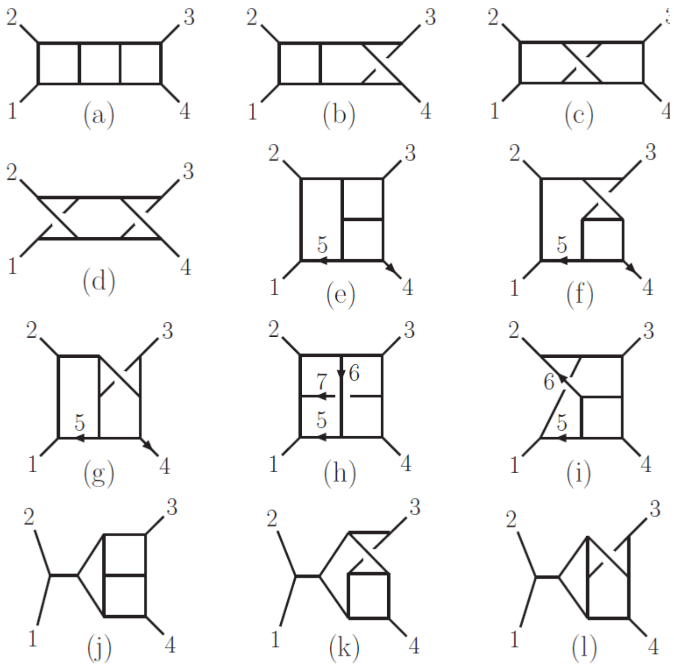
simply take

color factor $\longrightarrow$ kinematic numerator

Gravity loop integrands are free!

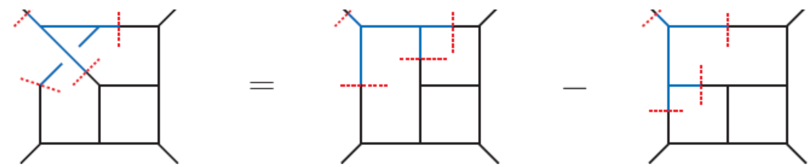
Explicit Three-Loop Construction

ZB, Carrasco, Johansson (2010)



$$c_i = c_j - c_k \;\Rightarrow\; n_i = n_j - n_k$$

For *N*=4 sYM we have the abililty to go to high loop orders. Go to 3 loops. (1 & 2 loops works.)



Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8}$ supergravity) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\left(-\tau_{35}+\tau_{45}+t\right)-t\left(\tau_{25}+\tau_{45}\right)+u\left(\tau_{25}+\tau_{35}\right)-s^2\right)/3$
(h)	$\left(s\left(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u\right)\right.$ $\left.+t\left(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}\right)+s^2\right)/3$
(i)	$\left(s\left(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t\right)\right.$ $\left.+t\left(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}\right)+u\,\tau_{25}+s^2\right)/3$
(j)-(l)	$s(t-u)/3$

$s\,t\,A^{\text{tree}}$ prefactor removed

$$\tau_{ij} = 2k_i \cdot l_j$$

- **Duality works!**
- **Double copy works!**

One diagram to rule them all

ZB, Carrasco, Johansson (2010)

$\mathcal{N} = 4$ super-Yang-Mills integrand

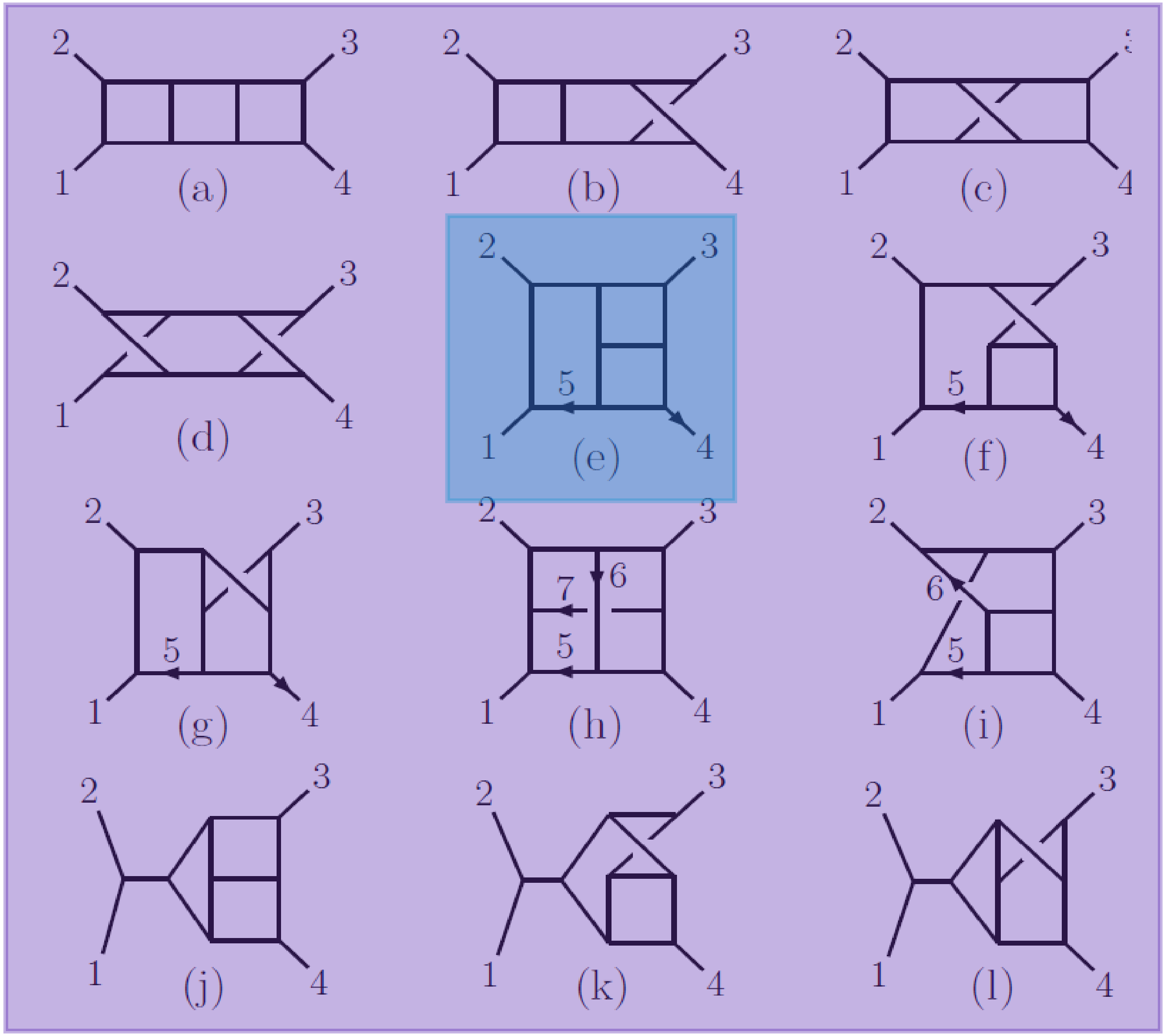


Diagram (e) is the master diagram.

Determine the master integrand in proper form and duality gives all others.

$\mathcal{N} = 8$ sugra given by double copy.

