First order gravity on the light front

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work in progress with
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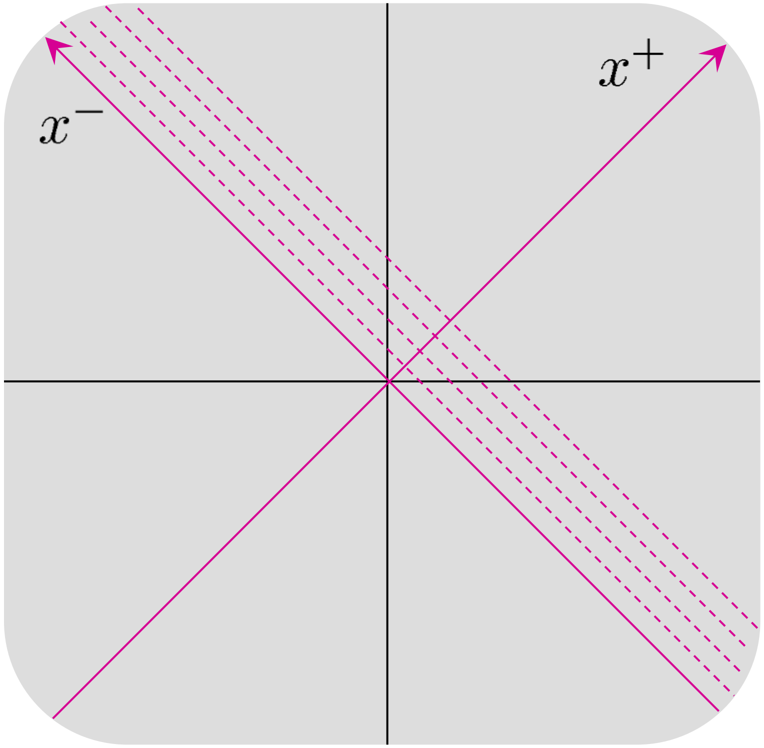
Light Front

Light cone coordinates:

$$x^+ = \frac{1}{\sqrt{2}}(t+x) \qquad x^- = \frac{1}{\sqrt{2}}(t-x)$$



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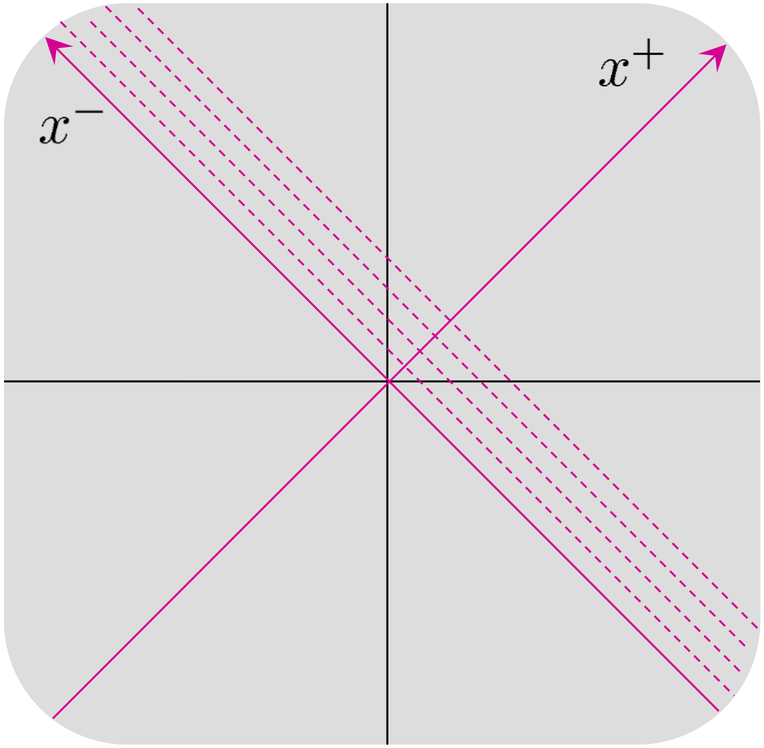
- Triviality of the vacuum $(p=0)$

$$p^+ = \frac{m^2 + p_\perp^2}{2p^-} > 0$$

p^- – energy

p^+ – momentum



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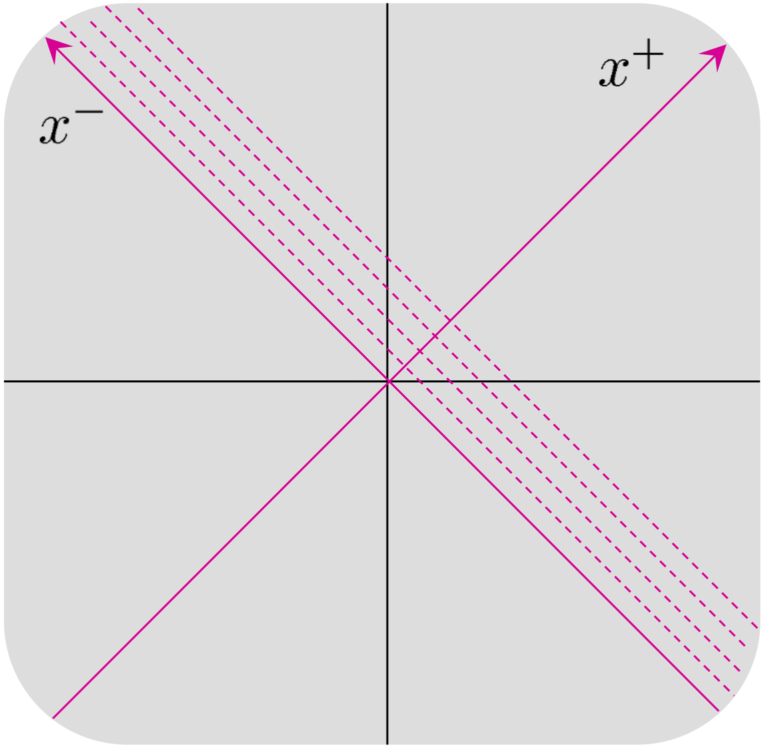

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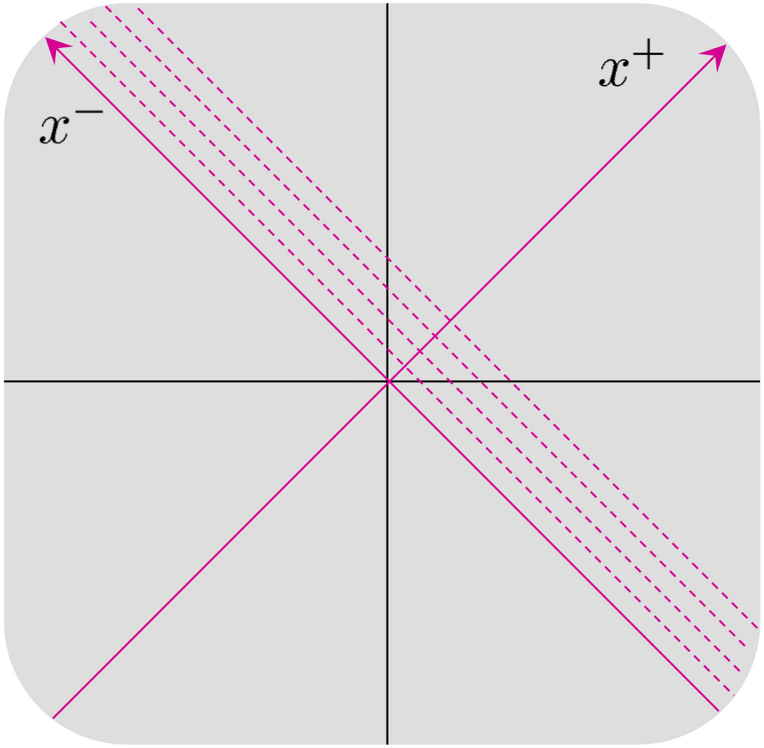

