Monodromy transform approach to solution of integrable reductions of Einstein's field equations in General Relativity and String gravity

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Plan of the talk:

- *Integrability of Eistein’s field equations (the hystory)*
- *Dynamical equations and equivalent spectral problems*
- *Monodromy transform approach*
- *Other methods in the context of the monodromy transform*
- *Some applications*

Integrability of Einstein's field equations

Vacuum: $R_{ik} = 0$

Symmetries: $g_{ik} \,||\, x^1, x^2$

First integrability conjectures:

-- R.Geroch –conjecture of integrability (1972)
-- W.Kinnersley&D.Citre – infinitesimal symmetries (1977…)
-- D.Maison - Lax pair +conjecture (1978)

Integrability began to work:

V.Belinski and V.Zakharov (1978)

-- Inverse Scattering Method
-- Soliton solutions on arbitrary backgrounds
-- Riemann – Hilbert problem
-- linear singular integral equations

Many "languages" of integrability:

-- Backlund and symm. transformations(K.Harrison 1978, G.Neugebauer 1979, HKX 1979)
-- Homogeneous Hilbert problem (I.Hauser & F.J.Ernst, 1979 + N.Sibgatullin 1984)
-- Monodromy transform + linear singular integral equations (GA 1985)
-- Finite-gap solutions (D.Korotkin&V.Matveev 1987, G.Neugebauer&R.Meinel 1993)
-- Boundary value problem for stationary fields (G.Neugebauer &R.Meinel 1996)
-- Charateristic init. value probl.(I.Hauser &F.J.Ernst 1988; GA 2001; GA&J.Griffiths 2001)

Electrovacuum Einstein - Maxwell fields

- -- *Infinite-dimensional algebra of symmetries (W.Kinnersley & D.Chitre 1977, …)*
- -- *Homogeneous Hilbert problem and singular integral equations for axisymmetric stationary fields with regular axis (I.Hauser & F.J.Ernst 1979 + N.Sibgatullin 1984)*
- -- *Inverse scattering method and Einstein – Maxwell solitons (GA 1980)*
- -- *Backlund transformations (K.Harrison 1983)*
- -- *Monodromy Transform and linear singular integral equations (GA 1985)*
- -- *Charateristic initial value problem (GA 2001; GA & J.Griffiths 2001, 2003)*

Einstein - Maxwell + Weyl neutrino fields:

- -- *Inverse scattering method (GA 1983)*
- -- *Generalization of the Hauser-Ernst approach (N.Sibgatullin 1984)*
- -- *Monodromy transform approach and linear singular integral equations (GA 1985)*

Gravity + stiff matter fluid

- -- *Inverse scattering method (V.Belinski 1979)*

Gravity in higher dimensions, string gravity and supergravity models

- -- *Vacuum equations in higher dimensions (V.Belinski & R.Ruffini 1980, A.Pomeranski 2006)*
- -- *D=4 gravity with axion and dilaton (Bakas 1996); D=4 EMDA (D.Gal'tsov, P.Letelier 1996)*
- -- *Bosonic dynamics of heterotic string effective action in D dimensions (GA 2009)*
- -- *D=5 minimal supergravity (Figueras, Jumsin, Rocha, Virmani 2010)*

Some applications

Solitons on arbitrary background:

D=4

- -- *Colliding plane waves (Khan&Penrose 1972, Y.Nutku&Khalil)*
- -- *Inhomogeneous cosmologies (V.Belinski 1979)*
- -- *Interacting black holes: 2 x Kerr (D.Kramer&G.Neugebauer 1980), 2 x Kerr-Newman (GA 1986) 2 x Reisner-Nordstrom (GA&V.Belinski 2007)*
- -- *Black holes in external fields: in Melvin universe (F.Ernst 1975), in Bertotti-Robinson space-time (GA&A.Garcia 1996)*
- -- *… … … ?*

D=5

- -- *black holes with non-simple rotation (A.Pomeransky 2006)*
- -- *black rings (R.Emparan & H.S.Reall, A.Pomeransky & R.Sen'kov)*
- -- *black Saturn (H. Elvang & P. Figueras 2007)*
- -- *… … …?*

Algebro-geometrical methods (D=4):

- -- *Finite-gap solutions for hyperelliptic curves (D.Korotkin & V.Matveev 1987)*
- -- *Solution for rigidly rotating thin disk of dust (G.Neugebauer & R.Meinel)*

Integral equation methods, boundary and initial value problems (D=4):

- -- *Solutions with rational monodromy (GA 1988,1992; N.Sibgatullin 1993;GA & J.Griffiths 2000)*
- -- *Boundary value problems for stationary axisymm. fields (G.Neugebauer&R.Meinel 1996)*
- -- *Characteristic initial value problems (I.Hauser&F.Ernst 1987; GA & J.Griffiths 2001)*

Dynamical equations and equivalent spectral problems

Dynamical equations and equivalent spectral problems

$$\mathcal{S}_E = -\frac{1}{16\pi}\int R\sqrt{-G}\, d^4x$$ --- vacuum

$$\mathcal{S}_{EM} = -\frac{1}{16\pi}\int \left(R + F_{ik}F^{ik}\right)\sqrt{-g}\, d^4x$$ --- Einstein – Maxwell fields

$$\mathcal{S}_{EMW} = -\frac{1}{16\pi}\int \left(R + F_{ik}F^{ik} + \Lambda(\psi_A)\right)\sqrt{-g}\, d^4x$$ --- Einstein – Maxwell --- Weyl fields

Einstein – Maxwell + axion + dilaton fields:

$$\mathcal{S}_{EM+a+d} = -\frac{1}{16\pi}\int \left(R - 2(\nabla\phi)^2 - \frac{1}{2}e^{4\phi}(\nabla a)^2 + e^{-2\phi}F_{ik}F^{ik} + a\, F_{ik}\overset{*}{F}{}^{ik}\right)\sqrt{-g}\, d^4x$$

Bosonic sector of heterotic string effective action:

$$\mathcal{S}_{HString} = c\int e^{-\widehat{\Phi}}\left\{\widehat{R}^{(D)} + \nabla_M\widehat{\Phi}\nabla^M\widehat{\Phi} - \frac{1}{12}H_{MNP}H^{MNP} - \frac{1}{2}\sum_{\mathfrak{p}=1}^{n} F_{MN}{}^{(\mathfrak{p})}F^{MN\,(\mathfrak{p})}\right\}\sqrt{-\widehat{G}}\, d^Dx$$

$$H_{MNP} = 3\left(\partial_{[M}B_{NP]} - \sum_{\mathfrak{p}=1}^{n} A_{[M}{}^{(\mathfrak{p})}F_{NP]}{}^{(\mathfrak{p})}\right),$$

$$F_{MN}{}^{(\mathfrak{p})} = 2\,\partial_{[M}A_{N]}{}^{(\mathfrak{p})}, \qquad B_{MN} = -B_{NM}.$$

$$M, N, \ldots = 1, 2, \ldots, D,$$
$$\mathfrak{p} = 1, \ldots, n$$

Symmety ansatz:

--- all field components depend on (x^1, x^2) only;

--- all "non-dynamical" degrees of freedom vanish.

Dynamical degrees of freedom:

$$G_{ab},\ \mathcal{B}_{ab},\ \mathcal{A}_b{}^{(\mathfrak{p})},\ \Phi\ \parallel\ x^1,\ x^2$$

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & G_{ab} \end{pmatrix}$$

$$B_{MN} = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{B}_{ab} \end{pmatrix}$$

$$A_M{}^{(I)} = \begin{pmatrix} 0 \\ \mathcal{A}_a{}^{(\mathfrak{p})} \end{pmatrix}$$

$$\|G_{ab}\| \quad - \quad d \times d\,(symmetric)$$

$$\|\mathcal{B}_{ab}\| \quad - \quad d \times d\,(antisymmetric)$$

$$\|\mathcal{A}_b{}^{(\mathfrak{p})}\| \quad - \quad d \times n$$

$$\Phi$$

$$d = D - 2$$

$$\mu, \nu, \ldots = 1, 2; \qquad a, b, \ldots = 3, 4, \ldots D;$$

Conformal factor:

$$g_{\mu\nu} = f \eta_{\mu\nu}$$

$$\eta_{\mu\nu} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}$$

$$\varepsilon^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Geometrically defined coordinates (α, β) and (ξ, η) :

$$\alpha: \quad \det \|G_{ab}\| = \epsilon \alpha^2, \qquad \eta^{\mu\nu}\partial_\mu \partial_\nu \alpha = 0$$

$$\beta: \quad \partial_\mu \beta = \epsilon \varepsilon_\mu{}^\nu \partial_\nu \alpha, \qquad \varepsilon_\mu{}^\nu = \eta_{\mu\gamma}\varepsilon^{\gamma\nu}$$

$$\epsilon_1 = \pm 1, \quad \epsilon_2 = \pm 1, \quad \epsilon = -\epsilon_1 \epsilon_2$$

$$\begin{cases} \xi = \beta + j\alpha, \\ \eta = \beta - j\alpha, \end{cases} \qquad j = \begin{cases} 1, & \epsilon = 1 \\ i, & \epsilon = -1 \end{cases}$$

-- **hyperbolic case**

-- **elliptic case**

$$\begin{cases} \xi = -x + t \\ \eta = -x - t \end{cases} \qquad \begin{cases} \xi = z + i\rho \\ \eta = z - i\rho \end{cases}$$

Ernst equation for vacuum

$$\begin{cases} (Re\mathcal{E})\,\eta^{\mu\nu}\left(\partial_\mu + \dfrac{\alpha_\mu}{\alpha}\right)\partial_\nu\mathcal{E} - \eta^{\mu\nu}\partial_\mu\mathcal{E}\partial_\nu\mathcal{E} = 0 \\ \eta^{\mu\nu}\partial_\mu\partial_\nu\alpha = 0 \end{cases}$$

$$\eta_{\mu\nu} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}$$

Ernst equations for electrovacuum

$$\begin{cases} (Re\mathcal{E} + \Phi\overline{\Phi})\,\eta^{\mu\nu}\left(\partial_\mu + \dfrac{\alpha_\mu}{\alpha}\right)\partial_\nu\mathcal{E} - \eta^{\mu\nu}(\partial_\mu\mathcal{E} + 2\overline{\Phi}\partial_\mu\Phi)\partial_\nu\mathcal{E} = 0 \\ (Re\mathcal{E} + \Phi\overline{\Phi})\,\eta^{\mu\nu}\left(\partial_\mu + \dfrac{\alpha_\mu}{\alpha}\right)\partial_\nu\Phi - \eta^{\mu\nu}(\partial_\mu\mathcal{E} + 2\overline{\Phi}\partial_\mu\Phi)\partial_\nu\Phi = 0 \\ \eta^{\mu\nu}\partial_\mu\partial_\nu\alpha = 0 \end{cases}$$