One diagram to rule them all

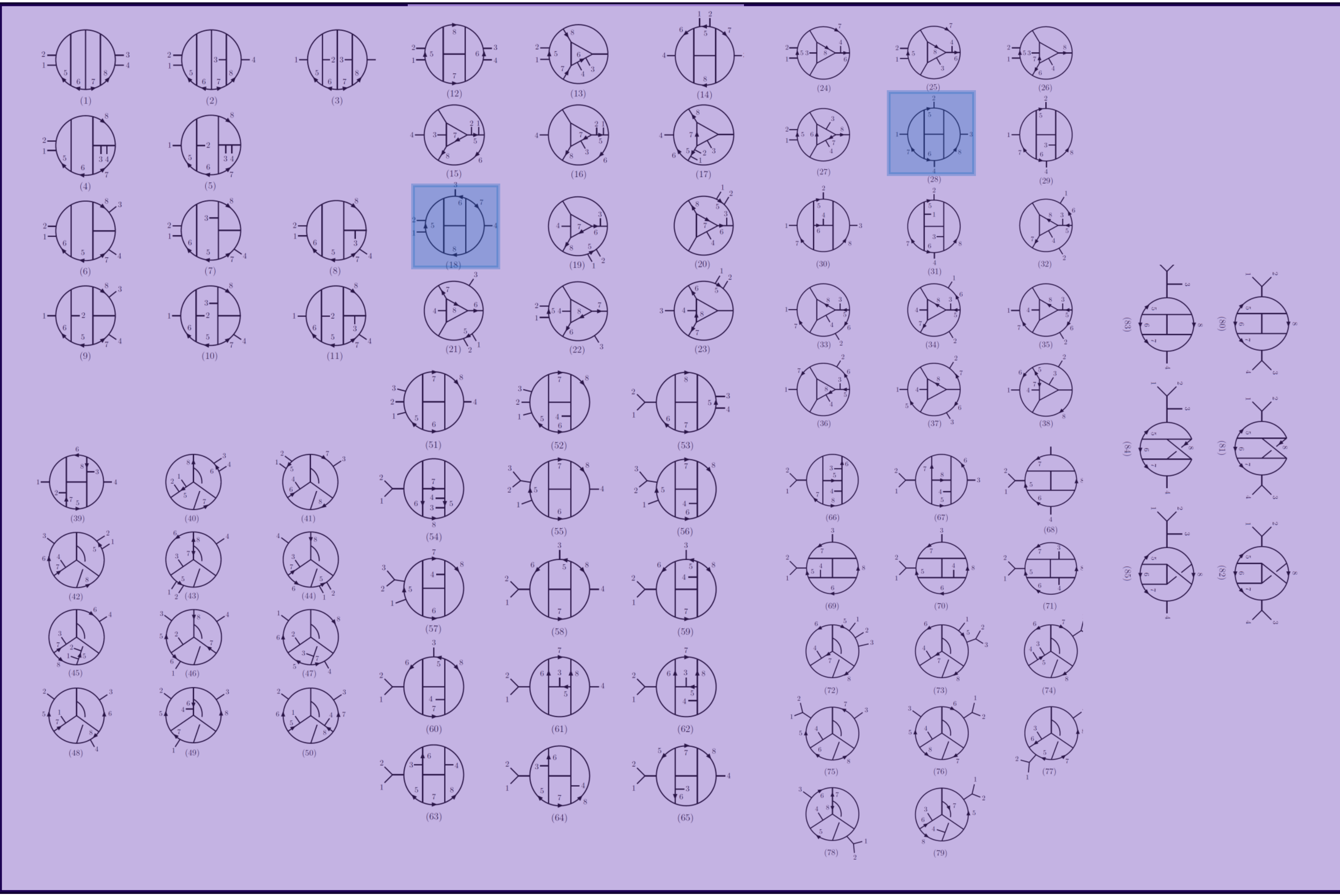
$$
\begin{aligned}
N^{(\mathrm{a})} &= N^{(\mathrm{b})}(k_1,k_2,k_3,l_5,l_6,l_7)\,,\\
N^{(\mathrm{b})} &= N^{(\mathrm{d})}(k_1,k_2,k_3,l_5,l_6,l_7)\,,\\
N^{(\mathrm{c})} &= N^{(\mathrm{a})}(k_1,k_2,k_3,l_5,l_6,l_7)\,,\\
N^{(\mathrm{d})} &= N^{(\mathrm{h})}(k_3,k_1,k_2,l_7,l_6,k_{1,3}-l_5+l_6-l_7) + N^{(\mathrm{h})}(k_3,k_2,k_1,l_7,l_6,k_{2,3}+l_5-l_7)\,,\\
N^{(\mathrm{f})} &= \boxed{N^{(\mathrm{e})}(k_1,k_2,k_3,l_5,l_6,l_7)\,,}\\
N^{(\mathrm{g})} &= \boxed{N^{(\mathrm{e})}(k_1,k_2,k_3,l_5,l_6,l_7)\,,}\\
N^{(\mathrm{h})} &= -N^{(\mathrm{g})}(k_1,k_2,k_3,l_5,l_6,k_{1,2}-l_5-l_7) - N^{(\mathrm{i})}(k_4,k_3,k_2,l_6-l_5,l_5-l_6+l_7-k_{1,2},l_6)\,,\\
N^{(\mathrm{i})} &= N^{(\mathrm{e})}(k_1,k_2,k_3,l_5,l_7,l_6) - N^{(\mathrm{e})}(k_3,k_2,k_1,-k_4-l_5-l_6,-l_6-l_7,l_6)\,,\\
N^{(\mathrm{j})} &= N^{(\mathrm{e})}(k_1,k_2,k_3,l_5,l_6,l_7) - N^{(\mathrm{e})}(k_2,k_1,k_3,l_5,l_6,l_7)\,,\\
N^{(\mathrm{k})} &= N^{(\mathrm{f})}(k_1,k_2,k_3,l_5,l_6,l_7) - N^{(\mathrm{f})}(k_2,k_1,k_3,l_5,l_6,l_7)\,,\\
N^{(\mathrm{l})} &= N^{(\mathrm{g})}(k_1,k_2,k_3,l_5,l_6,l_7) - N^{(\mathrm{g})}(k_2,k_1,k_3,l_5,l_6,l_7)\,,
\end{aligned}
$$

triangle subdiagrams vanish in *N* = 4 sYM

All numerators solved in terms of numerator (e)

N = 4 sYM Four Loops

ZB, Carrasco, Dixon, Johansson, Roiban (2012)



Generalized Gauge Invariance

BCJ
ZB, Dennen, Huang, Kiermaier
Tye and Zhang

gauge theory

$$\frac{(-i)^L}{g^{n-2+2L}}\mathcal{A}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

$$n_i \to n_i + \Delta_i \qquad\qquad \sum_j \int \frac{d^{DL} p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{\Delta_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2} = 0$$

$$(c_\alpha + c_\beta + c_\gamma) f(p_i) = 0$$

Above is just a definition of generalized gauge invariance

gravity

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

$$n_i \to n_i + \Delta_i \qquad\qquad \sum_j \int \frac{d^{DL} p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{\Delta_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2} = 0$$

- **Gravity inherits generalized gauge invariance from gauge theory!**
- **Double copy works even if only one of the two copies has duality manifest!**
- **Used to find expressions for $N \geq 4$ supergravity amplitudes at 1, 2 loops.** ZB, Boucher-Veronneau and Johansson; Boucher-Veroneau and Dixon

Symmetries?