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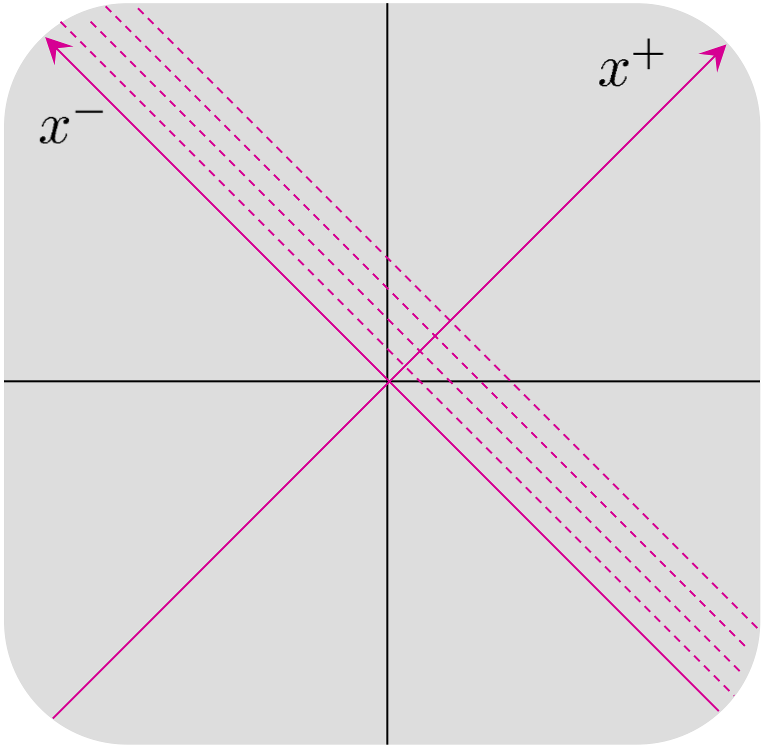

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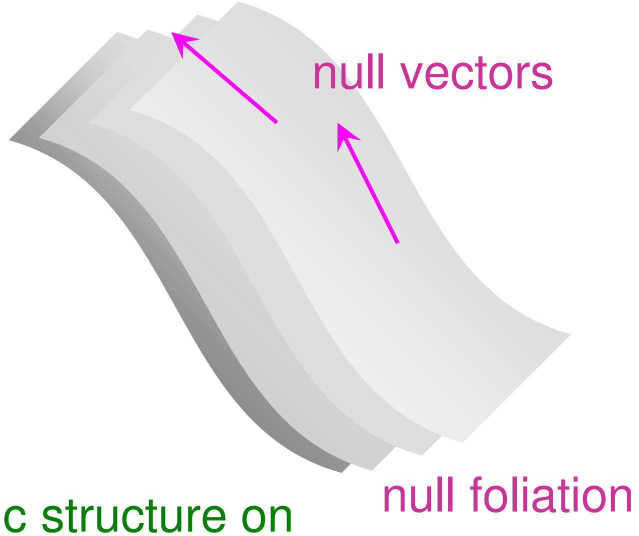
- Non-trivial physics of zero modes $\partial_-\phi_0 = 0$
- Importance of boundary conditions at $x^- \to \pm\infty$
- Presence of second class constraints

$$\int \mathrm{d}^n x\, \partial_+\phi \partial_-\phi$$

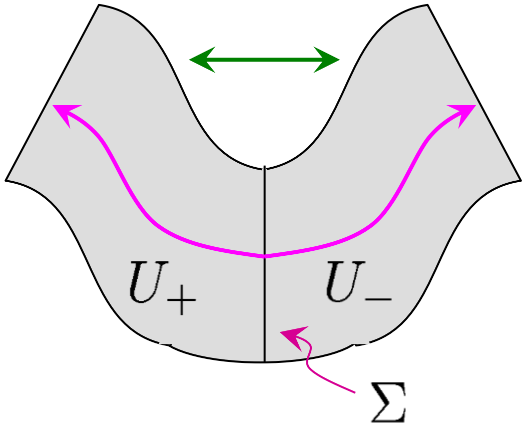
linear in velocities

Gravity on the light front

Null surfaces are natural in gravity (Penrose,…)



- Sachs(1962) – constraint free formulation



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  - intrinsic geometry of Σ
  - extrinsic curvature of Σ

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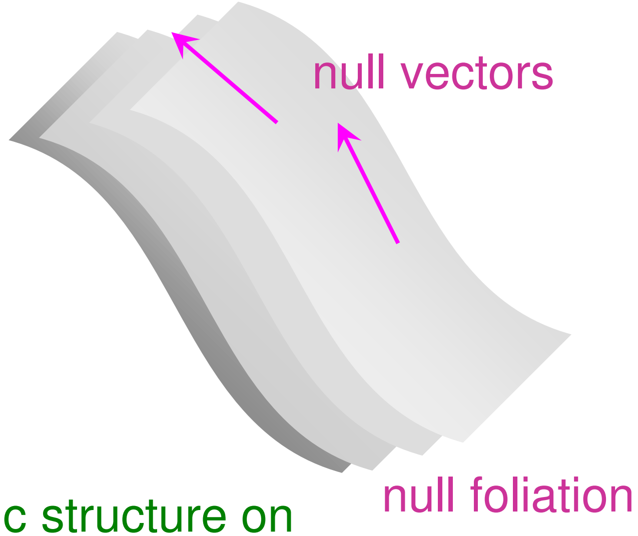

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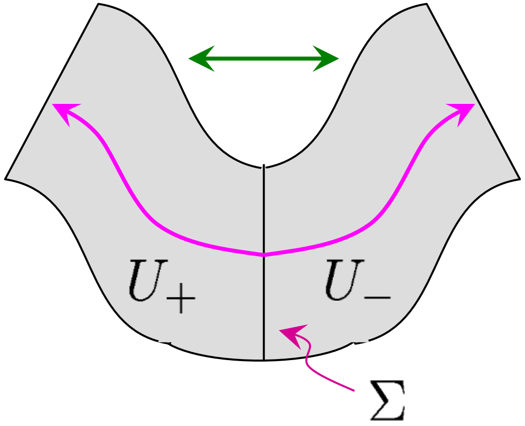