Ernst equations for Einstein - Maxwell - Weyl fields

$$\begin{cases} (Re\mathcal{E} + \Phi\overline{\Phi})\,\eta^{\mu\nu}\left(\partial_\mu + \dfrac{\partial_\mu(\alpha + i\delta)}{\alpha}\right)\partial_\nu\mathcal{E} - \eta^{\mu\nu}(\partial_\mu\mathcal{E} + 2\overline{\Phi}\partial_\mu\Phi)\partial_\nu\mathcal{E} = 0 \\ (Re\mathcal{E} + \Phi\overline{\Phi})\,\eta^{\mu\nu}\left(\partial_\mu + \dfrac{\partial_\mu(\alpha + i\delta)}{\alpha}\right)\partial_\nu\Phi - \eta^{\mu\nu}(\partial_\mu\mathcal{E} + 2\overline{\Phi}\partial_\mu\Phi)\partial_\nu\Phi = 0 \\ \eta^{\mu\nu}\partial_\mu\partial_\nu\alpha = 0, \qquad \eta^{\mu\nu}\partial_\mu\partial_\nu\delta = 0 \end{cases}$$

Bosonic sector of the heterotic string effective action

Matrix Ernst-like dynamical variables:

$$\begin{cases} \mathcal{E} = \|\mathcal{E}\|_{d\times d} \\ \mathcal{A} = \|\mathcal{A}\|_{d\times n} \\ \alpha \end{cases}$$

$$\mathcal{E} = \mathcal{G} + \mathcal{B} + \mathcal{A}\mathcal{A}^T$$

$$\mathcal{G}_{ab} = e^{2\Phi} G_{ab}$$

$$\det G = \epsilon\alpha^2$$

Matrix Ernst-like form of the dynamical equations

$$/\ \mathcal{G} = \frac{1}{2}(\mathcal{E} + \mathcal{E}^T) - \mathcal{A}\mathcal{A}^T /$$

$$\begin{cases} \eta^{\mu\nu}\partial_\mu(\alpha\partial_\nu\mathcal{E}) - \alpha\eta^{\mu\nu}(\partial_\mu\mathcal{E} - 2\partial_\mu\mathcal{A}\mathcal{A}^T)\mathcal{G}^{-1}\partial_\nu\mathcal{E} = 0 \\ \eta^{\mu\nu}\partial_\mu(\alpha\partial_\nu\mathcal{A}) - \alpha\eta^{\mu\nu}(\partial_\mu\mathcal{E} - 2\partial_\mu\mathcal{A}\mathcal{A}^T)\mathcal{G}^{-1}\partial_\nu\mathcal{A} = 0 \\ \eta^{\mu\nu}\partial_\mu\partial_\nu\alpha = 0 \end{cases}$$

$$\eta_{\mu\nu} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}$$

Examples for the choice of $\alpha(x^1, x^2),\ \beta(x^1, x^2)$

$\alpha = \rho,\ \beta = z$ **-- stationary axisymmetric fields**

$\alpha = f(u) + g(v),\ \beta = -f(u) + g(v)$ **-- colliding plane waves**

$\alpha = t,\ \beta = x$ **-- cosmological solutions**

Self-dual form of dynamical equations

Vacuum

$$\mathbf{U} = \begin{pmatrix} 1 & 0 \\ B_+ & 1 \end{pmatrix} \begin{pmatrix} i & -\mathcal{E}_\xi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B_+ & 1 \end{pmatrix}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ B_- & 1 \end{pmatrix} \begin{pmatrix} i & -\mathcal{E}_\eta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B_- & 1 \end{pmatrix}$$

$$h = g_{33}, \quad \Omega = \frac{g_{34}}{g_{33}}$$

$$B_+ = \Omega - \frac{ij\alpha}{h}$$

$$B_- = \Omega + \frac{ij\alpha}{h}$$

$$Re\mathcal{E} = h$$

$$d(Im\mathcal{E}) = -\alpha^{-1} h^2 {}^\star d\Omega$$

$$\begin{cases} \partial_\eta \mathbf{U} + \partial_\xi \mathbf{V} + \dfrac{[\mathbf{U}, \mathbf{V}]}{i(\xi - \eta)} = 0 \\ \partial_\eta \mathbf{U} - \partial_\xi \mathbf{V} = 0 \end{cases}$$

$$\mathbf{U} \cdot \mathbf{U} = i\,\mathbf{U}, \qquad \mathrm{tr}\,\mathbf{U} = i$$

$$\mathbf{V} \cdot \mathbf{V} = i\,\mathbf{V}, \qquad \mathrm{tr}\,\mathbf{V} = i$$

Einstein - Maxwell fields

$$\mathbf{U}=\begin{pmatrix}1&0&0\\B_+&1&0\\C_+&0&1\end{pmatrix}\begin{pmatrix}i&-\mathcal{E}_\xi&\Phi_\xi\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\-B_+&1&0\\-C_+&0&1\end{pmatrix}$$

$$\mathbf{V}=\begin{pmatrix}1&0&0\\B_-&1&0\\C_-&0&1\end{pmatrix}\begin{pmatrix}i&-\mathcal{E}_\eta&\Phi_\eta\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\-B_-&1&0\\-C_-&0&1\end{pmatrix}$$

$$h=g_{33},\quad \Omega=\frac{g_{34}}{g_{33}}$$

$$B_+=\Omega-\frac{ij\alpha}{h}$$

$$B_-=\Omega+\frac{ij\alpha}{h}$$

$$C_+=2(\overline{\widetilde{\Phi}}-\overline{\Phi}\,B_+)$$

$$C_-=2(\overline{\widetilde{\Phi}}-\overline{\Phi}\,B_-)$$

$$Re\Phi=A_3 \qquad d(Im\Phi)=-\alpha^{-1}h({}^\star dA_4-\Omega{}^\star dA_3)$$

$$Re\mathcal{E}=h-\Phi\overline{\Phi} \qquad d(Im\mathcal{E})=-\alpha^{-1}h^{2\star}d\Omega+i(\overline{\Phi}d\Phi-\Phi d\overline{\Phi})$$

$$\partial_\xi\widetilde{\Phi}=B_+\partial_\xi\Phi$$

$$\partial_\eta\widetilde{\Phi}=B_-\partial_\eta\Phi$$

$$\begin{cases}\partial_\eta\mathbf{U}+\partial_\xi\mathbf{V}+\dfrac{[\mathbf{U},\mathbf{V}]}{i(\xi-\eta)}=0\\[2mm]\partial_\eta\mathbf{U}-\partial_\xi\mathbf{V}=0\end{cases}\qquad\Bigg\|\qquad \begin{matrix}\mathbf{U}\cdot\mathbf{U}=i\,\mathbf{U}, & \operatorname{tr}\mathbf{U}=i\\ \mathbf{V}\cdot\mathbf{V}=i\,\mathbf{V}, & \operatorname{tr}\mathbf{V}=i\end{matrix}$$

Dynamics of the bosonic sector of string effective action --- (2 d+n)x(2 d+n)-matrices

$$\mathbf{U} = \begin{pmatrix} I_d & 0 & 0 \\ B_+ & I_d & 0 \\ C_+ & 0 & I_n \end{pmatrix} \begin{pmatrix} I_d & -\mathcal{E}_\xi & -2\mathcal{A}_\xi \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I_d & 0 & 0 \\ -B_+ & I_d & 0 \\ -C_+ & 0 & I_n \end{pmatrix}$$

$$\mathbf{V} = \begin{pmatrix} I_d & 0 & 0 \\ B_- & I_d & 0 \\ C_- & 0 & I_n \end{pmatrix} \begin{pmatrix} I_d & -\mathcal{E}_\eta & -2\mathcal{A}_\eta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I_d & 0 & 0 \\ -B_- & I_d & 0 \\ -C_- & 0 & I_n \end{pmatrix}$$

$$\mathcal{E} = \mathcal{G} + \mathcal{B} + \mathcal{A}\mathcal{A}^T$$

$$\alpha = (\xi - \eta)/2j$$

$$B_+ = \widetilde{B} - j\alpha\mathcal{G}^{-1}$$

$$B_- = \widetilde{B} + j\alpha\mathcal{G}^{-1}$$

$$C_+ = -2(\widetilde{\mathcal{A}}^T + \mathcal{A}^T B_+)$$

$$C_- = -2(\widetilde{\mathcal{A}}^T + \mathcal{A}^T B_-)$$

$$\partial_\xi \widetilde{\mathcal{B}} = -j\alpha\mathcal{G}^{-1}(\partial_\xi\mathcal{B} - 2\partial_\xi\mathcal{A}\,\mathcal{A}^T + 2\mathcal{A}\,\partial_\xi\mathcal{A}^T)\mathcal{G}^{-1},$$

$$\partial_\eta \widetilde{\mathcal{B}} = j\alpha\mathcal{G}^{-1}(\partial_\eta\mathcal{B} - 2\partial_\eta\mathcal{A}\,\mathcal{A}^T + 2\mathcal{A}\,\partial_\eta\mathcal{A}^T)\mathcal{G}^{-1}$$

$$\partial_\xi\widetilde{\mathcal{A}} = B_+\partial_\xi\mathcal{A}$$

$$\partial_\eta\widetilde{\mathcal{A}} = B_-\partial_\eta\mathcal{A}$$

$$\begin{cases} \partial_\eta\mathbf{U} + \partial_\xi\mathbf{V} + \dfrac{[\mathbf{U},\mathbf{V}]}{\xi-\eta} = 0 \\ \partial_\eta\mathbf{U} - \partial_\xi\mathbf{V} = 0 \end{cases}$$

$$\mathbf{U}\cdot\mathbf{U} = \mathbf{U}, \qquad \operatorname{tr}\mathbf{U} = d$$

$$\mathbf{V}\cdot\mathbf{V} = \mathbf{V}, \qquad \operatorname{tr}\mathbf{V} = d$$

Associated linear systems [1)]

Vacuum and electrovacuum (2x2 and 3x3 matrices)

$$\begin{cases} \partial_\eta \mathbf{U} + \partial_\xi \mathbf{V} + \dfrac{[\mathbf{U},\mathbf{V}]}{i(\xi-\eta)} = 0 \\ \partial_\eta \mathbf{U} - \partial_\xi \mathbf{V} = 0 \end{cases} \quad \Bigg\| \quad \begin{array}{ll} \mathbf{U}\cdot\mathbf{U} = i\,\mathbf{U}, & \mathrm{tr}\,\mathbf{U} = i \\ \mathbf{V}\cdot\mathbf{V} = i\,\mathbf{V}, & \mathrm{tr}\,\mathbf{V} = i \end{array}$$

$$\mathbf{\Psi}(\xi,\eta,w), \mathbf{U}(\xi,\eta), \mathbf{V}(\xi,\eta) \quad - \quad ? \qquad (w \in \mathbb{C})$$

$$\begin{cases} 2i(w-\xi)\partial_\xi \mathbf{\Psi} = \mathbf{U}(\xi,\eta)\,\mathbf{\Psi} \\ 2i(w-\eta)\partial_\eta \mathbf{\Psi} = \mathbf{V}(\xi,\eta)\,\mathbf{\Psi} \end{cases} \quad \Bigg\| \quad \begin{array}{ll} \mathrm{rank}\,\mathbf{U} = 1 & \mathrm{tr}\,\mathbf{U} = i \\ \mathrm{rank}\,\mathbf{V} = 1 & \mathrm{tr}\,\mathbf{V} = i \end{array}$$