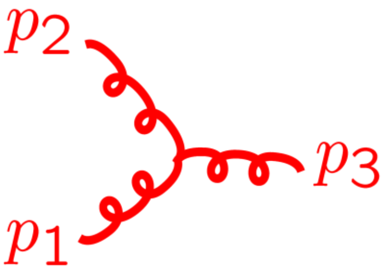
Monteiro and O'Connell

Consider self dual YM. Work in light-cone gauge.

$$\tfrac{i}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = F_{\mu\nu}$$

$$u = t - z$$
$$w = x + iy$$

Generators of diffeomorphism invariance:



$$L_k = e^{-ik\cdot x}(-k_w\partial_u + k_u\partial_w)$$

The Lie Algebra:

YM vertex

$$[L_{p_1}, L_{p_2}] = iX(p_1, p_2)L_{p_1+p_2} = iF_{p_1 p_2}{}^{k} L_k$$

The $X(p_1, p_2)$ are YM vertices, valid for self-dual configurations.

Explains why numerators satisfy Jacobi Identity

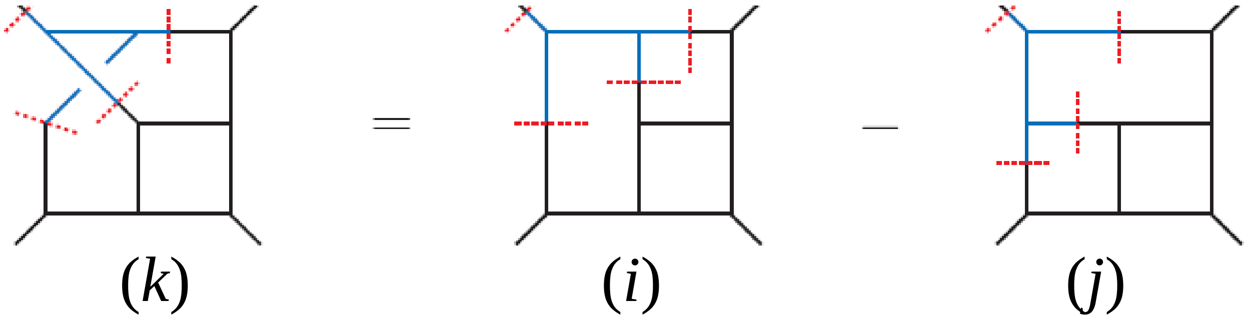
YM inherits diffeomorphism symmetry of gravity!

We need to understand general structure beyond this special case.

Recent progress from Bjerrum-Bohr, Damgaard, Monteiro and O'Connell

Operations on graphs

BCJ



$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

Questions for our math friends:

What infinite dimensional Lie algebra can be behind this (beyond the special self dual case)?

Is there a helpful machinery for solving the above functional graph operations?

One diagram to rule them all

$$
\begin{aligned}
N^{(a)} &= N^{(b)}(k_1, k_2, k_3, l_5, l_6, l_7)\,, \\
N^{(b)} &= N^{(d)}(k_1, k_2, k_3, l_5, l_6, l_7)\,, \\
N^{(c)} &= N^{(a)}(k_1, k_2, k_3, l_5, l_6, l_7)\,, \\
N^{(d)} &= N^{(h)}(k_3, k_1, k_2, l_7, l_6, k_{1,3} - l_5 + l_6 - l_7) + N^{(h)}(k_3, k_2, k_1, l_7, l_6, k_{2,3} + l_5 - l_7)\,, \\
N^{(f)} &= \boxed{N^{(e)}(k_1, k_2, k_3, l_5, l_6, l_7)\,,} \\
N^{(g)} &= \boxed{N^{(e)}(k_1, k_2, k_3, l_5, l_6, l_7)\,,} \\
N^{(h)} &= -N^{(g)}(k_1, k_2, k_3, l_5, l_6, k_{1,2} - l_5 - l_7) - N^{(i)}(k_4, k_3, k_2, l_6 - l_5, l_5 - l_6 + l_7 - k_{1,2}, l_6)\,, \\
N^{(i)} &= N^{(e)}(k_1, k_2, k_3, l_5, l_7, l_6) - N^{(e)}(k_3, k_2, k_1, -k_4 - l_5 - l_6, -l_6 - l_7, l_6)\,, \\
N^{(j)} &= N^{(e)}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(e)}(k_2, k_1, k_3, l_5, l_6, l_7)\,, \\
N^{(k)} &= N^{(f)}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(f)}(k_2, k_1, k_3, l_5, l_6, l_7)\,, \\
N^{(l)} &= N^{(g)}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(g)}(k_2, k_1, k_3, l_5, l_6, l_7)\,,
\end{aligned}
$$

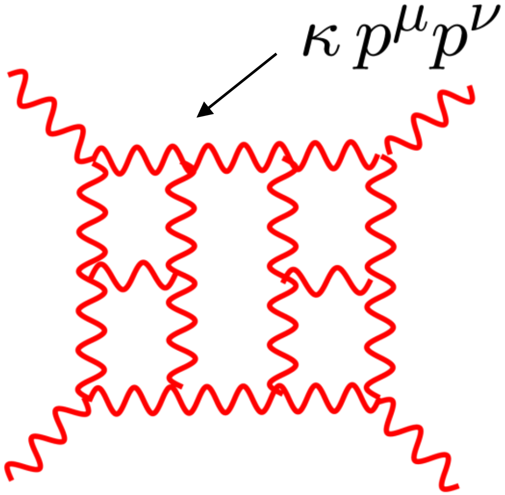
triangle subdiagrams vanish in *N* = 4 sYM

All numerators solved in terms of numerator (e)

Application: UV Properties of Gravity

Power Counting at High Loop Orders

$$\kappa = \sqrt{32\pi G_N}$$ ⟵ Dimensionful coupling



Gravity:
$$\int \prod_{i=1}^{L} \frac{dp_i^D}{(2\pi)^D} \frac{(\kappa\, p_j^\mu p_j^\nu)\cdots}{\text{propagators}}$$

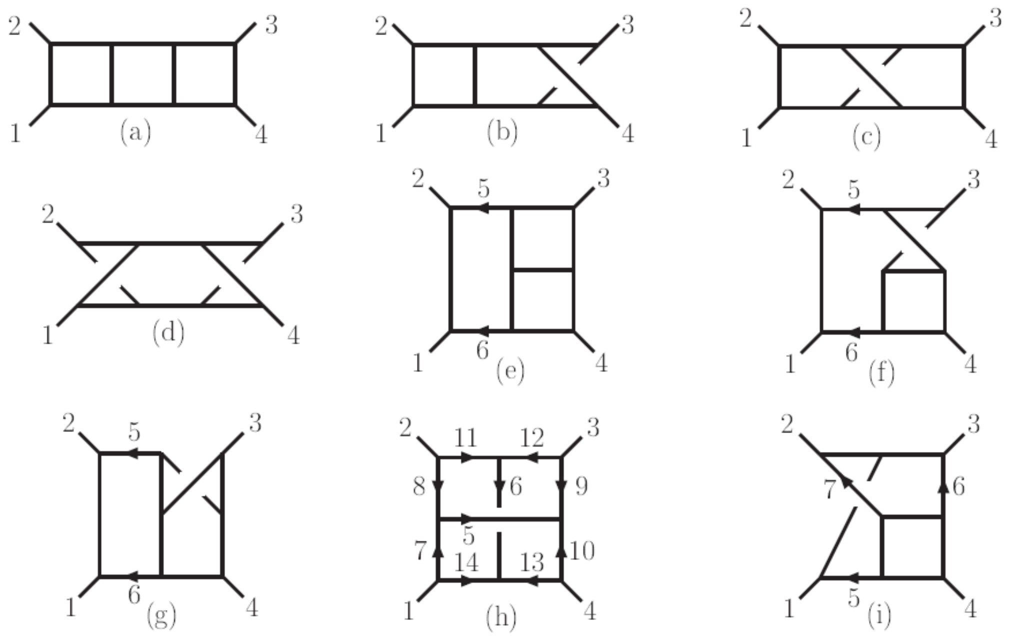
Gauge theory:
$$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g\, p_j^\nu)\cdots}{\text{propagators}}$$

Extra powers of loop momenta in numerator means integrals are badly behaved in the UV.