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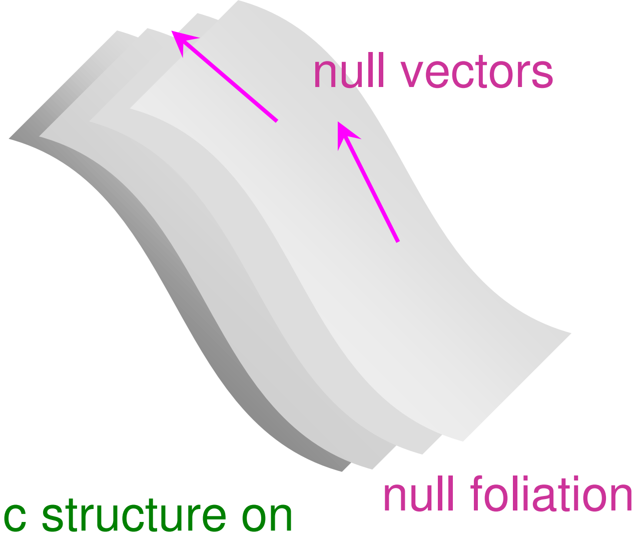

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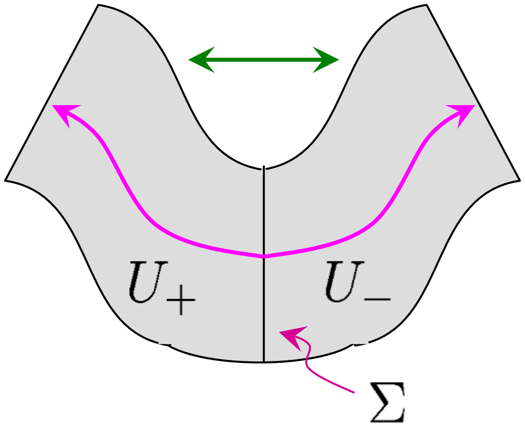

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- Veneziano *et al.* (recent) – light-cone averaging in cosmology

Motivation

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- In the first order formalism the null condition can be controlled by fields in the tangent space

Technical motivation

3+1 decomposition of the tetrad

$$
\begin{aligned}
e^0 &= \mathcal{N}\mathrm{d}t + \chi_i E^i_a \mathrm{d}x^a \\
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shift

$$\mathcal{N} = N + E^i_a \chi_i N^a$$

lapse

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$$g_{\mu\nu} = \begin{pmatrix} -gN^2 + g_{ab}N^aN^b & g_{ab}N^b \\ g_{ab}N^b & g_{ab} \end{pmatrix}$$

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spacelike	lightlike	timelike
$\chi^2 < 1$	$\chi^2 = 1$	$\chi^2 > 1$

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light front formulation

What happens at $\chi^2 = 1$?

Perform canonical analysis for the *real first order* formulation of general relativity on a lightlike foliation

Plan of the talk

1. Canonical formulation of field theories on the light front
2. A review of the canonical structure of first order gravity
3. Canonical analysis of first order gravity on the light front
4. The issue of zero modes

Massless scalar field in 2d

$$S = \frac{1}{2} \int \mathrm{d}t \mathrm{d}x \left((\partial_t \phi)^2 - (\partial_x \phi)^2 \right)$$

Solution: $\phi(t, x) = f(x^+) + g(x^-)$

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Primary constraint

$$\Phi = \pi - \partial_-\phi \approx 0$$

Hamiltonian

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first class

↓

$$\boxed{\lambda = \lambda_0(x^+)}$$

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Identification:

$$\phi(0, x^-) = g(x^-)$$

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Conclusions:

- the phase space is one-dimensional
- the lost dimension is encoded in the Lagrange multiplier

Massive theories

$$S = \int \mathrm{d}x^+ \mathrm{d}x^- \left(\partial_+ \phi \partial_- \phi - \frac{m^2}{2} \phi^2 \right)$$

One generates the same constraint but different Hamiltonian

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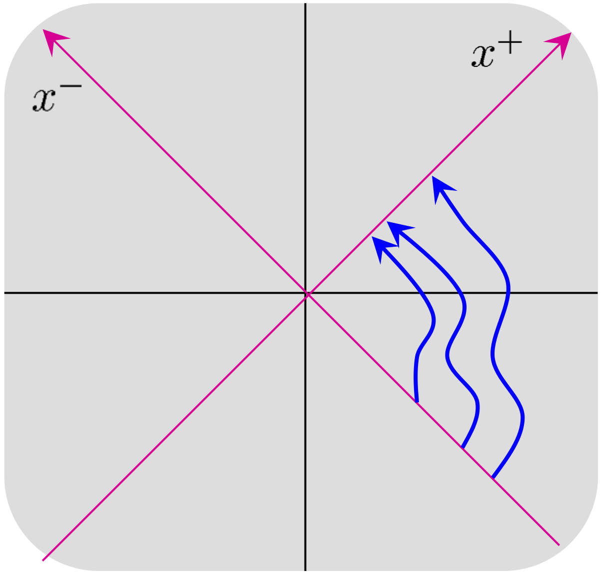

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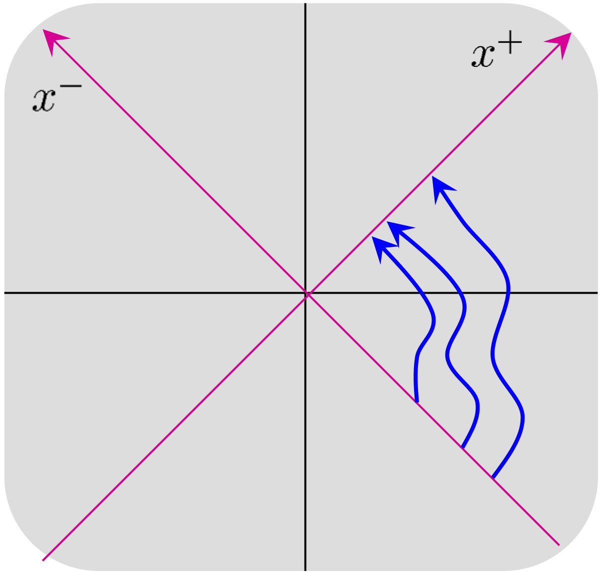

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In higher dimensions:

$$m_{\text{eff}}^2 = m^2 + k_\perp^2$$

behave like massive 2d case

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On the light front

dim. phase space = num. of deg. of freedom

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Fourier decompositions

$$\phi(x^+, x^-) = \int \mathrm{d}k^- \left[b(k^-) e^{-\mathrm{i}k^- x^- - \mathrm{i}\,\frac{m^2}{2k^-}\, x^+} \right] \qquad \phi(t, x) = \int \mathrm{d}k \left[a(k) e^{\mathrm{i}kx - \mathrm{i}\omega t} + a^*(k) e^{-\mathrm{i}kx + \mathrm{i}\omega t} \right]$$

$$b(-k) = b^*(k)$$

$$\frac{1}{\sqrt{2}} \left(k^- - \frac{m^2}{2k^-} \right) = k$$

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Symplectic structure is non-degenerate

Second class constraint Φ →

Dirac bracket

$$\{\phi(x^-), \phi(y^-)\}_D = \partial_-^{-1} \delta(x^-, y^-)$$

First order gravity (spacelike case)

$$S_{\mathrm{HP}}[e,\omega] = \frac{1}{4}\int_{\mathcal{M}} \varepsilon_{IJKL} e^I \wedge e^J \wedge \left(F^{KL}(\omega) + \frac{\Lambda}{24} e^K \wedge e^L \right)$$