1) *GA, JETP Lett.. (1980); Proc. Steklov Math. Inst. (1988); Physica D. (1999); Theor. Math. Phys. (2005)*

Bosonic dynamics in heterotic string gravity model ((2d+n)x(2d+n) matrices)[1)]

$$\begin{cases} \partial_\eta \mathbf{U} + \partial_\xi \mathbf{V} + \dfrac{[\mathbf{U},\mathbf{V}]}{(\xi-\eta)} = 0 \\ \partial_\eta \mathbf{U} - \partial_\xi \mathbf{V} = 0 \end{cases} \qquad \Bigg\| \qquad \begin{matrix} \mathbf{U}\cdot\mathbf{U} = \mathbf{U}, & \mathrm{tr}\,\mathbf{U} = d \\ \mathbf{V}\cdot\mathbf{V} = \mathbf{V}, & \mathrm{tr}\,\mathbf{V} = d \end{matrix}$$

$$\mathbf{\Psi}(\xi,\eta,w), \mathbf{U}(\xi,\eta), \mathbf{V}(\xi,\eta) \quad - \quad ? \qquad\qquad (w \in \mathbb{C})$$

$$\begin{cases} 2(w-\xi)\partial_\xi \mathbf{\Psi} = \mathbf{U}(\xi,\eta)\,\mathbf{\Psi} \\ 2(w-\eta)\partial_\eta \mathbf{\Psi} = \mathbf{V}(\xi,\eta)\,\mathbf{\Psi} \end{cases} \qquad \Bigg\| \qquad \begin{matrix} \mathbf{U}\cdot\mathbf{U} = \mathbf{U} & \mathrm{tr}\,\mathbf{U} = d \\ \mathbf{V}\cdot\mathbf{V} = \mathbf{V} & \mathrm{tr}\,\mathbf{V} = d \end{matrix}$$

[1)] *GA, Phys.Rev.D (2009)*

Equivalent spectral problems[1)]

Electrovacuum (3x3 matrices)

$$\mathbf{\Psi}(\xi,\eta,w), \mathbf{U}(\xi,\eta), \mathbf{V}(\xi,\eta), \mathbf{W}(\xi,\eta,w) \in GL(3,\mathbb{C})$$

$$\left\{\begin{array}{l} 2i(w-\xi)\partial_\xi \mathbf{\Psi} = \mathbf{U}(\xi,\eta)\,\mathbf{\Psi} \\ 2i(w-\eta)\partial_\eta \mathbf{\Psi} = \mathbf{V}(\xi,\eta)\,\mathbf{\Psi} \end{array}\right. \quad \Bigg\| \quad \begin{array}{ll} \mathrm{rank}\,\mathbf{U} = 1 & \mathrm{tr}\,\mathbf{U} = i \\ \mathrm{rank}\,\mathbf{V} = 1 & \mathrm{tr}\,\mathbf{V} = i \end{array}$$

$$\left\{\begin{array}{l} \mathbf{\Psi}^\dagger \mathbf{W} \mathbf{\Psi} = \mathbf{K}(w) \\ \mathbf{K}^\dagger(w) = \mathbf{K}(w) \end{array}\right. \quad \Bigg\| \quad \frac{\partial \mathbf{W}}{\partial w} = 4i\mathbf{\Omega}, \qquad \mathbf{\Omega} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{W} = 4i(w-(\xi+\eta)/2)\mathbf{\Omega} + \begin{pmatrix} -4\epsilon\alpha^2 \mathrm{g}^{ab} + 4\Phi^a\overline{\Phi}^b & -2\Phi^a \\ -2\overline{\Phi}^b & 1 \end{pmatrix}$$

1) *GA, JETP Lett.. (1980); Sov. Phys (1985);Proc. Steklov Inst. Math. (1988); Physica D. (1999)*

Bosonic dynamics in heterotic string gravity model ((2d+n)x(2d+n) matrices) [1]

$$\mathbf{\Psi}(\xi,\eta,w),\mathbf{U}(\xi,\eta),\mathbf{V}(\xi,\eta),\mathbf{W}(\xi,\eta,w)-?$$

$$\begin{cases} 2(w-\xi)\partial_\xi\mathbf{\Psi}=\mathbf{U}(\xi,\eta)\,\mathbf{\Psi}\\ 2(w-\eta)\partial_\eta\mathbf{\Psi}=\mathbf{V}(\xi,\eta)\,\mathbf{\Psi}\end{cases} \quad \Bigg\| \quad \begin{array}{ll}\mathbf{U}\cdot\mathbf{U}=\mathbf{U} & \mathrm{tr}\,\mathbf{U}=d\\ \mathbf{V}\cdot\mathbf{V}=\mathbf{V} & \mathrm{tr}\,\mathbf{V}=d\end{array}$$

$$\begin{cases}\mathbf{\Psi}^T\mathbf{W}\mathbf{\Psi}=\mathbf{K}(w)\\ \mathbf{K}^T(w)=\mathbf{K}(w)\end{cases} \quad \Bigg\| \quad \frac{\partial\mathbf{W}}{\partial w}=\mathbf{\Omega},\qquad \mathbf{\Omega}=\begin{pmatrix}0 & \mathbf{I}_d & 0\\ \mathbf{I}_d & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\Longrightarrow\qquad \mathbf{W}=(w-\frac{\xi+\eta}{2})\Omega+\begin{pmatrix}\epsilon\alpha^2\mathcal{G}^{-1}-\widetilde{\mathcal{B}}\mathcal{G}\widetilde{\mathcal{B}}+\widetilde{\mathcal{A}}\widetilde{\mathcal{A}}^T & \widetilde{\mathcal{B}}\mathcal{G}+\widetilde{\mathcal{A}}\mathcal{A}^T & \widetilde{\mathcal{A}}\\ -\mathcal{G}\widetilde{\mathcal{B}}+\mathcal{A}\widetilde{\mathcal{A}}^T & \mathcal{G}+\mathcal{A}\mathcal{A}^T & \mathcal{A}\\ \widetilde{\mathcal{A}}^T & \mathcal{A}^T & I_n\end{pmatrix}$$

[1] GA, Phys. Rev. D. (2009)

Monodromy transform approach

The space of local solutions:

$g_{ik},\ F_{ik}\ldots \| x^1, x^2$

↔

Free space of functional parameters -- "coordinates" in the space of local solutions

$\{u_\pm(w), v_\pm(w)\ldots\},\ w\in\overline{\mathbb{C}}$

(Constraint: field equations)

(No constraints)

"Direct" problem: $\{g_{ik}(x),\ F_{ik(x)}\ldots\}\ \to\ \{u_\pm(w), v_\pm(w)\ldots\}$

(linear ordinary differential equations)

"Inverse" problem: $\{u_\pm(w), v_\pm(w)\ldots\}\ \to\ \{g_{ik}(x),\ F_{ik}(x)\ldots\}$

(linear integral equations)

Interpretation: $\{u_\pm(w), v_\pm(w)\ldots\}\ \leftrightarrow$ Monodromy data for $\Psi(\xi,\eta,w)$

The spectral problem with constant Jordan forms of it's coefficients

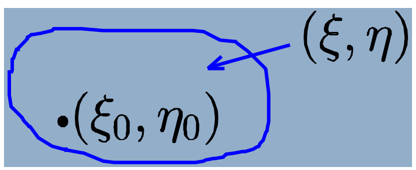
$$\mathbf{\Psi}(\xi,\eta,w),\mathbf{U}(\xi,\eta),\mathbf{V}(\xi,\eta)-?$$

$(2\mathrm{d}+\mathrm{n})\times(2\mathrm{d}+\mathrm{n})$

$$\begin{cases} 2(w-\xi)\partial_\xi \mathbf{\Psi} = \mathbf{U}(\xi,\eta)\,\mathbf{\Psi} \\ 2(w-\eta)\partial_\eta \mathbf{\Psi} = \mathbf{V}(\xi,\eta)\,\mathbf{\Psi} \end{cases}$$

$$\mathbf{U}\cdot\mathbf{U}=\mathbf{U} \qquad \mathrm{tr}\,\mathbf{U}=d$$

$$\mathbf{V}\cdot\mathbf{V}=\mathbf{V} \qquad \mathrm{tr}\,\mathbf{V}=d$$



$$(\xi,\eta)\in(\Omega_{\xi_o}\times\Omega_{\eta_o}),\qquad w\in\overline{\mathbb{C}}$$

$$\mathbf{\Psi}(\xi_o,\eta_o,w)\equiv\mathbf{I}$$

$$\mathbf{U}(\xi,\eta)=\mathcal{F}_+\mathbf{U}_{(o)}\mathcal{F}_+^{-1}$$

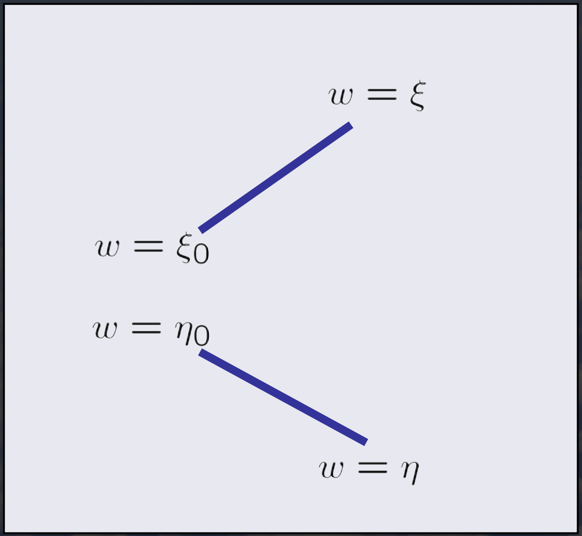
$$\mathbf{V}(\xi,\eta)=\mathcal{F}_-\mathbf{V}_{(o)}\mathcal{F}_-^{-1}$$

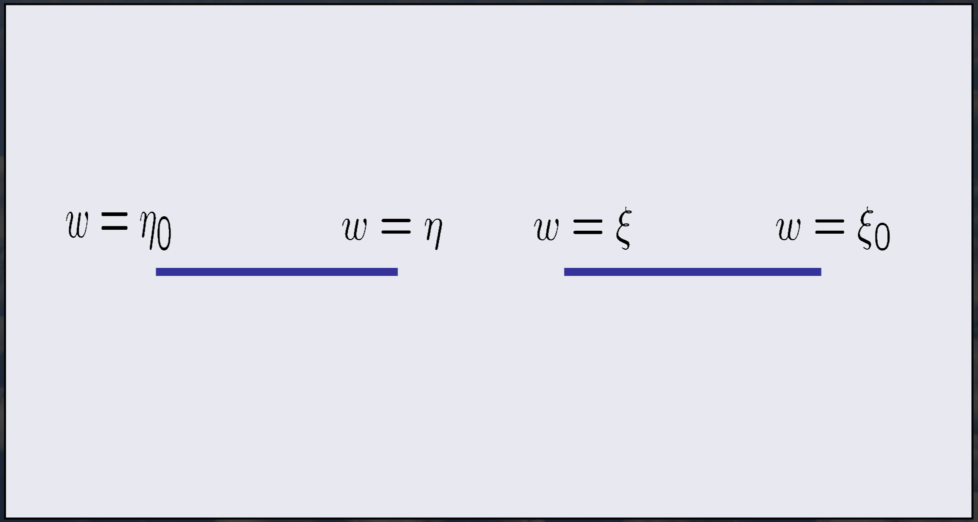
$$\mathbf{U}_{(o)}=\mathrm{diag}\,\{\underbrace{1,\ldots,1}_{d},\underbrace{0,\ldots,0}_{d+n}\},$$

$$\mathbf{V}_{(o)}=\mathrm{diag}\,\{\underbrace{1,\ldots,1}_{d},\underbrace{0,\ldots,0}_{d+n}\}.$$