Non-renormalizable by power counting.

Reasons to focus on *N* = 8 supegravity:
- **With more susy expect better UV properties.**
- **High symmetry implies technical simplicity.**

Ultraviolet properties



Our problem is simple to state: Given a collection of Feynman integrals composing a (super)gravity amplitude is it finite or is it infinite?

Answering this is highly nontrivial when:

1) integrals are not manifestly finite.

2) We are at high loop orders.

Finiteness of *N* = 8 Supergravity?

We are interested in possible UV finiteness of *N* = 8 supergravity because it would imply a new symmetry or non-trivial dynamical mechanism. No previously known symmetry can render a UV theory finite.

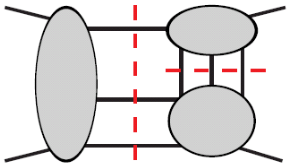
The discovery of either would have a fundamental impact on our understanding of gravity.

Note: Perturbative finiteness is not the only issue for consistent gravity: Nonperturbative completions? High energy behavior of theory? Realistic models?

Constructing Multiloop Amplitudes

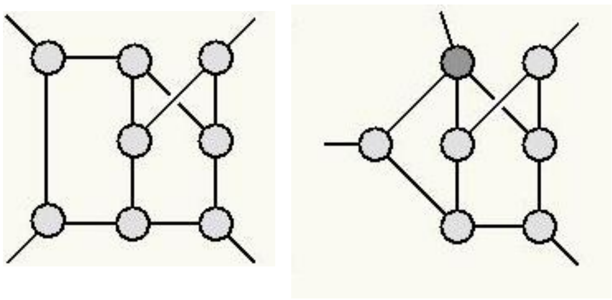
We do have powerful tools for complete calculations including nonplanar contributions and for discovering new structures:

See also talks from Arkani-Hamed and Trnka



- **Unitarity Method.**

  ZB, Dixon, Dunbar, Kosower



- **Method of Maximal Cuts.**

  ZB, Carrasco, Johansson, Kosower

- **Duality between color and kinematics.**

  ZB, Carrasco and Johansson

- **Advanced loop integration technology.**

Chetyrkin, Kataev and Tkachov; A.V. Smirnov; V. A. Smirnov, Vladimirov; Marcus, Sagnotti; Cazkon; etc

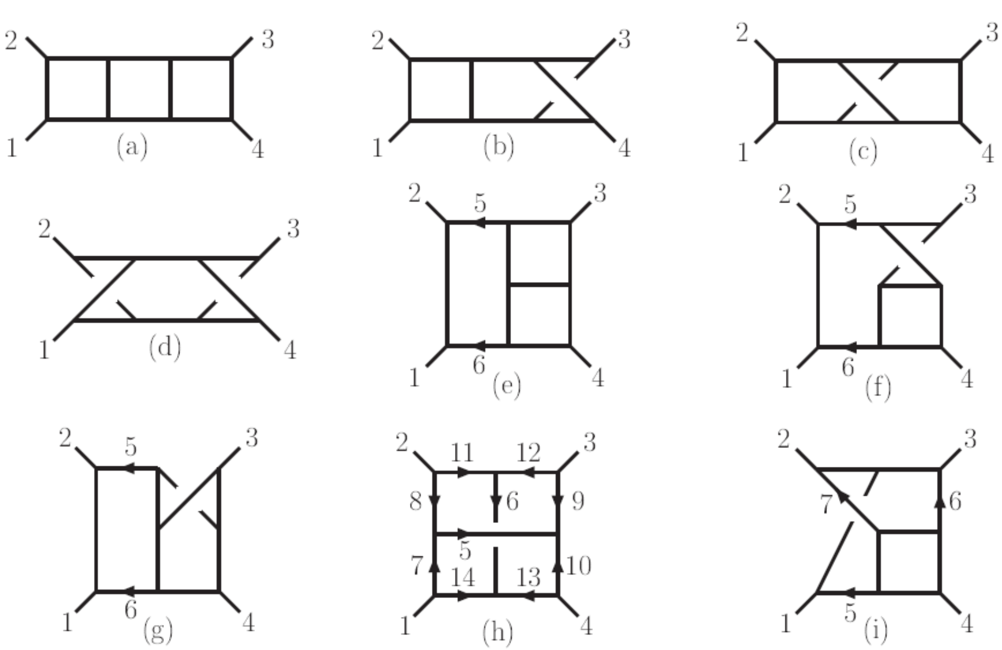
Talks from Smirnov; Weinzeirl; Kosower

In this talk we will explain how the duality between color and kinematics allows us to uncover surprising UV properties in gravity theories.

Complete Three-Loop Result

Analysis of unitarity cuts shows highly nontrivial all-loop cancellations. ZB, Dixon and Roiban (2006); ZB, Carrasco, Forde, Ita, Johansson (2007)

To test completeness of cancellations, we decided to directly calculate potential three-loop divergence.



ZB, Carrasco, Dixon, Johansson, Kosower, Roiban (2007)

Three loops is not only ultraviolet finite it is "superfinite"— finite for $D < 6$.

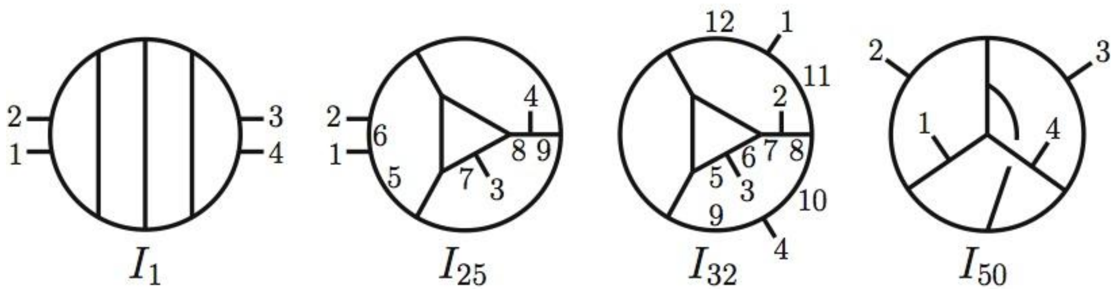
It is very finite!

Obtained via on-shell unitarity method:

Four-Loop Amplitude Construction

ZB, Carrasco, Dixon, Johansson, Roiban

Get 50 distinct diagrams or integrals (ones with two- or three-point subdiagrams not needed).



$$M_4^{\text{4-loop}} = \left(\frac{\kappa}{2}\right)^{10} stu M_4^{\text{tree}} \sum_{S_4} \sum_{i=1}^{50} c_i I_i$$

Integral

symmetry factor

leg perms

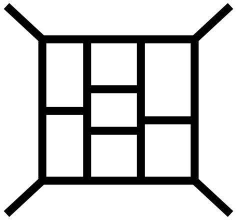
UV finite for $D < 11/2$
It's very finite!

Originally took more than a year.
Today with the double copy we can reproduce it in a couple of days! Another non-trivial example.

Recent Status of Divergences

Consensus that in $N = 8$ supergravity trouble starts at 5 loops and by 7 loops we have valid UV counterterm in $D = 4$ under all known symmetries, suggesting divergences.