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Fix $\begin{matrix} x_+ = (1, \chi^i) \\ x_+^2 < 0 \end{matrix}$ – normal to the foliation

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Linear simplicity constraints

$$\Phi^{a}_{I}=\varepsilon_{IJ}{}^{KL}x^{J}_{+}\widetilde{P}^{a}_{KL}=0$$

Hamiltonian is a linear combination of constraints

$$\Phi^{a}_{I}\qquad \mathcal{G}_{IJ}=D_{a}\widetilde{P}^{a}_{IJ}\qquad \mathcal{C}_{a}=-\widetilde{P}^{b}_{IJ}F^{IJ}_{ab}$$

$$\mathcal{H}=2\widetilde{P}^{a}_{IK}\widetilde{P}^{b,K}_{J}\left(F^{IJ}_{ab}-\frac{\Lambda}{12}\,\varepsilon_{abc}\varepsilon^{IJKL}\widetilde{P}^{c}_{KL}\right)$$

First order gravity (spacelike case)

$$S_{\rm HP}[e,\omega]=\frac{1}{4}\int_{\mathcal{M}}\varepsilon_{IJKL}e^I\wedge e^J\wedge\left(F^{KL}(\omega)+\frac{\Lambda}{24}\,e^K\wedge e^L\right)$$

$$e^0=\left(N+E^i_a\chi_iN^a\right)\mathrm{d}t+\chi_iE^i_a\mathrm{d}x^a$$

$$e^i=\left(N^a+E^a_i\chi^iN\right)E^i_a\mathrm{d}t+E^i_a\mathrm{d}x^a$$

Fix $x_+=(1,\chi^i)$, $x_+^2<0$ – normal to the foliation

Hamiltonian is a linear combination of constraints

$$\begin{tikzcd} \Phi^a_I \arrow[r, leftrightarrow, "3"'] \arrow[d, leftrightarrow, "6"'] & \mathcal{G}_{IJ}=D_a\tilde{P}^a_{IJ} & \mathcal{C}_a=-\tilde{P}^b_{IJ}F^{IJ}_{ab} \\ \mathcal{H}=2\tilde{P}^a_{IK}\tilde{P}^{b,K}_J\left(F^{IJ}_{ab}-\frac{\Lambda}{12}\,\varepsilon_{abc}\varepsilon^{IJKL}\tilde{P}^c_{KL}\right) & & \end{tikzcd}$$

Canonical variables:

$$\omega^{IJ}_a \qquad \tilde{P}^a_{IJ}=\frac{1}{4}\,\varepsilon^{abc}\varepsilon_{IJKL}e^K_be^L_c$$

⬇

Linear simplicity constraints

$$\Phi^a_I=\varepsilon_{IJ}{}^{KL}x^J_+\tilde{P}^a_{KL}=0$$

First order gravity (spacelike case)

$$S_{\mathrm{HP}}[e,\omega] = \frac{1}{4}\int_{\mathcal{M}} \varepsilon_{IJKL}\, e^I \wedge e^J \wedge \left(F^{KL}(\omega) + \frac{\Lambda}{24}\, e^K \wedge e^L \right)$$

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Canonical variables:

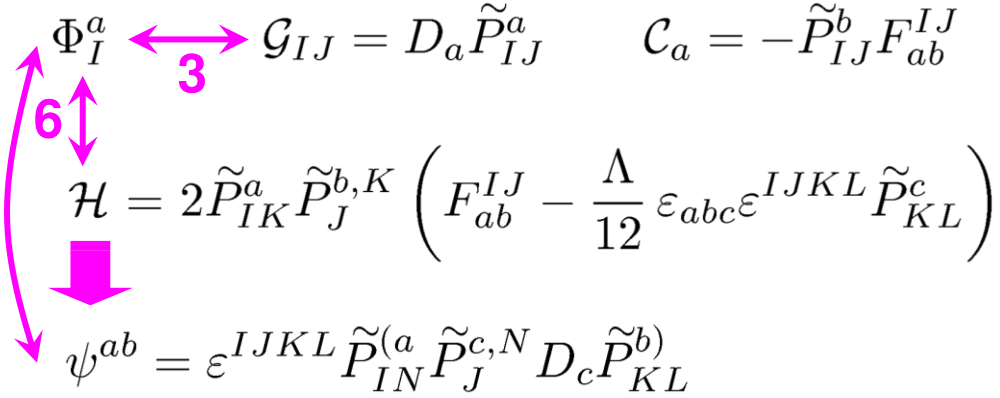
$$\omega_a^{IJ} \qquad \tilde{P}^a_{IJ} = \frac{1}{4}\, \varepsilon^{abc} \varepsilon_{IJKL} e^K_b e^L_c$$

Linear simplicity constraints

$$\Phi^a_I = \varepsilon_{IJ}{}^{KL} x^J_+ \tilde{P}^a_{KL} = 0$$

Fix $x_+ = (1, \chi^i)$, $x_+^2 < 0$ – normal to the foliation

Hamiltonian is a linear combination of constraints



$$\Phi^a_I \qquad \mathcal{G}_{IJ} = D_a \tilde{P}^a_{IJ} \qquad \mathcal{C}_a = -\tilde{P}^b_{IJ} F^{IJ}_{ab}$$

$$\mathcal{H} = 2 \tilde{P}^a_{IK} \tilde{P}^{b,K}_J \left(F^{IJ}_{ab} - \frac{\Lambda}{12}\, \varepsilon_{abc} \varepsilon^{IJKL} \tilde{P}^c_{KL} \right)$$

$$\psi^{ab} = \varepsilon^{IJKL} \tilde{P}^{(a}_{IN} \tilde{P}^{c,N}_J D_c \tilde{P}^{b)}_{KL}$$

secondary constraints

First order gravity (spacelike case)

$$S_{\mathrm{HP}}[e,\omega]=\frac{1}{4}\int_{\mathcal{M}}\varepsilon_{IJKL}e^I\wedge e^J\wedge\left(F^{KL}(\omega)+\frac{\Lambda}{24}\,e^K\wedge e^L\right)$$

$$e^0=\left(N+E^i_a\chi_i N^a\right)\mathrm{d}t+\chi_i E^i_a\mathrm{d}x^a$$
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Canonical variables:

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Linear simplicity constraints

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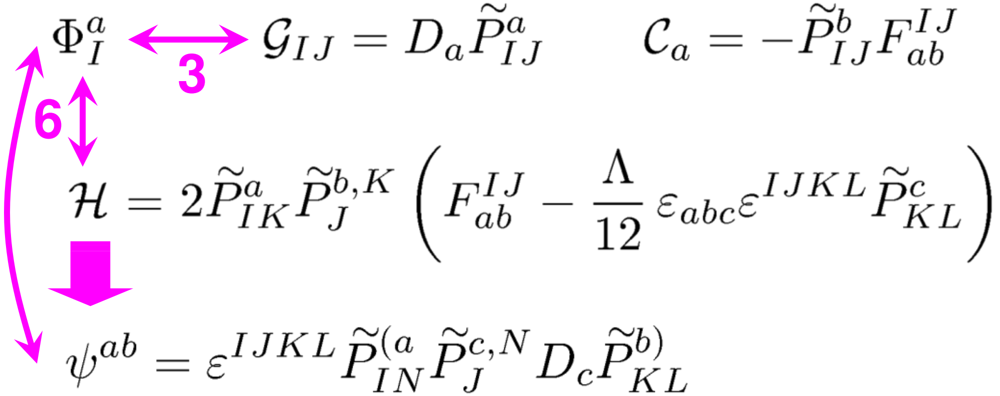
Hamiltonian is a linear combination of constraints