Normalization: $\mathbf{\Psi}(\xi,\eta,w)\;\rightarrow\;\mathbf{\Psi}(\xi,\eta,w)\cdot\mathbf{\Psi}^{-1}(\xi_0,\eta_0,w)$

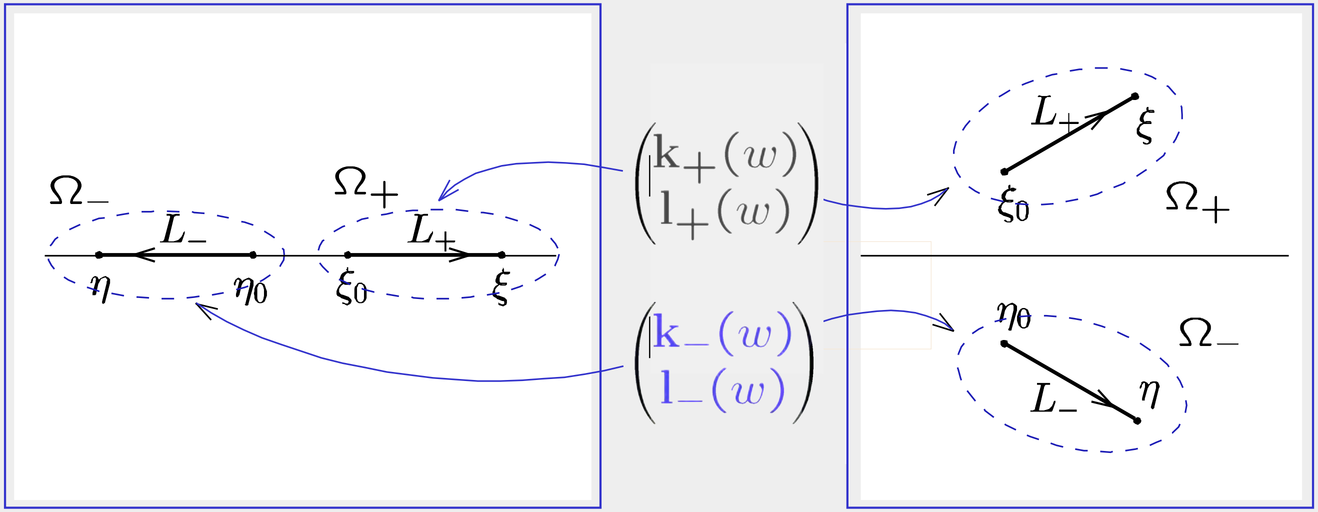
Analytical structure of on the w-plane [1)]





1) ***GA, Sov. Phys (1985) ;***

Analytical structure of $\Psi(\xi,\eta,w)$ on the spectral plane w



Hyperbolic case $(\epsilon = 1)$ Elliptic case $(\epsilon = -1)$

$$\lambda_+ = \sqrt{\frac{w-\xi}{w-\xi_0}}$$

$$\lambda_- = \sqrt{\frac{w-\eta}{w-\eta_0}}$$

$$\lambda_\pm(w=\infty) = 1$$

Ω_+ :

$$\boldsymbol{\Psi} = \lambda_+^{-1}\,\psi_+(\xi,\eta,w)\otimes \mathbf{k}_+(w) + \mathbf{M}_+(\xi,\eta,w)$$

$$\boldsymbol{\Psi}^{-1} = \lambda_+\,\mathbf{l}_+(w)\otimes \varphi_+(\xi,\eta,w) + \mathbf{N}_+(\xi,\eta,w)$$

$\mathrm{M}_+\cdot \mathrm{l}_+ = 0$

$\mathrm{k}_+\cdot \mathrm{N}_+ = 0$

$\varphi_+\cdot \mathrm{M}_+ = 0$

$\mathrm{N}_+\cdot \psi_+ = 0$

Ω_- :

$$\boldsymbol{\Psi} = \lambda_-^{-1}\,\psi_-(\xi,\eta,w)\otimes \mathbf{k}_-(w) + \mathbf{M}_-(\xi,\eta,w)$$

$$\boldsymbol{\Psi}^{-1} = \lambda_-\,\mathbf{l}_-(w)\otimes \varphi_-(\xi,\eta,w) + \mathbf{N}_-(\xi,\eta,w)$$

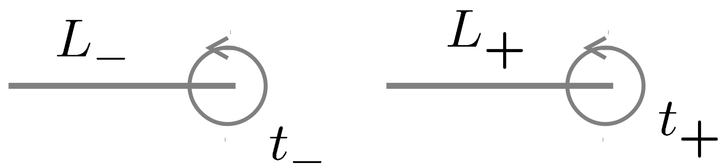
$\mathrm{M}_-\cdot \mathrm{l}_- = 0$

$\mathrm{k}_-\cdot \mathrm{N}_- = 0$

$\varphi_-\cdot \mathrm{M}_- = 0$

$\mathrm{N}_-\cdot \psi_- = 0$

Monodromy data of a given solution



$$\Psi \xrightarrow{t_\pm} \Psi \cdot \mathbf{T}_\pm(w), \quad \mathbf{T}^2_\pm(w) = \mathbf{I}$$

$$\mathbf{T}_+(w) = \mathbf{I} - 2\,\mathbf{l}_+(w) \otimes (\mathbf{k}_+(w) \cdot \mathbf{l}_+(w))^{-1} \otimes \mathbf{k}_+(w),$$
$$\mathbf{T}_-(w) = \mathbf{I} - 2\,\mathbf{l}_-(w) \otimes (\mathbf{k}_-(w) \cdot \mathbf{l}_-(w))^{-1} \otimes \mathbf{k}_-(w).$$

``Extended'' monodromy data:

$$\mathbf{k}_\pm(w) = \{\mathbf{I}_d,\ \mathbf{u}_\pm(w),\ \mathbf{v}_\pm(w)\},$$
$$\mathbf{l}_\pm(w) = \{\mathbf{I}_d,\ \mathbf{p}_\pm(w),\ \mathbf{q}_\pm(w)\}^T,$$

$$\mathbf{u}_\pm(w),\ \mathbf{p}^T_\pm(w) \in M_{d\times d}$$
$$\mathbf{v}_\pm(w),\ \mathbf{q}^T_\pm(w) \in M_{d\times n}$$

Monodromy data constraint: $\mathbf{l}_\pm(w)(w) = \mathcal{W}_0(w) \cdot \mathbf{k}^\dagger_\pm(w)$

Monodromy data for solutions of reduced Einstein's field equations:

$$\{\mathbf{u}^{d\times d}_\pm(w), \mathbf{v}^{d\times n}_\pm(w)\}$$

Simple example for solution of the direct problem of the monodromy transform

Let us take a symmetric vacuum Kazner solution:

$$ds^2 = \frac{1}{\sqrt{t}}(dt^2 - dx^2) - t(dy^2 + dz^2)$$

For this solution the matrix $\Psi(\xi, \eta, w)$ derived as a solution of the spectral problem linear equations takes the form

$$\mathbf{\Psi} = \frac{1}{2}\left(\frac{w-\xi_0}{w-\xi}\right)^{1/2}\begin{pmatrix}1\\ it_0\end{pmatrix}\otimes\left(1, -\frac{i}{t_0}\right) + \frac{1}{2}\left(\frac{w-\eta_0}{w-\eta}\right)^{1/2}\begin{pmatrix}1\\ -it_0\end{pmatrix}\otimes\left(1, \frac{i}{t_0}\right)$$

$$\mathbf{\Psi}^{-1} = \frac{1}{2}\left(\frac{w-\xi}{w-\xi_0}\right)^{1/2}\begin{pmatrix}1\\ it_0\end{pmatrix}\otimes\left(1, -\frac{i}{t_0}\right) + \frac{1}{2}\left(\frac{w-\eta}{w-\eta_0}\right)^{1/2}\begin{pmatrix}1\\ -it_0\end{pmatrix}\otimes\left(1, \frac{i}{t_0}\right)$$

This allows to calculate immediately the monodromy data functions

$$\mathbf{u}_+(w) = -\frac{i}{t_0}, \qquad \mathbf{u}_-(w) = \frac{i}{t_0}$$

Inverse problem of the monodromy transform [1)]

Free space of the monodromy data
$\{\mathbf{u}_\pm(w), \mathbf{v}_\pm(w)\dots\}$

→

Space of solutions
$\mathrm{g}_{\mathrm{ik}}, \mathbf{F}_{\mathrm{ik}}\dots\|\mathrm{x}^1, \mathrm{x}^2$

<u>**Theorem 1.**</u> For any local solution $(\mathcal{E}, \Phi)$ holomorphic near (ξ_0, η_0)

- $\boldsymbol{\Psi}(\xi,\eta,w)$ Is holomorphic on $\overline{\mathbb{C}}\backslash L \quad (L = L_+ \cup L_-)$,
- $\boldsymbol{\Psi}(\xi_0,\eta_0,w) = \mathbf{I}$ and $\boldsymbol{\Psi}(\xi,\eta,w=\infty) = \mathbf{I}$
- the \`\`jumps'' of $\boldsymbol{\Psi}(\xi,\eta,w)$ on the cuts $L_\pm$ satisfy the Hölder condition and are integrable near the endpoints.
- $\boldsymbol{\Psi}^{-1}(\xi,\eta,w)$ posess the same properties

→

$$\boldsymbol{\Psi} = \mathbf{I} + \frac{1}{i\pi}\int_L \frac{[\boldsymbol{\Psi}]_\zeta}{\zeta - w} d\zeta, \qquad \boldsymbol{\Psi}^{-1} = \mathbf{I} + \frac{1}{i\pi}\int_L \frac{[\boldsymbol{\Psi}^{-1}]_\zeta}{\zeta - w} d\zeta$$

1) *GA, Sov.Phys.Dokl. 1985;Proc. Steklov Inst. Math. 1988; Theor.Math.Phys. 2005*

Theorem 2. *) For any local solution $(\mathcal{E}, \Phi)$ holomorphic near (ξ_0, η_0) $\boldsymbol{\Psi}(\xi,\eta,w)$ and $\boldsymbol{\Psi}^{-1}(\xi,\eta,w)$ possess the local structures

$$\boldsymbol{\Psi} = \begin{cases} \lambda_+^{-1} \boldsymbol{\psi}_+(\xi,\eta,w) \otimes \mathbf{k}_+(w) + \mathbf{M}_+(\xi,\eta,w), & w \in \Omega_+ \\ \lambda_-^{-1} \boldsymbol{\psi}_-(\xi,\eta,w) \otimes \mathbf{k}_-(w) + \mathbf{M}_-(\xi,\eta,w), & w \in \Omega_- \end{cases}$$

$$\boldsymbol{\Psi}^{-1} = \begin{cases} \lambda_+ \mathbf{l}_+(w) \otimes \boldsymbol{\varphi}_+(\xi,\eta,w) + \mathbf{N}_+(\xi,\eta,w), & w \in \Omega_+ \\ \lambda_- \mathbf{l}_-(w) \otimes \boldsymbol{\varphi}_-(\xi,\eta,w) + \mathbf{N}_-(\xi,\eta,w), & w \in \Omega_- \end{cases}$$

where $\mathbf{k}_\pm, \mathbf{l}_\pm, \boldsymbol{\varphi}_\pm, \boldsymbol{\psi}_\pm, \mathbf{M}_\pm, \mathbf{N}_\pm$ are holomorphic on $\Omega_\pm$ respectively.