Bossard, Howe, Stelle; Elvang, Freedman, Kiermaier; Green, Russo, Vanhove ; Green and Bjornsson ; Bossard , Hillmann and Nicolai; Ramond and Kallosh; Broedel and Dixon; Elvang and Kiermaier; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger



For $N = 8$ sugra in $D = 4$:

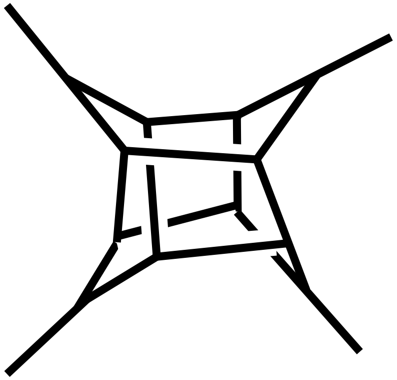
- **All counterterms ruled out until 7 loops!**
- **D^8R^4 counterterm apparently available at 7 loops (1/8 BPS) under all known symmetries. (No known nonrenormalization theorem)**

Bossard, Howe, Stelle and Vanhove

Based on this a reasonable person would conclude that $N = 8$ supergravity almost certainly diverges at 7 loops in $D = 4$

N = 8 Sugra 5 Loop Calculation

ZB, Carrasco, Dixon, Johannson, Roiban



~500 such diagrams with ~100s terms each

Being reasonable and being right are not the same

Place your bets:

- **At 5 loops in D = 24/5 does N = 8 supergravity diverge?**
- **At 7 loops in D = 4 does N = 8 supergravity diverge?**

$D^8 R^4$ **counterterms**

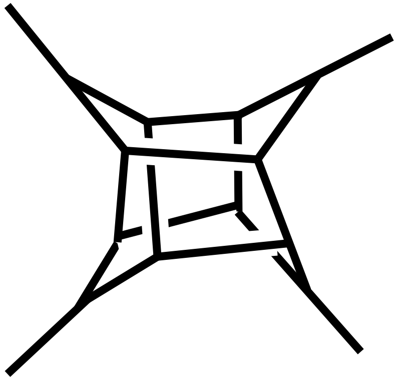
5 loops

Kelly Stelle:
English wine
"It will diverge"

Zvi Bern:
California wine
"It won't diverge"

$\mathcal{N} = 8$ Sugra 5 Loop Calculation

ZB, Carrasco, Dixon, Johannson, Roiban



~500 such diagrams with ~100s terms each

Being reasonable and being right are not the same

Place your bets:

- **At 5 loops in $D = 24/5$ does $\mathcal{N} = 8$ supergravity diverge?**
- **At 7 loops in $D = 4$ does $\mathcal{N} = 8$ supergravity diverge?**

$D^8 R^4$ **counterterms**

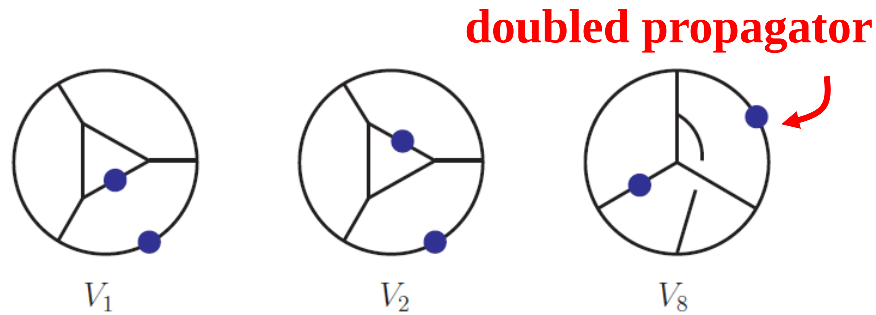
7 loops

David Gross: California wine "It will diverge"

Zvi Bern: California wine "It won't diverge"

New Four-Loop Surprise

ZB, Carrasco, Dixon, Johansson, Roiban (2012)



Critical dimension D =11/2.
Express UV divergences
in terms of vacuum like integrals.
Double copy gives formula with manifest power counting.

gauge theory

$$\mathcal{A}_4^{(4)}(1,2,3,4)\Big|^{SU(N_c)}_{\text{pole}} = -6\, g^{10}\, \mathcal{K}\, N_c^2 \Big(N_c^2\, V_1 + 12\, \boxed{(V_1 + 2\, V_2 + V_8)} \Big)$$
$$\times \Big(s\,(\mathrm{Tr}_{1324} + \mathrm{Tr}_{1423}) + t\,(\mathrm{Tr}_{1243} + \mathrm{Tr}_{1342}) + u\,(\mathrm{Tr}_{1234} + \mathrm{Tr}_{1432}) \Big)$$

color trace

Surprise: same divergence

gravity

$$\mathcal{M}_4^{(4)}\Big|_{\text{pole}} = -\frac{23}{8}\left(\frac{\kappa}{2}\right)^{10} stu(s^2+t^2+u^2)^2\, M_4^{\text{tree}}\, \boxed{(V_1 + 2V_2 + V_8)}$$

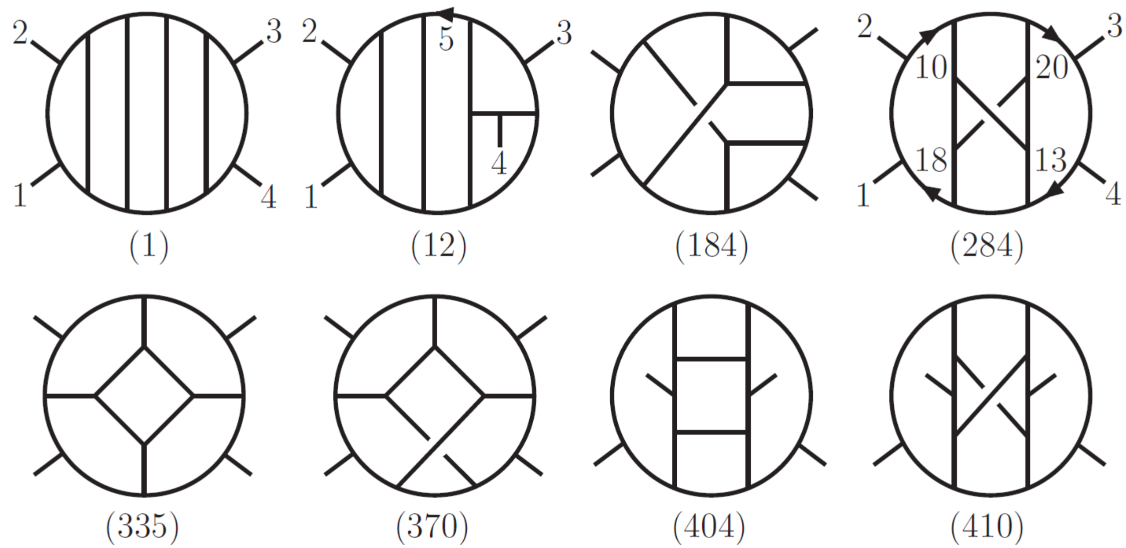
- **Gravity UV divergence is directly proportional to subleading-color single-trace divergence of N = 4 super-Yang-Mills theory.**
- **Same happens at 1-3 loops.**

Calculation of *N* = 4 sYM 5 Loop Amplitude

ZB, Carrasco, Johansson, Roiban (2012)

Key step for *N* = 8 supergravity is construction of complete 5 loop integrand of *N* = 4 sYM theory.