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$$\psi^{ab}=\varepsilon^{IJKL}\tilde{P}^{(a}_{IN}\tilde{P}^{c,N}_J D_c\tilde{P}^{b)}_{KL}$$

secondary constraints

1st class

$$\varepsilon_{IJ}{}^{KL}x^J_+\mathcal{G}_{KL}\qquad \mathcal{C}_a\qquad \mathcal{H}$$

2d class

$$x^J_+\mathcal{G}_{IJ}\qquad \Phi^a_I\qquad \psi^{ab}$$

First order gravity (spacelike case)

$$S_{\mathrm{HP}}[e,\omega]=\frac{1}{4}\int_{\mathcal{M}}\varepsilon_{IJKL}e^I\wedge e^J\wedge\left(F^{KL}(\omega)+\frac{\Lambda}{24}e^K\wedge e^L\right)$$

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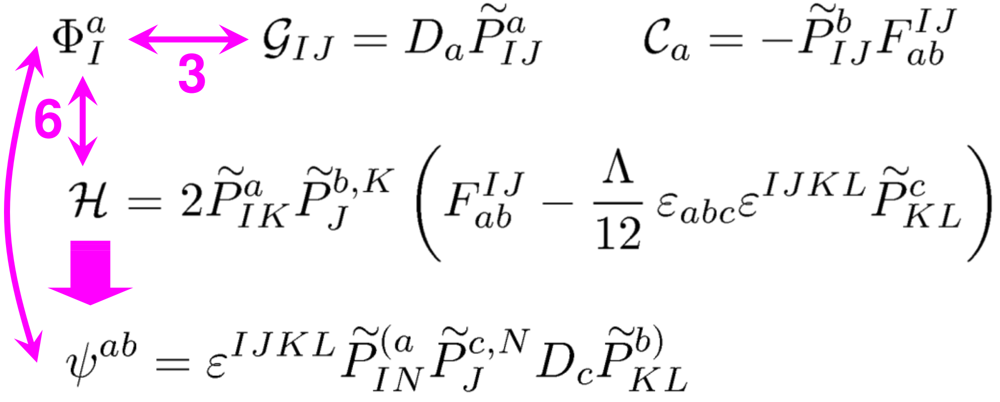
Linear simplicity constraints

$$\Phi^a_I=\varepsilon_{IJ}{}^{KL}x^J_+\tilde{P}^a_{KL}=0$$

Hamiltonian is a linear combination of constraints



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2d class

$$x^J_+\mathcal{G}_{IJ} \qquad \Phi^a_I \qquad \psi^{ab}$$

dim. of phase space = 2×18 – 2(3+3+1)-(3+9+6)=4

Cartan equations

$$S_{\mathrm{HP}}[e,\omega] = \frac{1}{4}\int_{\mathcal{M}} \varepsilon_{IJKL} e^I \wedge e^J \wedge \left(F^{KL}(\omega) + \frac{\Lambda}{24}\, e^K \wedge e^L \right)$$

$\delta\omega_\mu^{IJ}$

Cartan equations

$$\boxed{D_{[\mu}e^I_{\nu]} = \partial_{[\mu}e^I_{\nu]} + \omega^I_{[\mu,J}e^J_{\nu]} = 0}$$

Cartan equations

$$S_{\mathrm{HP}}[e,\omega] = \frac{1}{4}\int_{\mathcal{M}} \varepsilon_{IJKL} e^I \wedge e^J \wedge \left(F^{KL}(\omega) + \frac{\Lambda}{24}\, e^K \wedge e^L \right)$$

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Cartan equations

$$\boxed{D_{[\mu} e^I_{\nu]} = \partial_{[\mu} e^I_{\nu]} + \omega^I_{[\mu,J} e^J_{\nu]} = 0}$$

$$D_{[a} e^I_{b]} \qquad D_{[0} e^I_{a]}$$

do not contain time derivatives and Lagrange multipliers

12 constraints

$$\mathcal{G}_{IJ} \qquad \psi^{ab}$$

Cartan equations

$$S_{\mathrm{HP}}[e,\omega] = \frac{1}{4}\int_{\mathcal{M}} \varepsilon_{IJKL}\, e^I \wedge e^J \wedge \left(F^{KL}(\omega) + \frac{\Lambda}{24}\, e^K \wedge e^L\right)$$

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Cartan equations

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$D_{[a}e^I_{b]}$ $\qquad$ $D_{[0}e^I_{a]}$

for fixed x^I_+

$\eta_{IJ}x^I_+ D_{[0}e^J_{a]}$

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$$S_{\mathrm{HP}}[e,\omega]=\frac{1}{4}\int_{\mathcal{M}}\varepsilon_{IJKL}e^I\wedge e^J\wedge\left(F^{KL}(\omega)+\frac{\Lambda}{24}e^K\wedge e^L\right)$$

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$D_{[a}e^I_{b]}$ $D_{[0}e^I_{a]}$

do not contain time derivatives and Lagrange multipliers

for fixed x_+^I

$\eta_{IJ}x_+^I D_{[0}e^J_{a]}$

do not contain time derivatives

$x_+^2\neq 0$

12 constraints

$\mathcal{G}_{IJ}$ ψ^{ab}

fix 3 components of ω_0^{0i}

($x_+^J\mathcal{G}_{IJ}$ – 2d class)

Cartan equations

$$S_{\text{HP}}[e,\omega]=\frac{1}{4}\int_{\mathcal{M}}\varepsilon_{IJKL}e^I\wedge e^J\wedge\left(F^{KL}(\omega)+\frac{\Lambda}{24}e^K\wedge e^L\right)$$

$\delta\omega_\mu^{IJ}$

Cartan equations

$$\boxed{D_{[\mu}e^I_{\nu]}=\partial_{[\mu}e^I_{\nu]}+\omega^I_{[\mu,J}e^J_{\nu]}=0}$$

$D_{[a}e^I_{b]}$ $\quad D_{[0}e^I_{a]}$

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12 constraints

$\mathcal{G}_{IJ} \quad \psi^{ab}$

for fixed x^I_+

$\eta_{IJ}x^I_+D_{[0}e^J_{a]}$

do not contain time derivatives

$x^2_+\neq 0$

fix 3 components of ω_0^{0i}

($x^J_+\mathcal{G}_{IJ}$ – 2d class)

$x^2_+=0$

fix 2 components of ω_0^{0i} and the lapse N

$$\boxed{E^a_i\chi^i\left(\partial_a\log N-\omega_a^{0j}\chi_j\right)=0}$$

Cartan equations

$$S_{\mathrm{HP}}[e,\omega] = \frac{1}{4}\int_{\mathcal{M}} \varepsilon_{IJKL} e^I \wedge e^J \wedge \left(F^{KL}(\omega) + \frac{\Lambda}{24} e^K \wedge e^L \right)$$