Fragments of these structures satisfy in $\Omega_\pm$ the algebraic constraints

$\mathbf{M}_+ \cdot \mathbf{l}_+ = 0$	$\boldsymbol{\varphi}_+ \cdot \mathbf{M}_+ = 0$	$\mathbf{M}_- \cdot \mathbf{l}_- = 0$	$\boldsymbol{\varphi}_- \cdot \mathbf{M}_- = 0$
$\mathbf{k}_+ \cdot \mathbf{N}_+ = 0$	$\mathbf{N}_+ \cdot \boldsymbol{\psi}_+ = 0$	$\mathbf{k}_- \cdot \mathbf{N}_- = 0$	$\mathbf{N}_- \cdot \boldsymbol{\psi}_- = 0$

$$\{\boldsymbol{\Psi}\}_{L_\pm} = \mathbf{M}_\pm, \qquad [\boldsymbol{\Psi}]_{L_\pm} = [\lambda_\pm^{-1}]\, \boldsymbol{\psi}_\pm \otimes \mathbf{k}_\pm,$$

$$\{\boldsymbol{\Psi}^{-1}\}_{L_\pm} = \mathbf{N}_\pm, \qquad [\boldsymbol{\Psi}]_{L_\pm}^{-1} = [\lambda_\pm]\, \mathbf{l}_\pm \otimes \boldsymbol{\varphi}_\pm.$$

and the relations in boxes give rise to the linear singular integral equations.

Theorem 3. For any local solution of the ``null curvature'' equations with the above Jordan conditions the fragments of the local structures of $\boldsymbol{\Psi}(\xi,\eta,w)$ and $\boldsymbol{\Psi}^{-1}(\xi,\eta,w)$ on the cuts $L_{\pm}$ should satisfy

$$\frac{1}{\pi i}\int\limits_{L}\frac{\mathcal{K}(\xi,\eta,\tau,\zeta)}{\zeta-\tau}\cdot\boldsymbol{\varphi}\,(\xi,\eta,\zeta)\,d\zeta=\mathbf{k}(\tau)$$

$$\frac{1}{\pi i}\int\limits_{L}\boldsymbol{\psi}\,(\xi,\eta,\zeta)\cdot\frac{\widetilde{\mathcal{K}}(\xi,\eta,\tau,\zeta)}{\zeta-\tau}\,d\zeta=\mathbf{l}(\tau)$$

where the dot means the matrix product and the kernals are

$$\mathcal{K}(\xi,\eta,\tau,\zeta)=-[\lambda]_{\zeta}\mathcal{H}(\tau,\zeta)$$

$$\widetilde{\mathcal{K}}(\xi,\eta,\tau,\zeta)=-[\lambda^{-1}]_{\zeta}\mathcal{H}(\zeta,\tau)$$

$$\mathcal{H}(x,y)\equiv(\mathbf{k}(x)\cdot\mathbf{l}(y))$$

where the parameters ζ and τ run over the contour $L=L_{+}\cup L_{-}$

$$\mathbf{k}(\tau)=\begin{cases}\mathbf{k}_{+}(\tau), & \tau\in L_{+}\\ \mathbf{k}_{-}(\tau), & \tau\in L_{-}\end{cases}\qquad \boldsymbol{\varphi}\,(\xi,\eta,\tau)=\begin{cases}\boldsymbol{\varphi}_{+}(\xi,\eta,\tau), & \tau\in L_{+}\\ \boldsymbol{\varphi}_{-}(\xi,\eta,\tau), & \tau=\in L_{-}\end{cases}$$

Theorem 4. For arbitrarily chosen extended monodromy data – two pairs of vectors (N=2,3) or two pairs of dx(2d+n) and (2d+n)xd matrix (N=2d+n) functions $\mathbf{k}_+(w), \mathbf{l}_+(w)$ and $\mathbf{k}_-(w), \mathbf{l}_-(w)$ holomorphic respectively in some neighborhoods Ω_+ and Ω_- of the points $w = \xi_0$ and $w = \eta_0$ on the spectral plane, there exists some neighborhood $\Omega^2 = \Omega_{\xi_0} \times \Omega_{\eta_0}$ of the initial point $P_0(\xi_0, \eta_0)$ such that the solutions $\boldsymbol{\varphi}_\pm(\xi, \eta, w)$ and $\boldsymbol{\psi}_\pm(\xi, \eta, w)$ of the integral equations given in Theorem 3 exist and are unique in $\Omega^2 \times \Omega_+$ and $\Omega^2 \times \Omega_-$ respectively.

The matrix functions $\boldsymbol{\Psi}(\xi, \eta, w)$ and $\widetilde{\boldsymbol{\Psi}}(\xi, \eta, w)$ are defined as

$$\boldsymbol{\Psi} = \mathbf{I} + \frac{1}{i\pi} \int\limits_L \frac{[\lambda^{-1}]_\zeta}{\zeta - w} \boldsymbol{\psi}(\xi, \eta, \zeta) \otimes \mathbf{k}(\zeta) d\zeta,$$

$$\widetilde{\boldsymbol{\Psi}} = \mathbf{I} + \frac{1}{i\pi} \int\limits_L \frac{[\lambda e^{i\sigma}]_\zeta}{\zeta - w} \mathbf{l}(\zeta) \otimes \boldsymbol{\varphi}(\xi, \eta, \zeta) d\zeta$$

$$\boldsymbol{\Psi} \cdot \widetilde{\boldsymbol{\Psi}} = \mathbf{I}$$

$\boldsymbol{\Psi}(\xi, \eta, w)$ is a normalized fundamental solution of the associated linear system with the Jordan conditions.

Inverse problem of the monodromy transform [1)]

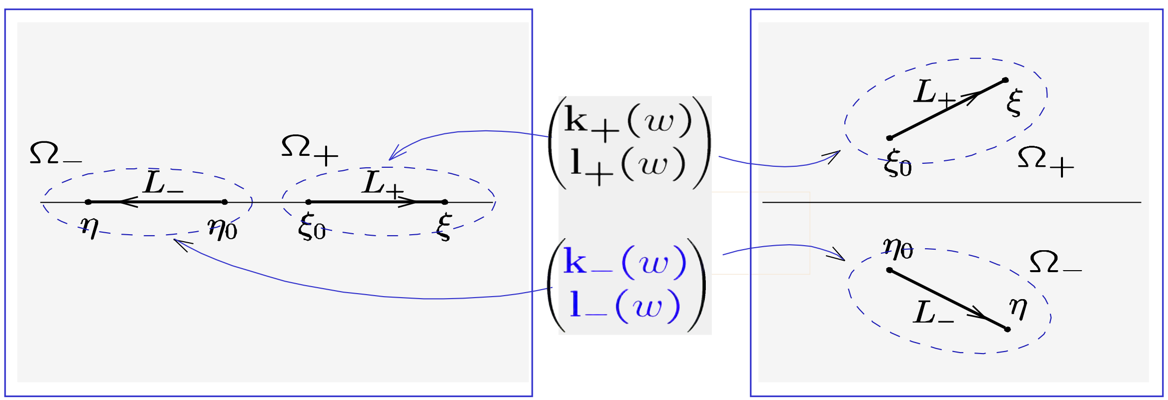
$$\frac{1}{i\pi}\int\limits_{L_+}\frac{[\lambda_+]_{\zeta_+}}{\zeta_+-\tau_+}\mathcal{H}(\tau_+,\zeta_+)\boldsymbol{\varphi}_+(\zeta_+)d\zeta_+ + \frac{1}{i\pi}\int\limits_{L_-}\frac{[\lambda_-]_{\zeta_-}}{\zeta_--\tau_+}\mathcal{H}(\tau_+,\zeta_-)\boldsymbol{\varphi}_-(\zeta_-)d\zeta_- + \mathbf{k}_+(\tau_+)=0$$

$$\frac{1}{i\pi}\int\limits_{L_+}\frac{[\lambda_+]_{\zeta_+}}{\zeta_+-\tau_-}\mathcal{H}(\tau_-,\zeta_+)\boldsymbol{\varphi}_+(\zeta_+)d\zeta_+ + \frac{1}{i\pi}\int\limits_{L_-}\frac{[\lambda_-]_{\zeta_-}}{\zeta_--\tau_-}\mathcal{H}(\tau_-,\zeta_-)\boldsymbol{\varphi}_-(\zeta_-)d\zeta_- + \mathbf{k}_-(\tau_-)=0$$

$$\mathcal{H}(\tau_\pm,\zeta_\pm)\equiv(\mathbf{k}(\tau_\pm)\cdot\mathbf{l}(\zeta_\pm))$$

$$\mathbf{k}(\tau_\pm)=\begin{pmatrix}1, & \mathbf{u}(\tau_\pm), & \mathbf{v}(\tau_\pm)\end{pmatrix}$$

$$\mathbf{l}(\tau_\pm)=\begin{pmatrix}1+i\epsilon_0(\tau_\pm-\beta_0)\mathbf{u}^\dagger(\tau_\pm)\\ -i\epsilon_0(\tau_\pm-\beta_0)+\epsilon\alpha_0^2\mathbf{u}^\dagger(\tau_\pm)\\ 4\epsilon_0\left[(\tau_\pm-\beta_0)^2-\epsilon\alpha_0^2\right]\mathbf{v}^\dagger(\tau_\pm)\end{pmatrix}$$



Hyperbolic case $(\epsilon=1)$ Elliptic case $(\epsilon=-1)$

$$\lambda_+=\sqrt{\frac{w-\xi}{w-\xi_0}}$$

$$\lambda_-=\sqrt{\frac{w-\eta}{w-\eta_0}}$$

$$\lambda_\pm(w=\infty)=1$$

1) *GA, Sov.Phys.Dokl. 1985;Proc. Steklov Inst. Math. 1988; Theor.Math.Phys. 2005*