416 such diagrams with ~100s terms each



$$s = (k_1 + k_2)^2$$

diagram numerators

$$N_1 = s^4 \,, \qquad N_{12} = 2s^3 k_3 \cdot l_5 \,,$$
$$N_{284} = 2s^2((l_{10}\cdot l_{20})^2 + (l_{13}\cdot l_{18})^2)$$

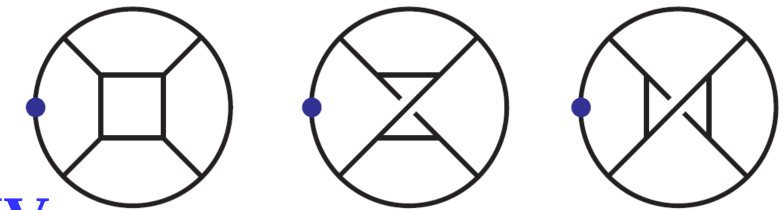
We are trying to figure out a BCJ form. If we can get it we should have supergravity finished very soon!

UV divergences $N = 4$ sYM 5 Loop Amplitude

ZB, Carrasco, Johansson, Roiban (2012)

Critical dimension where divergences first occur: $D = 26/5$

- **Expand in small external momenta.**
- **Apply integral consistency equations.**
- **Find all integral identities.**
- **Get astonishingly simple formula for UV**



(a) (b) (c)

$$A_4^{(5)}\Big|_{\rm div} = \frac{144}{5} g^{12} K \boxed{N_c^3 \Big(N_c^2 V^{(a)} + 12(V^{(\rm a)} + 2V^{(\rm b)} + V^{(\rm c)})\Big)}$$
$$\times \Big(s(\text{Tr}_{1234} + \text{Tr}_{1423}) + t(\text{Tr}_{1243} + \text{Tr}_{1342}) + u(\text{Tr}_{1234} + \text{Tr}_{1432})\Big)$$

Surprising similarity to four loops

color trace

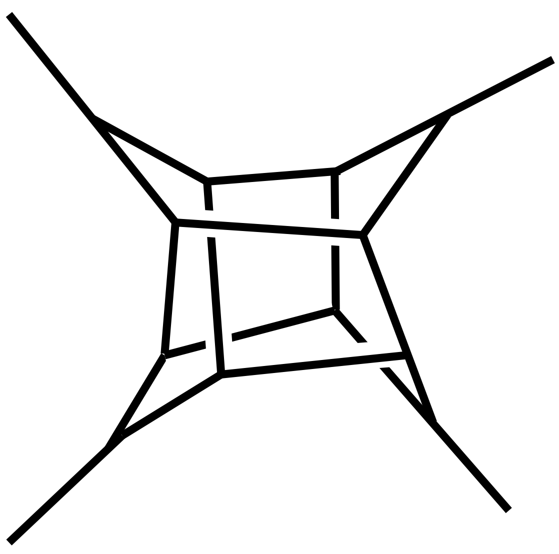
Use FIESTA: $V^{(\rm a)} = \frac{0.331K}{\epsilon}, \quad V^{(\rm b)} = \frac{0.310K}{\epsilon}, \quad V^{(\rm c)} = \frac{0.291K}{\epsilon}$

- **Proves known finiteness bound is saturated.**
- **No double color trace. No $O(N_c)$ single trace.**
- **Note amazing similarity to four loops.**

Stay tuned.

We now have key information needed to calculate the UV properties of *N* = 8 supergravity at five loops.

How long this will take depends on how quickly we can find a representation for *N* = 4 sYM amplitude where duality between color and kinematics holds.



Fine, but do we have any examples where a vanishing divergence is surprising, based on known understanding of the known symmetries ?

Yes.

Two examples in half-maximal supergravity :

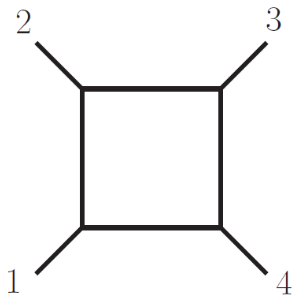
- **D = 5 at 2 loops.** ZB, Davies, Dennen, Huang
- **D = 4 at 3 loops.**

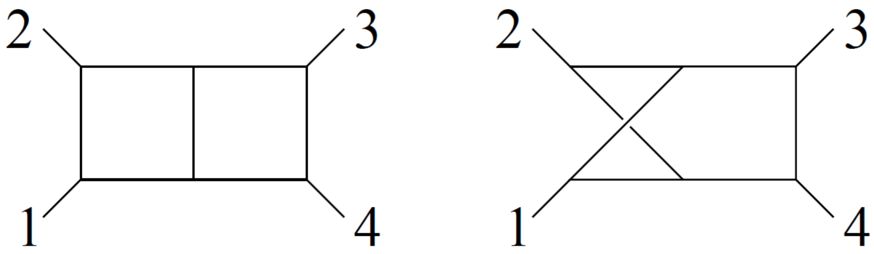
Of course people are searching for symmetry explanations:

- **If a full off-shell superspace exists, these divergences must vanish.**
  Bossard, Howe and Stelle (yesterday)
- **A hidden superconformal symmetry might lead to all-loop finiteness.**
  Ferrara, Kallosh, Van Proeyen
- **Link to gauge-theory cancellations of forbidden color terms.**
  ZB, Davies, Dennen, Huang

Half-Maximal Supergravity

ZB, Davies, Dennen, Huang





No surprises at one loop:

- **Finite for $D < 8$**
- **R^4 divergence in $D = 8$**
- **F^4 four-matter multiplet amplitude diverges in $D = 4$.**

Very instructive to understand from double-copy vantage point

A two-loop surprise:

- **Finite in $D = 5$.**

Tourkine and Vanhove

A three loop surprise:

- **Finite for $D = 4$.**

We now go through these examples

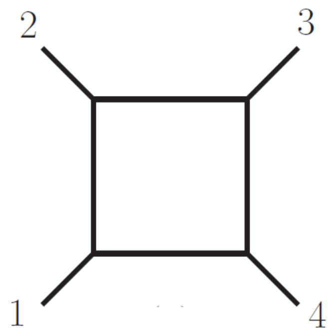
One-Loop Warmup in Half-Maximal Sugra

ZB, Davies, Dennen, Huang

Generic color decomposition:

$$\mathcal{A}_Q^{(1)} = ig^4\Big[c_{1234}^{(1)}A_Q^{(1)}(1,2,3,4) + c_{1342}^{(1)}A_Q^{(1)}(1,3,4,2) + c_{1423}^{(1)}A_Q^{(1)}(1,4,2,3)\Big]$$

Q = # supercharges **Q = 0 is pure non-susy YM**



To get Q +16 supergravity take 2nd copy *N* = 4 sYM

***N* = 4 sYM numerators independent of loop momenta**

$$n_{1234} = n_{1342} = n_{1423} = st A_{Q=16}^{\text{tree}}(1,2,3,4) \qquad c_{1234}^{(1)} \to n_{1234}$$

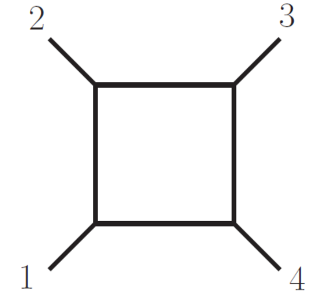
$$\mathcal{M}_{Q+16}^{(1)} = i\left(\frac{\kappa}{2}\right)^4 st A_{Q=16}^{\text{tree}}(1,2,3,4)\Big[A_Q^{(1)}(1,2,3,4) + A_Q^{(1)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3)\Big]$$

Note *exactly* the same combination as in *U*(1) decoupling identity of tree amplitudes.