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Cartan equations

$$\boxed{D_{[\mu} e^I_{\nu]} = \partial_{[\mu} e^I_{\nu]} + \omega^I_{[\mu,J} e^J_{\nu]} = 0}$$

$D_{[a} e^I_{b]}$ $D_{[0} e^I_{a]}$

do not contain time derivatives and Lagrange multipliers

for fixed x_+^I

$\eta_{IJ} x_+^I D_{[0} e^J_{a]}$

do not contain time derivatives

$x_+^2 \neq 0$

$x_+^2 = 0$

12 constraints

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fix 3 components of ω_0^{0i}

($x_+^J \mathcal{G}_{IJ}$ – 2d class)

fix 2 components of ω_0^{0i} and the lapse N

$$\boxed{E^a_i \chi^i \left(\partial_a \log N - \omega_a^{0j} \chi_j \right) = 0}$$

We expect that the Hamiltonian constraint becomes second class

Hamiltonian analysis on the light front

$$S_{\mathrm{HP}}[e,\omega] = \frac{1}{4}\int_{\mathcal{M}} \varepsilon_{IJKL} e^I \wedge e^J \wedge \left(F^{KL}(\omega) + \frac{\Lambda}{24} e^K \wedge e^L \right)$$

$$e^0 = \left(N + E^i_a \chi_i N^a\right) \mathrm{d}t + \chi_i E^i_a \mathrm{d}x^a$$

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Light front condition $x_+^2 = 0$

Canonical variables:

ω_a^{IJ} $\qquad \tilde{P}^a_{IJ} = \frac{1}{4}\varepsilon^{abc}\varepsilon_{IJKL} e^K_b e^L_c$

⬇

Linear simplicity constraints

$$\Phi^a_I = \varepsilon_{IJ}{}^{KL} x^J_+ \tilde{P}^a_{KL} = 0$$

Hamiltonian analysis on the light front

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$$\mathcal{H} = \frac{1}{2} \tilde{p}^a_I \tilde{p}^b_J \left(F^{IJ}_{ab} + \frac{\Lambda}{12} \varepsilon_{abc} \varepsilon^{IJKL} \widetilde{P}^c_{KL} \right)$$

where $x_- = (-1, \chi^i)$

$$\tilde{p}^a_I = -(x^J_+ - x^J_-) \widetilde{P}^a_{IJ} = (0, \widetilde{E}^a_i)$$

Hamiltonian analysis on the light front

$$S_{\mathrm{HP}}[e,\omega] = \frac{1}{4}\int_{\mathcal{M}} \varepsilon_{IJKL} e^I \wedge e^J \wedge \left(F^{KL}(\omega) + \frac{\Lambda}{24} e^K \wedge e^L \right)$$

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Linear simplicity constraints

$$\Phi^a_I = \varepsilon_{IJ}{}^{KL} x^J_+ \tilde{P}^a_{KL} = 0$$

Hamiltonian is a linear combination of constraints

$$\begin{tikzcd} \Phi^a_I \arrow[r, leftrightarrow, "2"'] \arrow[d, leftrightarrow, "7"'] & \mathcal{G}_{IJ} = D_a \tilde{P}^a_{IJ} \\ \mathcal{H} = \frac{1}{2} \tilde{p}^a_I \tilde{p}^b_J \left(F^{IJ}_{ab} + \frac{\Lambda}{12} \varepsilon_{abc} \varepsilon^{IJKL} \tilde{P}^c_{KL} \right) & \end{tikzcd}$$

$$\mathcal{C}_a = -\tilde{P}^b_{IJ} F^{IJ}_{ab}$$

where $x_- = (-1, \chi^i)$

$$\tilde{p}^a_I = -(x^J_+ - x^J_-)\tilde{P}^a_{IJ} = (0, \tilde{E}^a_i)$$

Hamiltonian analysis on the light front

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Canonical variables:

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where $x_-=(-1,\chi^i)$

$$\tilde{p}^a_I=-(x^J_+-x^J_-)\tilde{P}^a_{IJ}=(0,\tilde{E}^a_i)$$

$$\psi^{ab}=\varepsilon^{IJKL}\tilde{p}^{(a}_I\tilde{p}^c_JD_c\tilde{P}^{b)}_{KL}$$

secondary constraints

+

equation fixing the lapse

$$\boxed{E^a_i\chi^i\left(\partial_a\log N-\omega^{0j}_a\chi_j\right)=0}$$

Tertiary constraints

The crucial observation:

$$\{\psi^{ab}, \mathcal{G}_{IJ}\} \approx \{\psi^{ab}, \mathcal{C}_a\} \approx 0$$

and

$$\{\psi^{ab}, \Phi_I^c\} \sim \mathcal{M}^{ab,cd} = \varepsilon^{(acf}\varepsilon^{b)dg} g_{fg}$$

induced metric on the foliation

has 2 null eigenvectors

Tertiary constraints

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$$\{\psi^{ab}, \mathcal{G}_{IJ}\} \approx \{\psi^{ab}, \mathcal{C}_a\} \approx 0$$

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induced metric on the foliation

has 2 null eigenvectors

There are two *tertiary constraints*

Projector on the null eigenvectors

$$\begin{aligned} \Upsilon_{ab} &= \Pi_{ab,cd}\{\psi^{cd}, \mathcal{H}\} \\ &= \frac{1}{2}\Pi_{ab,cd}\varepsilon^{cfg}F^{IJ}_{fg}(\omega)x_{-,I}\widetilde{p}^d_J \end{aligned}$$

Tertiary constraints

The crucial observation:

$$\{\psi^{ab},\mathcal{G}_{IJ}\}\approx\{\psi^{ab},\mathcal{C}_a\}\approx 0$$

and

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Stabilization procedure stops due to

$$\det\{\Upsilon,(\Pi\phi)\}\neq 0 \qquad \det\{(\Pi\psi),(\Pi\psi)\}\neq 0$$

Summary

List of constraints:

Gauss preserving x_+^I	Gauss rotating x_+^I	spatial diffeos	Hamiltonian	primary simplicity	secondary simplicity	tertiary
$\mathcal{G}^{\parallel}_{IJ}$	$\mathcal{G}^{\perp}_{IJ}$	$\mathcal{C}_a$	$\mathcal{H}$	Φ^a_I	ψ^{ab}	Υ_{ab}

Summary

List of constraints:

Gauss preserving x_+^I	Gauss rotating x_+^I	spatial diffeos	Hamiltonian	primary simplicity	secondary simplicity	tertiary

First class

$\mathcal{G}^{\|\|}_{IJ}$	$\mathcal{C}_a$
4	**3**

Second class

$\mathcal{G}^{\perp}_{IJ}$	$\mathcal{H}$	Φ_I^a	ψ^{ab}	Υ_{ab}
2	**1**	**9**	**6**	**2**

Summary

List of constraints:

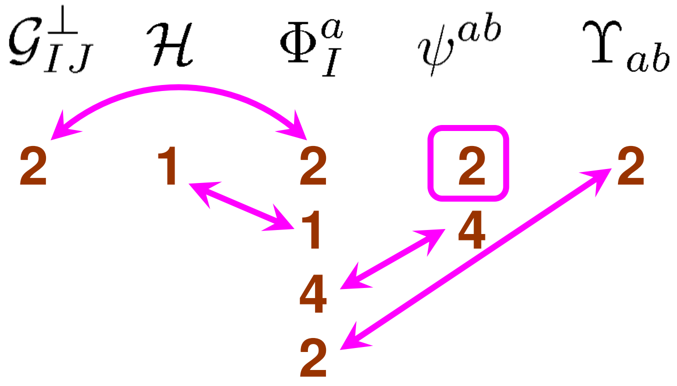
Gauss preserving x^I_+	Gauss rotating x^I_+	spatial diffeos	Hamiltonian	primary simplicity	secondary simplicity	tertiary

First class

$\mathcal{G}^{\|\|}_{IJ}$	$\mathcal{C}_a$
4	**3**

Lie algebra

Second class



Summary

List of constraints:

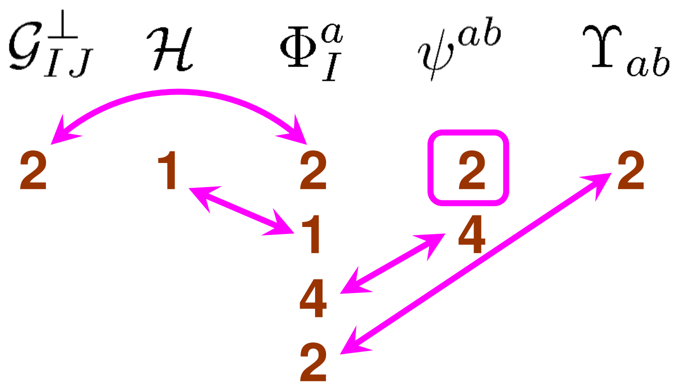
- Gauss preserving x_+^I
- Gauss rotating x_+^I
- spatial diffeos
- Hamiltonian
- primary simplicity
- secondary simplicity
- tertiary

First class

$\mathcal{G}^{\parallel}_{IJ}$	$\mathcal{C}_a$
4	3

Lie algebra

Second class



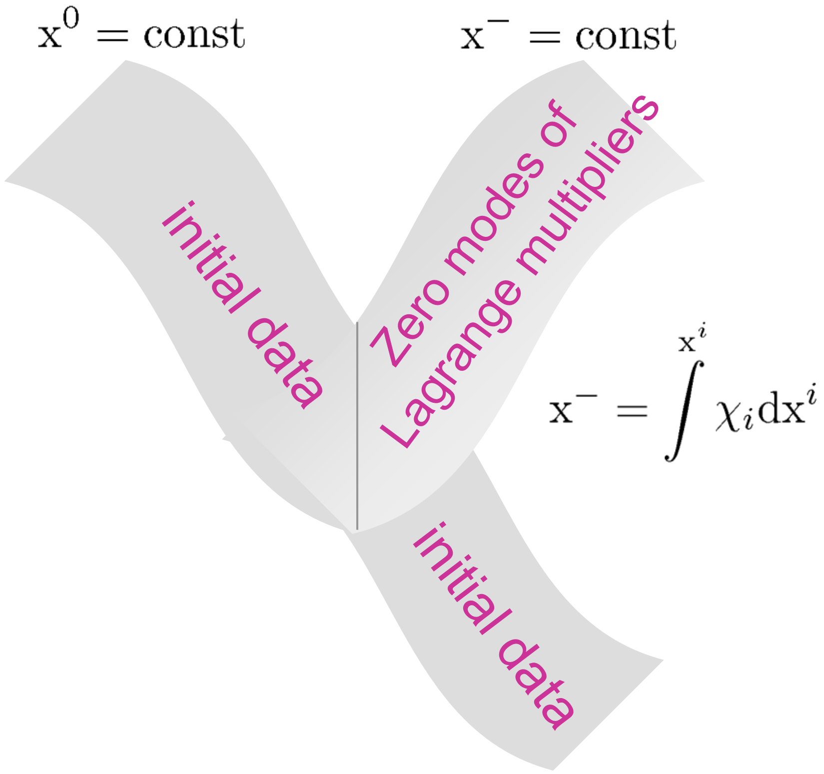
$\mathcal{G}^{\perp}_{IJ}$	$\mathcal{H}$	Φ^a_I	ψ^{ab}	Υ_{ab}
2	1	2 1 4 2	$\boxed{2}$ 4	2

dim. of phase space = 2×18 – 2(4+3)-(2+1+9+6+2)=2

as it should be on the light front

Zero modes

The zero modes of constraints are determined by equations fixing Lagrange multipliers



Zero modes

The zero modes of constraints are determined by equations fixing Lagrange multipliers

Potential first class constraints:

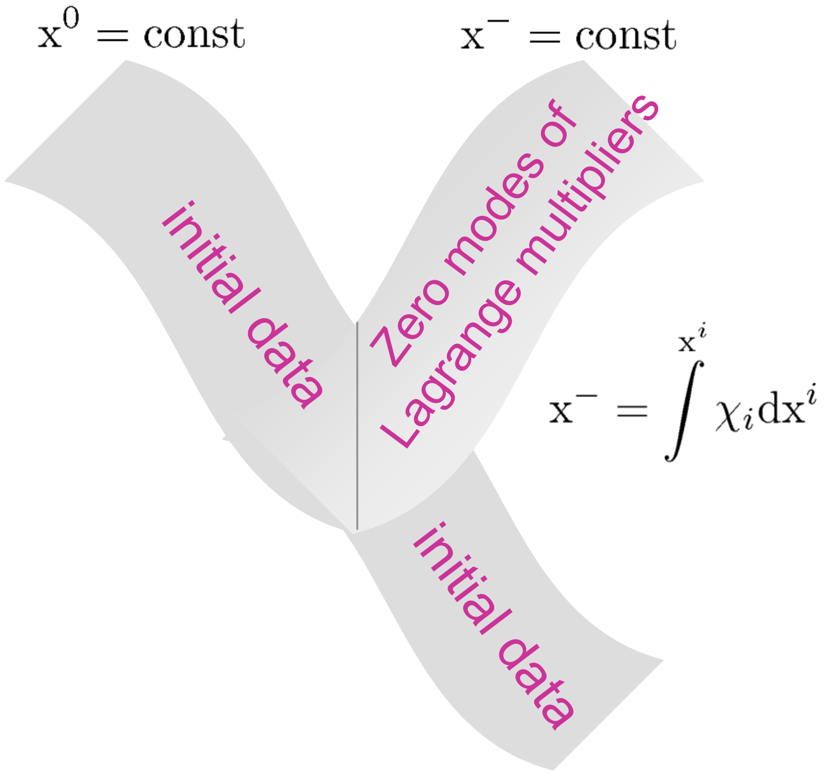
$$\int \mathrm{dx}^- \mathcal{H}$$

$$\int \mathrm{dx}^- \varepsilon_{abc} \widetilde{p}^a_I (\widetilde{p}^b_J x^J_+) \Phi^c_I$$