Inverse problem of the monodromy transform *(compact form)*

$$\mathbf{k}_{\pm}(w) = \{I_d,\ \|\mathbf{u}_{\pm}\|_{d\times d}(w),\ \|\mathbf{v}_{\pm}\|_{d\times n}(w)\}$$

$$\mathbf{l}_{\pm}(w) = (w-\xi_o)(w-\eta_o)\,\mathbf{W}_o^{-1}\cdot\mathbf{k}_{\pm}{}^{T}(w)$$

[redacted]

$$\mathbf{U} = \overset{\circ}{\mathbf{U}} + 2\partial_\xi \mathbf{R},$$

$$\mathbf{V} = \overset{\circ}{\mathbf{V}} + 2\partial_\eta \mathbf{R}$$

$$\mathbf{R} = \frac{1}{i\pi}\int_L [\lambda]_\zeta\ \mathbf{l}(w)\cdot\boldsymbol{\varphi}\,(\xi,\eta,w)d\zeta$$

$$\mathbf{W} = \overset{\circ}{\mathbf{W}} - \boldsymbol{\Omega}\cdot\mathbf{R} - \mathbf{R}^T\cdot\boldsymbol{\Omega}$$

$$\mathbf{W} = (w - \frac{\xi+\eta}{2})\boldsymbol{\Omega} + \begin{pmatrix} \epsilon\alpha^2\mathcal{G}^{-1} - \widetilde{\mathcal{B}}\mathcal{G}\widetilde{\mathcal{B}} + \widetilde{\mathcal{A}}\widetilde{\mathcal{A}}^T & \widetilde{\mathcal{B}}\mathcal{G} + \widetilde{\mathcal{A}}\mathcal{A}^T & \widetilde{\mathcal{A}} \\ -\mathcal{G}\widetilde{\mathcal{B}} + \mathcal{A}\widetilde{\mathcal{A}}^T & \mathcal{G} + \mathcal{A}\mathcal{A}^T & \mathcal{A} \\ \widetilde{\mathcal{A}}^T & \mathcal{A}^T & I_n \end{pmatrix}$$

Properties of the monodromy data

Map of some known solutions

$$\begin{cases} \mathbf{u}(w) = 0 \\ \mathbf{v}(w) = 0 \end{cases}$$ Minkowski space-time

$$\begin{cases} \mathbf{u}(w) = u_0 \\ \mathbf{v}(w) = 0 \end{cases}$$ Rindler metric

$$\begin{cases} \mathbf{u}_+(w) = ik_0 \\ \mathbf{u}_-(w) = -ik_0 \\ \mathbf{v}_\pm(w) = 0 \end{cases}$$ Symmetric Kasner space-time

$$\begin{cases} \mathbf{u}(w) = 0 \\ \mathbf{v}(w) = v_0 \end{cases}$$ Bertotti – Robinson solution for electromagnetic universe, Bell – Szekeres solution for colliding plane electromagnetic waves

$$\begin{cases} \mathbf{u}(w) = u_0 + u_1 w \\ \mathbf{v}(w) = v_0 \end{cases}$$ Melvin magnetic universe

$$\begin{cases} \mathbf{u}(w) = \dfrac{u_0}{w-h} \\ \mathbf{v}(w) = \dfrac{v_0}{w-h} \end{cases}$$ Kerr – Newman black hole

$$\begin{cases} \mathbf{u}_+(w) = ik_0 \dfrac{w-a_+}{w-b_+} \\ \mathbf{u}_-(w) = -ik_0 \dfrac{w-a_-}{w-b_-} \\ \mathbf{v}_\pm(w) = 0 \end{cases}$$ Khan-Penrose and Nutku – Halil solutions for colliding plane gravitational waves

$$\begin{cases} \mathbf{u}(w) = u_0 + \dfrac{u_0}{w-h} \\ \mathbf{v}(w) = v_0 + \dfrac{v_0}{w-h} \end{cases}$$ Kerr – Newman black hole in the external electromagnetic field

Monodromy data map of some classes of solutions

- Solutions with diagonal metrics: static fields, waves with linear polarization:

$$\mathbf{u}_\pm^\dagger(w) = -\mathbf{u}_\pm(w), \quad \mathbf{v}_\pm^\dagger(w) = -\mathbf{v}_\pm(w)$$

- Stationary axisymmetric fields with the regular axis of symmetry are described by analytically matched monodromy data::

$$\mathbf{u}_+(w) = \mathbf{u}_-(w) = \mathbf{u}(w), \quad \mathbf{v}_+(w) = \mathbf{v}_-(w) \equiv \mathbf{v}(w),$$

- For asymptotically flat stationary axisymmetric fields

$$\mathbf{u}(w) = \frac{u_0}{w} + \frac{u_1}{w^2} + \ldots, \quad \mathbf{v}(w) = \frac{v_0}{w} + \frac{v_1}{w^2} + \ldots \quad \text{for} \ \ w \to \infty$$

with the coefficients expressed in terms of the multipole moments.

- For stationary axisymmetric fields with a regular axis of symmetry the values of the Ernst potentials on the axis near the point of normalization are

$$\mathcal{E}(z) = \mathcal{E}(z_0) - 2i(z - z_0)\mathbf{u}(z), \quad \Phi(z) = 2i(z - z_0)\mathbf{v}(z)$$

- For arbitrary rational and analytically matched monodromy data the solution can be found explicitly.

Solutions for analitically matched, rational monodromy data

Generic data:

$$\mathbf{u}(w) = \begin{cases} \mathbf{u}_+(\zeta), & \zeta \in L_+ \\ \mathbf{u}_-(\zeta), & \zeta \in L_- \end{cases}$$

$$\mathbf{v}(w) = \begin{cases} \mathbf{v}_+(\zeta), & \zeta \in L_+ \\ \mathbf{v}_-(\zeta), & \zeta \in L_- \end{cases}$$

Analytically matched data:

$$\mathbf{u}(w) = \mathbf{u}_+(w) = \mathbf{u}_-(w)$$
$$\mathbf{v}(w) = \mathbf{v}_+(w) = \mathbf{v}_-(w)$$

Unknowns:

$$\boldsymbol{\varphi}(w) = \boldsymbol{\varphi}_+(w) = \boldsymbol{\varphi}_-(w)$$

Rational, analytically matched data:

$$\mathbf{u}(w) = \frac{U(w)}{Q(w)} \qquad \mathbf{v}(w) = \frac{V(w)}{Q(w)}$$

$$U(w) = u_0 + u_1 w + \ldots + u_{N_u} w^{N_u}$$
$$U(w) = u_0 + u_1 w + \ldots + u_{N_u} w^{N_u}$$
$$Q(w) = 1 + q_1 w + \ldots + q_{N_q} w^{N_q}$$

$$N_o = \max\{N_u, N_v, N_q\}$$

Modified integral equation

$$-\frac{1}{\pi i}\int_L \frac{[\lambda]_\zeta}{\zeta-\tau}\mathcal{H}(\tau,\zeta)\boldsymbol{\varphi}(\xi,\eta,\zeta)\,d\zeta = \mathbf{k}(\tau)$$

$$\mathbf{k}(\tau) = \Big(Q(\tau),\ U(\tau),\ V(\tau)\ \Big)$$

$$\mathcal{H}(\tau,\zeta) = \frac{\mathbf{P}(\tau,\zeta)}{Q(\tau)Q^\dagger(\zeta)}$$

Auxiliary polynomials

$$P(\tau,\zeta) = Q(\tau)Q^\dagger(\zeta) + i\epsilon_0(\zeta-\beta_0)[Q(\tau)U^\dagger(\zeta) - Q^\dagger(\zeta)U(\tau)] + \epsilon\alpha_0^2 U(\tau)U^\dagger(\zeta) + 4\epsilon_0[(\zeta-\beta_0)^2 - \epsilon\alpha_0^2]V(\tau)V^\dagger(\zeta)$$

$$R(\tau,\zeta) = \left(Q^\dagger(\zeta) + i\epsilon_0\zeta U^\dagger(\zeta)\right)\left(\frac{Q(\tau)-Q(\zeta)}{\zeta-\tau}\right) - i\epsilon_0\zeta Q^\dagger(\zeta)\left(\frac{U(\tau)-U(\zeta)}{\zeta-\tau}\right) + 4\epsilon_0\zeta^2 V^\dagger(\zeta)\left(\frac{V(\tau)-V(\zeta)}{\zeta-\tau}\right) = \sum_{k=0}^{N_0-1} R_k(\zeta)\tau^k,$$

Auxiliary functions

$$\Delta_{kl} = \delta_{kl} + \frac{1}{i\pi}\int_L \frac{[\lambda]_\chi}{P(\chi,\chi)} R_k(\chi)L_l(\chi)\,d\chi \qquad L_k(\tau) \equiv \frac{1}{i\pi}\int_L \frac{[\lambda^{-1}]_\chi \chi^k}{\chi-\tau}\,d\chi$$

$$\begin{pmatrix}\mathbf{J}_k\\ \tilde{\mathbf{J}}_k\end{pmatrix} = \frac{1}{i\pi}\int_{\mathcal{L}} \frac{[\lambda]_\chi}{P(\chi,\chi)}\mathbf{l}'(\chi)L_k(\chi)d\chi = \frac{1}{i\pi}\int_L \frac{[\lambda]_\chi}{P(\chi,\chi)}\begin{pmatrix}Q^\dagger(\chi) + i\epsilon_0\chi U^\dagger(\chi)\\ -i\epsilon_0\chi Q^\dagger(\chi)\end{pmatrix} L_k(\chi)d\chi$$

Solution of the integral equation

$$\boldsymbol{\varphi}\,(\xi,\eta,\tau) = -\frac{Q^{\dagger}(\tau)}{P(\tau,\tau)} \sum_{k,l=0}^{N_0} L_k(\tau)\,\Delta^{-1}_{kl} \begin{pmatrix} q_l \\ u_l \\ v_l \end{pmatrix}$$

Calculation of the solution components

$$\mathbf{R} \equiv \frac{1}{i\pi} \int\limits_L [\lambda]_\chi\, \mathbf{l}(\chi) \otimes \boldsymbol{\varphi}\,(\chi)\, d\,\chi = -\sum_{k,l=0}^{N_0} \Delta^{-1}_{kl} \mathbf{J}_k \otimes \{q_l,\, u_l,\, v_l\}$$

$$g_{33} = \epsilon_0 - i({R_3}^4 - {\overline{R}_3}^4) + \Phi_3 \overline{\Phi}_3$$

$$g_{34} = -i(\beta - \beta_0) + i({R_3}^3 + {\overline{R}_4}^4) + \Phi_3 \overline{\Phi}_4$$

$$g_{44} = \epsilon_0 \epsilon \alpha_0^2 + i({R_4}^3 - {\overline{R}_4}^3) + \Phi_4 \overline{\Phi}_4$$

$$\begin{pmatrix} \Phi_3 \\ \Phi_4 \end{pmatrix} = 2i \begin{pmatrix} {R_3}^5 \\ {R_4}^5 \end{pmatrix}$$