One-loop divergences in pure YM

ZB, Davies, Dennen, Huang

Go to a basis of color factors:



$C_A = 2N_c$ for $SU(N_c)$

$b_1^{(0)}$ and $b_2^{(0)}$: tree color tensors

$b_1^{(1)}$: independent 1-loop color tensor

$$\mathcal{A}_Q^{(1)} = ig^4\Big[b_1^{(1)}\Big(A_Q^{(1)}(1,2,3,4) + A_Q^{(1)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3)\Big) - \frac{1}{2}C_A b_1^{(0)} A_Q^{(1)}(1,3,4,2) - \frac{1}{2}C_A b_2^{(0)} A_Q^{(1)}(1,4,2,3)\Big]$$

Q supercharges (mainly interested in $Q = 0$)

D **= 4** F^2 **counterterm: 1-loop color tensor *not* allowed.**

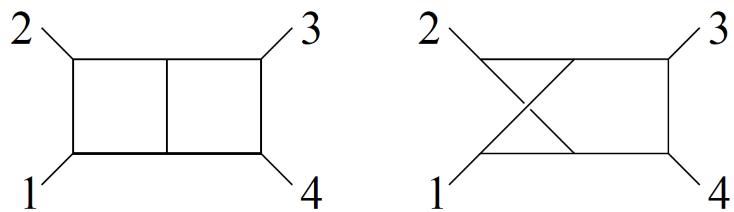
D **= 6** F^3 **counterterm: 1-loop color tensor *not* allowed.**

$$F^3 = f^{abc} F^{a\mu}_{\nu} F^{b\nu}_{\sigma} F^{c\sigma}_{\mu}$$

$$A_Q^{(1)}(1,2,3,4) + A_Q^{(2)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3)\Big|_{D=4,6\ \mathrm{div.}} = 0$$

$$\boxed{M_{Q+16}^{(1)}(1,2,3,4)\Big|_{D=4,6\ \mathrm{div.}} = 0}$$

Two Loop Half Maximal Sugra in $D = 5$



ZB, Davies, Dennen, Huang

$$\mathcal{A}_Q^{(2)} = -g^6 \Big[c^{\rm P}_{1234} A^{\rm P}_Q(1,2,3,4) + c^{\rm P}_{3421} A^{\rm P}_Q(3,4,2,1) + c^{\rm NP}_{1234} A^{\rm NP}_Q(1,2,3,4) + c^{\rm NP}_{3421} A^{\rm NP}_Q(3,4,2,1) + \text{ cyclic} \Big]$$

***D* = 5 *F*³ counterterm: 1,2-loop color tensors forbidden!**
Demand this and plug into double copy:

1) **Go to color basis.**
2) **Demand no forbidden color tensors in pure YM divegence.**
3) **Plug into the BCJ double-copy formula.**

$$\mathcal{M}^{(2)}_{16+Q}(1,2,3,4)\Big|_{D=5\,\text{div.}} = 0$$

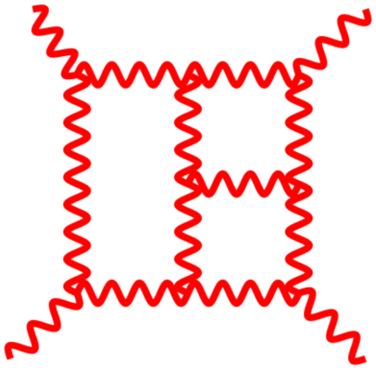
Half-maximal supergravity four-point divergence vanishes because forbidden color tensor cancels in pure YM theory.

$N = 4$ Supergravity in $D = 4$

Fine, but do we have a $D = 4$ example?

$N = 4$ sugra at 3 loops ideal $D = 4$ test case to study.

ZB, Davies, Dennen, Huang



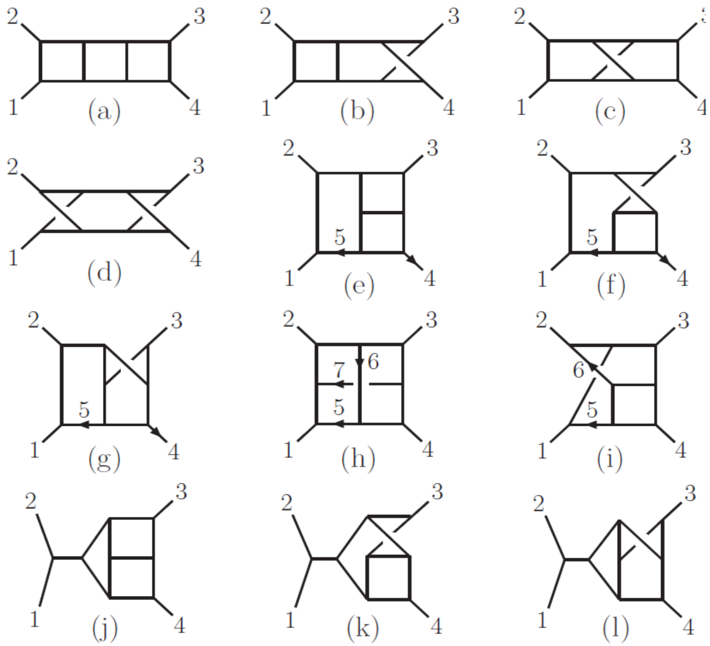
Consensus had it that a valid R^4 counterterm exists for this theory in $D = 4$. Analogous to 7 loop counterterm of $N = 8$.

Bossard, Howe, Stelle; Bossard, Howe, Stelle, Vanhove

Three-loop Construction

ZB, Davies, Dennen, Huang

$\mathcal{N}$ = 4 sugra : ($\mathcal{N}$ = 4 sYM) x ($\mathcal{N}$ = 0 YM)



- **For $\mathcal{N}$ = 4 sYM copy use known BCJ representation.**
- **We will use a Feynman diagram representation for pure YM side.**

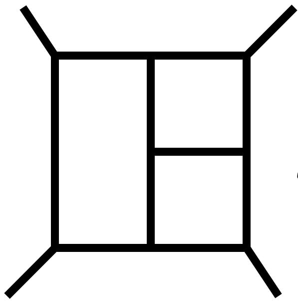
Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\left(-\tau_{35}+\tau_{45}+t\right)-t\left(\tau_{25}+\tau_{45}\right)+u\left(\tau_{25}+\tau_{35}\right)-s^2\right)/3$
(h)	$\left(s\left(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u\right)\right.$ $\left.+t\left(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}\right)+s^2\right)/3$
(i)	$\left(s\left(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t\right)\right.$ $\left.+t\left(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}\right)+u\,\tau_{25}+s^2\right)/3$
(j)-(l)	$s(t-u)/3$