$$\int \mathrm{dx}^- (\Pi\phi)_{ab}$$

$$\int \mathrm{dx}^- (\Pi\psi)_{ab}$$

$$\int \mathrm{dx}^- \Upsilon_{ab}$$



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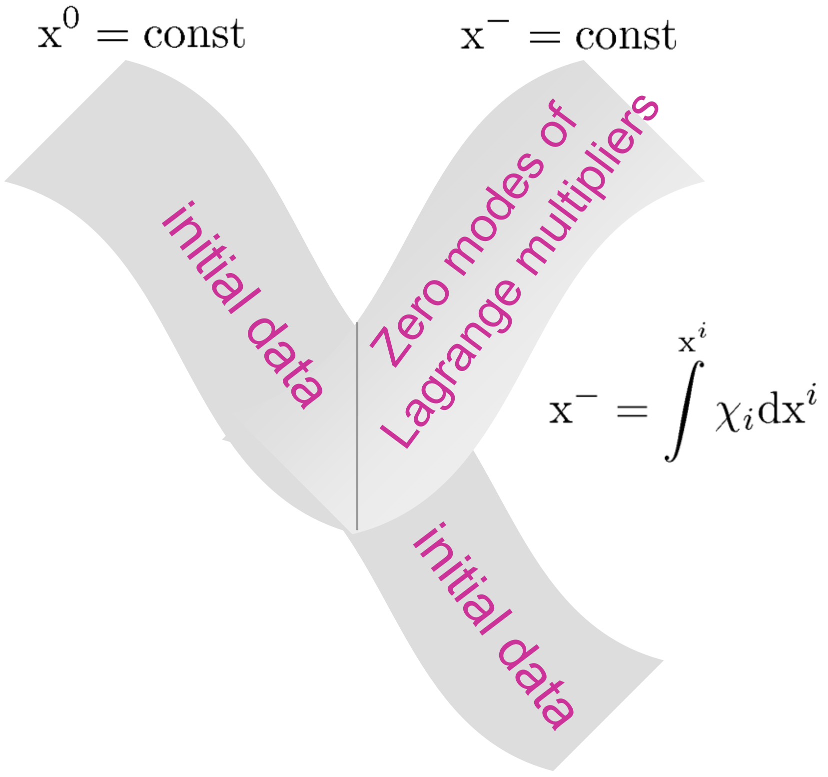
$$\int \mathrm{dx}^-(\Pi\phi)_{ab}$$

$$\int \mathrm{dx}^-(\Pi\psi)_{ab}$$

$$\int \mathrm{dx}^-\Upsilon_{ab}$$

$$\boxed{\begin{aligned}\widetilde{E}^a_i\chi^i\partial_a N &= N\widetilde{E}^a_i\chi^i\omega^{0j}_a\chi_j\\ \partial_a\left(\kappa\widetilde{E}^a_i\chi^i\right) &= -\kappa\widetilde{E}^a_i\chi^i\omega^{0j}_0\chi_j\end{aligned}}$$

homogeneous equations



Zero modes

The zero modes of constraints are determined by equations fixing Lagrange multipliers

Potential first class constraints:

$$\int \mathrm{dx}^- \mathcal{H}$$

$$\int \mathrm{dx}^- \varepsilon_{abc}\widetilde{p}^a_I(\widetilde{p}^b_J x^J_+)\Phi^c_I$$

$$\int \mathrm{dx}^- (\Pi\phi)_{ab}$$

$$\int \mathrm{dx}^- (\Pi\psi)_{ab}$$

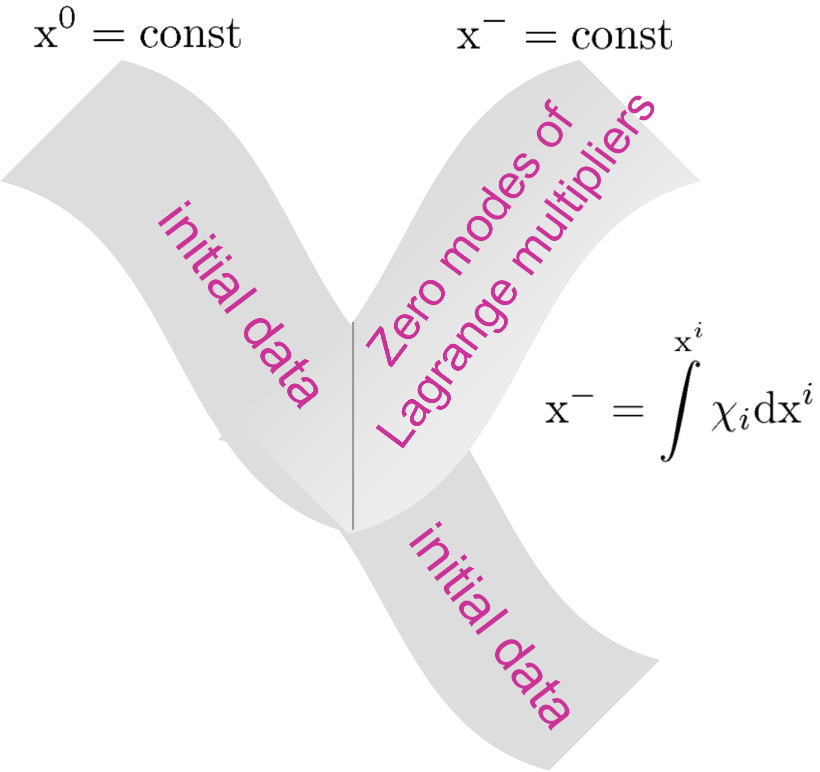
$$\int \mathrm{dx}^- \Upsilon_{ab}$$

$$\widetilde{E}^a_i \chi^i \partial_a N = N \widetilde{E}^a_i \chi^i \omega^{0j}_a \chi_j$$

$$\partial_a\left(\kappa \widetilde{E}^a_i \chi^i\right) = -\kappa \widetilde{E}^a_i \chi^i \omega^{0j}_0 \chi_j$$

homogeneous equations

$$\widetilde{E}^a_i \chi^i \partial_a \lambda = (\cdots)\lambda + \text{inhomogeneous terms}$$



Conjecture

**A zero mode can exist only if
the corresponding Lagrange multiplier satisfies
a homogeneous equation**

In our case there are two homogenous equations
for N and κ

One may expect only two zero modes

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Gravity behaves like 2d massless theory

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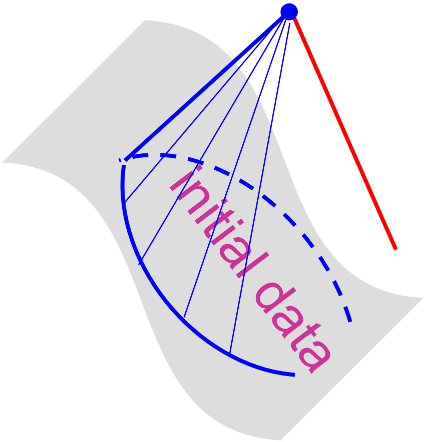

exist

Gravity behaves like 2d massless theory

do not exist

Initial data on one null hypersurface fix the solution

Open problems

- Do the zero modes in gravity really exist?
  If yes, what is the geometric meaning of the zero modes?

- What are the appropriate boundary conditions along x^- ?

- How do singularities appear in this formalism?

- Can one solve (at least formally) all constraints?

- What is the right symplectic structure (Dirac bracket)?

- Can this formulation be applied to quantum gravity problems?