Determinant form of solution

$$G_{ik} = \Delta_{ik} + 2i\epsilon_0 u_i (\mathbf{J}_k \cdot \mathbf{e}_1)$$

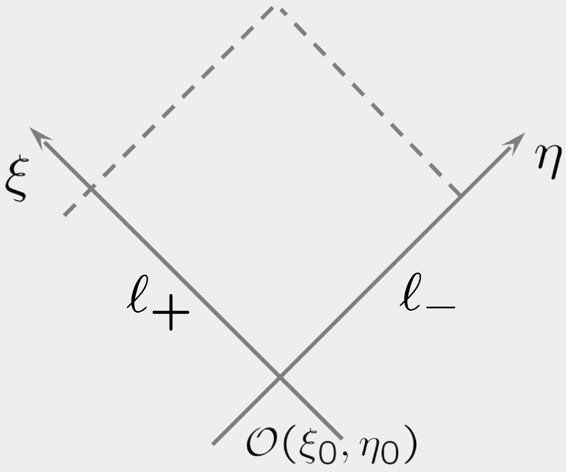
$$F_{ik} = \Delta_{ik} - 2i v_i (\mathbf{J}_k \cdot \mathbf{e}_1)$$

$$\mathcal{E} = \epsilon_0 \frac{\det \|G_{ik}\|}{\det \|\Delta_{ik}\|}, \quad \Phi = \frac{\det \|F_{ik}\|}{\det \|\Delta_{ik}\|}$$

$$m, n = 0, 1, \ldots, N_0 \qquad N_0 = \max\{N_u, N_v, N_q\}$$

Principle algorithm for solution of boundary, initial and characteristic initial value problems

Characteristic initial value problem for the hyperbolic Ernst equations[1)]



Analytical data:

$$\ell_+ : \begin{cases} \mathcal{E} = \mathcal{E}_+(\xi) \\ \Phi = \Phi_+(\xi) \end{cases} \qquad \ell_- : \begin{cases} \mathcal{E} = \mathcal{E}_-(\eta) \\ \Phi = \Phi_-(\eta) \end{cases}$$

$$\mathcal{O} : \quad \mathcal{E}(\xi_0,\eta_0) = -1, \;\; \Phi(\xi_0,\eta_0) = 0$$

$$\ell_+ : \begin{cases} \dfrac{d}{d\xi}\mathbf{\Psi}_+ = \dfrac{\mathbf{U}(\xi,\eta_0)}{2i(w-\xi)}\mathbf{\Psi}_+ \\ \mathbf{\Psi}_+(\xi_0,w) = \mathbf{I} \end{cases} \qquad \begin{cases} \dfrac{d}{d\eta}\mathbf{\Psi}_- = \dfrac{\mathbf{V}(\xi_0,\eta)}{2i(w-\eta)}\mathbf{\Psi}_- \\ \mathbf{\Psi}_-(\eta_0,w) = \mathbf{I} \end{cases}$$

$$\mathbf{\Psi}_+(\xi,w) \Rightarrow \mathbf{u}_+(w) \qquad \mathbf{\Psi}_-(\eta,w) \Rightarrow \mathbf{u}_-(w)$$

$$\Rightarrow \boxed{\frac{1}{\pi i}\int_L \frac{\mathcal{K}(\xi,\eta,\tau,\zeta)}{\zeta-\tau}\cdot \boldsymbol{\varphi}\; d\zeta = \mathbf{k}(\tau)} \Rightarrow \quad \mathbf{R} = \frac{1}{i\pi}\int_L [\lambda]_\zeta \mathbf{l}(\zeta)\otimes \boldsymbol{\varphi}\,(\xi,\eta,\zeta)d\zeta$$

$$\mathbf{U} = \partial_\xi \mathbf{R}, \;\; \mathbf{V} = \partial_\eta \mathbf{R}$$

Solution generating transformations and integral equation methods in the context of the monodromy transform

Soliton generating transformations in terms of the monodromy data

$\overset{o}{\mathbf{u}}_\pm(w),\ \overset{o}{\mathbf{v}}_\pm(w)$ -- the monodromy data of arbitrary seed solution.

$\mathbf{u}_\pm(w),\ \mathbf{v}_\pm(w)$ -- the monodromy data of N-soliton solution.

Belinskii-Zakharov vacuum N-soliton solution:

$$\mathbf{u}_\pm(w) = \frac{\mathcal{U}_0(w) + \mathcal{U}_1(w)\,\overset{o}{\mathbf{u}}_\pm(w)}{\mathcal{Q}_0(w) + \mathcal{Q}_1(w)\,\overset{o}{\mathbf{u}}_\pm(w)}$$

Electrovacuum N-soliton solution:

$$\mathbf{u}_\pm(w) = \frac{\mathcal{U}_0(w) + \mathcal{U}_1(w)\,\overset{o}{\mathbf{u}}_\pm(w) + \mathcal{U}_2(w)\,\overset{o}{\mathbf{v}}_\pm(w)}{\mathcal{Q}_0(w) + \mathcal{Q}_1(w)\,\overset{o}{\mathbf{u}}_\pm(w) + \mathcal{Q}_2(w)\,\overset{o}{\mathbf{v}}_\pm(w)},$$

$$\mathbf{v}_\pm(w) = \frac{\mathcal{V}_0(w) + \mathcal{V}_1(w)\,\overset{o}{\mathbf{u}}_\pm(w) + \mathcal{V}_2(w)\,\overset{o}{\mathbf{v}}_\pm(w)}{\mathcal{Q}_0(w) + \mathcal{Q}_1(w)\,\overset{o}{\mathbf{u}}_\pm(w) + \mathcal{Q}_2(w)\,\overset{o}{\mathbf{v}}_\pm(w)}$$

$$\mathcal{U}_1(w), \mathcal{U}_2(w), \mathcal{U}_3(w), \mathcal{V}_1(w), \mathcal{V}_2(w), \mathcal{V}_3(w), \mathcal{Q}_1(w), \mathcal{Q}_2(w), \mathcal{Q}_3(w)$$

-- polynomials in w of the orders $\leq N$ (the number of solitons)

Sibgatullin's integral equations in the monodromy transform context

The Sibgatullin's reduction of the Hauser & Ernst matrix integral equations (vacuum case, for simplicity):

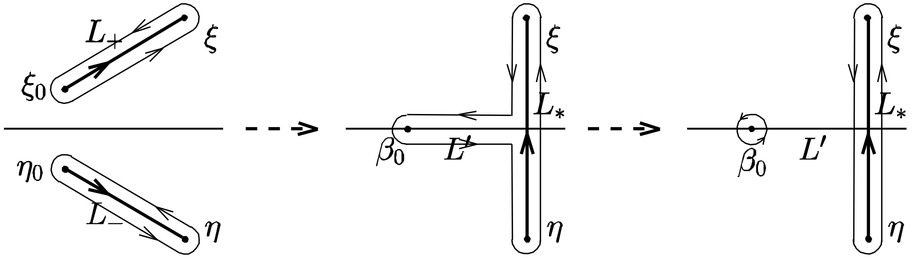
$$\fint_{-1}^{1} \frac{\mu(\sigma)(e(\zeta)+\tilde{e}(\zeta))}{(\sigma-\tau)\sqrt{1-\sigma^2}} d\sigma = 0, \qquad \frac{1}{\pi}\int_{-1}^{1} \frac{\mu(\sigma) d\sigma}{\sqrt{1-\sigma^2}} = 1, \quad \zeta = z + i\sigma\rho$$

To derive the Sibgatullin's equations from the monodromy transform,

(1) restrict the monodromy data by the regularity axis condition:

$$\mathbf{u}_+(w) = \mathbf{u}_-(w) \equiv \mathbf{u}(w)$$

(2) chose the first component of the monodromy transform equations for $\psi(\xi,\eta,w)$ In this case, the contour can be transformed as shown below:

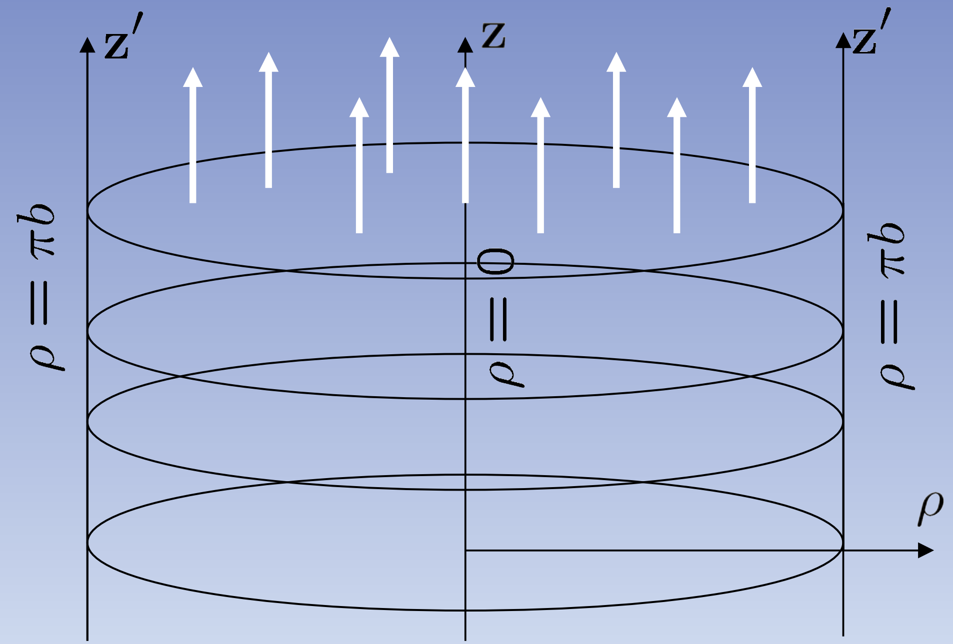


Then we obtain just the above equation on the reduced contour and the pole at $w=\beta_0$ gives rise to the above normalization condition.