BCJ form of the $\mathcal{N}$ = 4 sYM integrand

Three-Loop Construction

N **= 4 sugra : (*N* = 4 sYM) x (*N* = 0 YM)**

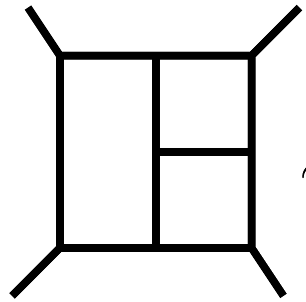
N = 4 sYM



$\sim l \cdot k\, s^2 t A_4^{\text{tree}}$

BCJ representation

pure YM



$\sim (\varepsilon_i \cdot l)^4\; l^4$

Feynman representation

N = 4 sugra linear divergent

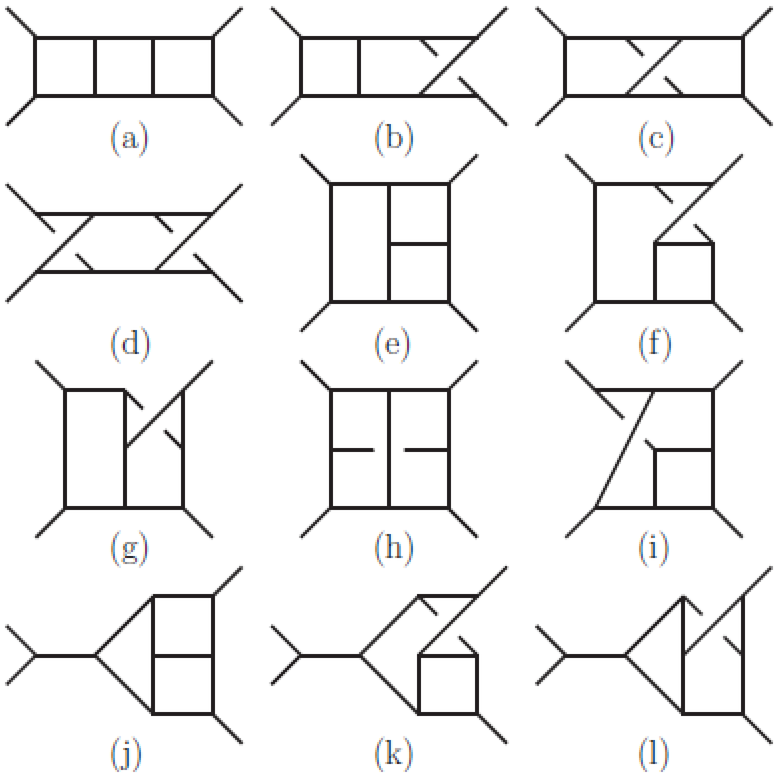
$$\int (d^D l)^3 \frac{k^7 l^9}{l^{20}}$$

simple to see finite for *N*=5,6 sugra

Integrals have subdivergences which causes complications. But this was well understood 30 years ago by Vladimirov and by Marcus and Sagnotti.

The N = 4 Supergravity UV Cancellation

ZB, Davies, Dennen, Huang



Graph	$(\text{divergence})/(\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^8)$
(a)-(d)	0
(e)	$\frac{263}{768}\frac{1}{\epsilon^3}+\frac{205}{27648}\frac{1}{\epsilon^2}+\left(-\frac{5551}{768}\zeta_3+\frac{326317}{110592}\right)\frac{1}{\epsilon}$
(f)	$-\frac{175}{2304}\frac{1}{\epsilon^3}-\frac{1}{4}\frac{1}{\epsilon^2}+\left(\frac{593}{288}\zeta_3-\frac{217571}{165888}\right)\frac{1}{\epsilon}$
(g)	$-\frac{11}{36}\frac{1}{\epsilon^3}+\frac{2057}{6912}\frac{1}{\epsilon^2}+\left(\frac{10769}{2304}\zeta_3-\frac{226201}{165888}\right)\frac{1}{\epsilon}$
(h)	$-\frac{3}{32}\frac{1}{\epsilon^3}-\frac{41}{1536}\frac{1}{\epsilon^2}+\left(\frac{3227}{2304}\zeta_3-\frac{3329}{18432}\right)\frac{1}{\epsilon}$
(i)	$\frac{17}{128}\frac{1}{\epsilon^3}-\frac{29}{1024}\frac{1}{\epsilon^2}+\left(-\frac{2087}{2304}\zeta_3-\frac{10495}{110592}\right)\frac{1}{\epsilon}$
(j)	$-\frac{15}{32}\frac{1}{\epsilon^3}+\frac{9}{64}\frac{1}{\epsilon^2}+\left(\frac{101}{12}\zeta_3-\frac{3227}{1152}\right)\frac{1}{\epsilon}$
(k)	$\frac{5}{64}\frac{1}{\epsilon^3}+\frac{89}{1152}\frac{1}{\epsilon^2}+\left(-\frac{377}{144}\zeta_3+\frac{287}{432}\right)\frac{1}{\epsilon}$
(l)	$\frac{25}{64}\frac{1}{\epsilon^3}-\frac{251}{1152}\frac{1}{\epsilon^2}+\left(-\frac{835}{144}\zeta_3+\frac{7385}{3456}\right)\frac{1}{\epsilon}$

Spinor helicity used to clean up table, but calculation for all states

***All* three-loop divergences cancel completely!**
Cancellations much less trivial than from susy.

4-point 3-loop *N* = 4 sugra UV finite contrary to expectations

Tourkine and Vanhove have understood this result by extrapolating from two-loop heterotic string amplitudes.

Obvious Next Steps

Still want to understand gravity better at even high loops.

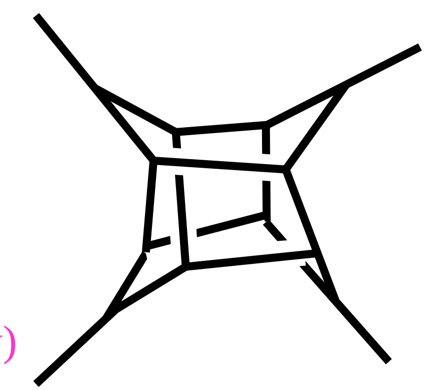
1) Five Loops: Compute the coefficient of the D^8R^4 five-loop counterterm of $N = 8$ supergravity in $D = 24/5$.

ZB, Carrasco, Johansson, Roiban

2) Four Loops: Find the coefficient of the D^2R^4 four-loop counterterm of half-maximal supergravity in $D = 4$.

ZB, Davies, Dennen, Smirnov & Smirnov

3) Can we prove that in general, surprising supergravity cancellations are tied to cancellations of forbidden color factors in gauge-theory divergences?



4) How many of the surprises might be explainable by more standard symmetries?

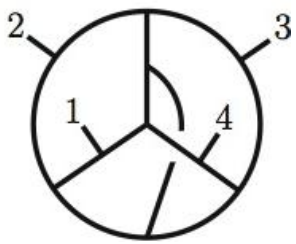
Bossard, Howe and Stelle (yesterday)

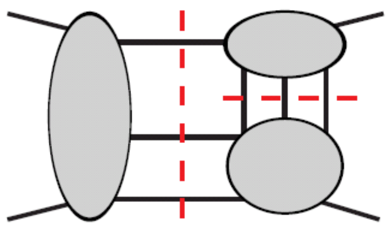
Summary

- A new duality conjectured between color and kinematics (not well understood, mathematical structure??)
- When duality between color and kinematics manifest, gravity integrands follow *immediately* from gauge-theory ones.
- We now have extremely powerful way to explore UV properties of gauge and gravity theories.
- Surprises, contrary to previous symmetry expectations and evidence of new structures.

Duality between color and kinematics is powerful but not well understood. We sure could use some help to unravel the underlying mathematical structure.

Workshop Announcement: “Physics and Mathematics of Scattering Amplitudes”





Starts final week of August 2013, for period of 4 months

“A primary aim of this workshop is to bring together mathematicians and physicists for a significant period of time to make fundamental progress in understanding scattering amplitudes in quantum field theory.”

Organizers: Zvi Bern, Lance Dixon, Michael Douglas, Sasha Goncharov, Lionel Mason

bern@physics.ucla.edu
lance@slac.stanford.edu