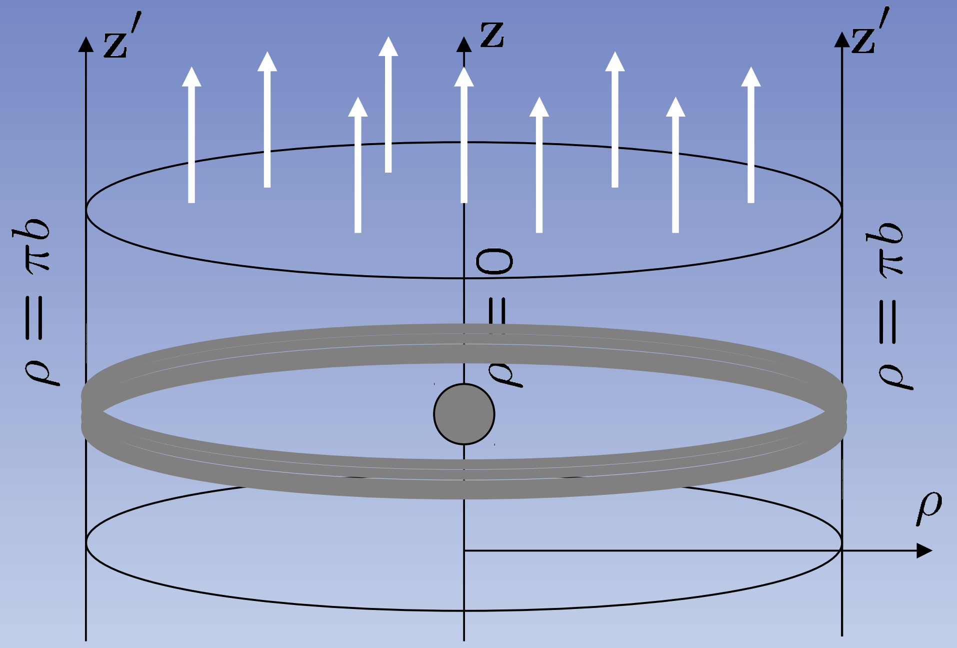
Aspects of black hole dynamics in the external gravitational and electromagnetic fields

- **Schwarzschild black hole in a homogeneous magnetic field**
  GA&A.Garcia, PRD (1996)
- **Schwarzschild black hole hovering in the field of a charged naked singularity**
  GA&V. Belinski, Nuov.Cim. (2007)
- **Equilibrium configurations of two charged massive sources of the Reissner – Nordstrom type**
  GA&V. Belinski, PRD (2007)
- **"Geodesic" motion of a Schwarzschild black hole in the external gravitational field (Bertotti – Robinson universe)**
- **Charged black hole accelerated by homogeneous electric field**

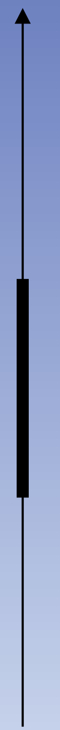
Space-time with a homogeneous magnetic / electric field (Bertotti - Robinson universe)



Schwarzschild black hole in a homogeneous magnetic field



In Weyl coordinates :



Bipolar coordinates:

Weyl coordinates:

Metric components and electromagnetic potential

$$g_{tt} = \frac{(x_1^2+1)(x_2+y_1)^2}{x_2^2-1}, \qquad \Phi = x_1 + \frac{m}{b} y_2$$

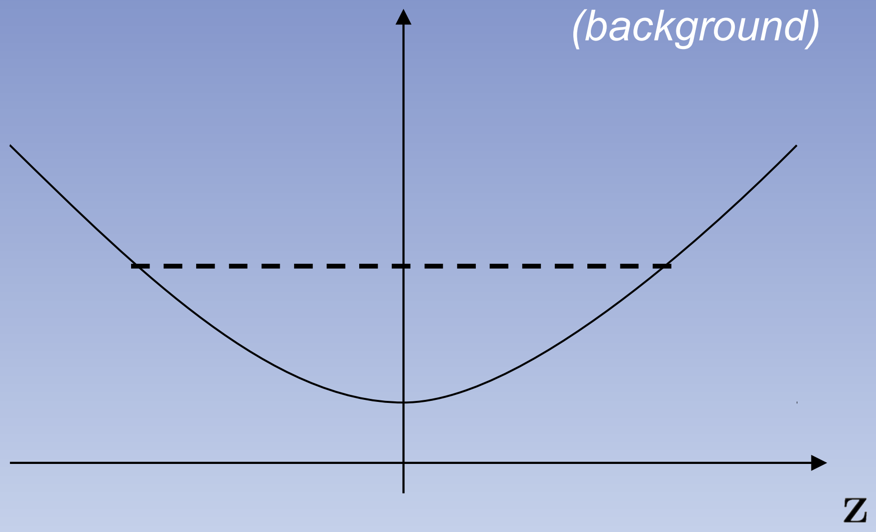
$$g_{\rho\rho} = g_{zz} = \frac{f_0 (x_2+y_1)^2}{(x_1^2+y_1^2)(x_2^2-y_2^2)} \left[\frac{m(x_2-y_2)+bx_1-(\ell+m)y_1}{m(x_2+y_2)+bx_1-(\ell-m)y_1} \right]^2$$

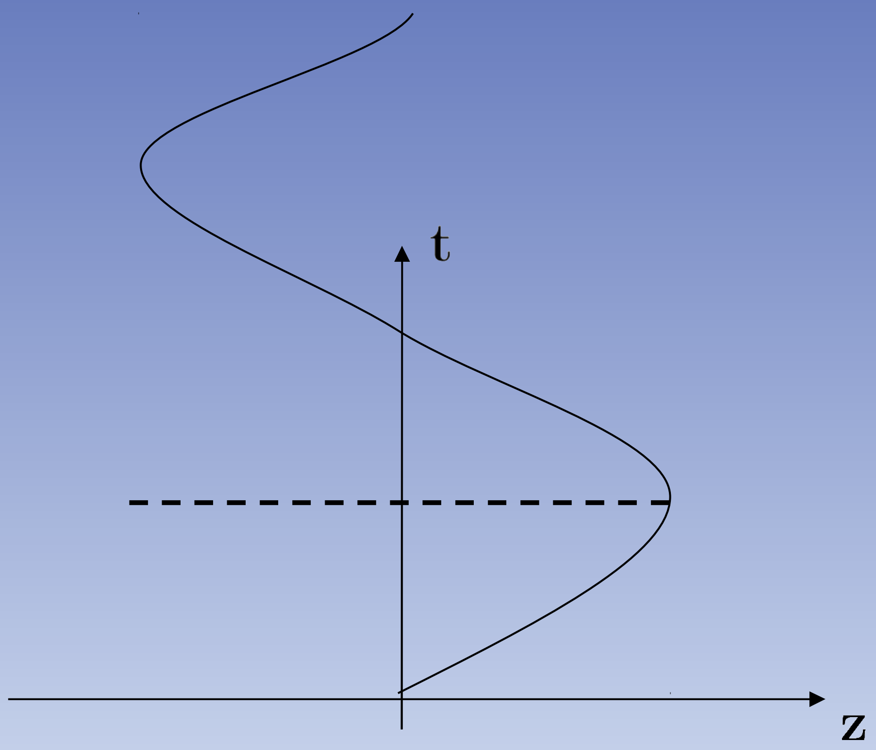
1) *GA & A.Garcia, PRD 1996*

"Geodesic" motion of a Schwarzschild black hole in the external gravitational field (Bertotti – Robinson universe)[1)]

"Geodesic" motion of a black hole

(background)





Schwarzschild black hole hovering in the field of a naked singularity[1)]

1) *GA and V.Belinski Nuovo Cimento (2007)*

Equilibrium configurations of two charged massive sources of the Reissner – Nordstrom type [1)]

1) *GA and V.Belinski Phys.Rev. D (2007)*

5-parametric solution for the superposed field of two arbitrary Reissner-Norstrom sources[1)]

(solution with two simple poles in the monodromy data)

$$H=\frac{\mathcal{D}^2-\mathcal{G}^2+\mathcal{F}^2}{(\mathcal{D}+\mathcal{G})^2},\qquad \Phi=\frac{\mathcal{F}}{\mathcal{D}+\mathcal{G}},\qquad f=\frac{(\mathcal{D}+\mathcal{G})^2}{16S_0^2(x_1^2-\sigma_1^2y_1^2)(x_2^2-\sigma_2^2y_2^2)}$$

$$\begin{aligned}\mathcal{D}&=A_0(x_1^2+x_2^2-\sigma_1^2y_1^2-\sigma_2^2y_2^2)\\&\quad+2B_0x_1x_2+2C_0y_1y_2\\ \mathcal{G}&=g_1x_1+g_2x_2+\widetilde{g}_1y_1+\widetilde{g}_2y_2\\ \mathcal{F}&=f_1x_1+f_2x_2+\widetilde{f}_1y_1+\widetilde{f}_2y_2\end{aligned}$$

$$\begin{aligned}m_+&=M_1+M_2\\ m_-&=M_1-M_2\\ e_+&=Q_1+Q_2\\ e_-&=Q_1-Q_2\end{aligned}$$

$$\sigma_1^2=M_1^2-Q_1^2-\frac{2Q_1(M_1Q_2-M_2Q_1)}{\ell+M_1+M_2}$$

$$\sigma_2^2=M_2^2-Q_2^2+\frac{2Q_2(M_1Q_2-M_2Q_1)}{\ell+M_1+M_2}$$

$$\begin{aligned}A_0&=S_0-(\ell+m_+)^2(\ell^2-m_+^2+e_+^2)\\ B_0&=S_0+(\ell+m_+)^2(\ell^2-m_+^2+e_+^2)\\ C_0&=\frac{1}{2}(\ell^2-\sigma_1^2-\sigma_2^2)B_0-(\ell^2-m_+^2+e_+^2)S_0\end{aligned}$$

$$\begin{aligned}g_1&=2m_+S_0-2m_-(\ell+m_+)^2(\ell^2-m_+^2+e_+^2)\\ g_2&=2m_+S_0+2m_-(\ell+m_+)^2(\ell^2-m_+^2+e_+^2)\\ \widetilde{g}_1&=(\ell^2+\sigma_1^2-\sigma_2^2)H_1-4\sigma_1^2\ell(\ell+m_+)^2(m_+-m_-)\\ \widetilde{g}_2&=(\ell^2-\sigma_1^2+\sigma_2^2)H_2+4\sigma_2^2\ell(\ell+m_+)^2(m_++m_-)\end{aligned}$$

$$\begin{aligned}S_0&=(\ell+m_+)^2(\ell^2-m_-^2)+(\ell e_-+m_-e_+)^2\\ H_1&=(m_-e_+-m_+e_-)\Big[(e_++e_-)\ell+(m_++m_-)e_+\Big]\\ H_2&=(m_-e_+-m_+e_-)\Big[(e_+-e_-)\ell+(m_+-m_-)e_+\Big]\\ S_1&=-(\ell e_-+m_-e_+)^2+(\ell+m_+)(\ell e_+^2+m_+e_-^2)\\&\quad-(m_-e_+-m_+e_-)^2+\ell e_+(m_++m_-)(e_+-e_-)\\ S_2&=(\ell e_-+m_-e_+)^2-(\ell+m_+)(\ell e_+^2+m_+e_-^2)\\&\quad+(m_-e_+-m_+e_-)^2-\ell e_+(m_+-m_-)(e_++e_-)\end{aligned}$$

$$\begin{aligned}f_1&=2e_+S_0-2(\ell+m_+)(\ell^2-m_+^2+e_+^2)(\ell e_-+m_-e_+)\\ f_2&=2e_+S_0+2(\ell+m_+)(\ell^2-m_+^2+e_+^2)(\ell e_-+m_-e_+)\\ \widetilde{f}_1&=-(m_++m_-)\Big[(\ell+m_+)^2-e_+^2\Big]\Big[m_+(\ell e_++m_-e_-)\\&\quad-m_-(\ell e_-+m_-e_+)\Big]+\ell(e_++e_-)S_1\\ \widetilde{f}_2&=(m_+-m_-)\Big[(\ell+m_+)^2-e_+^2\Big]\Big[m_+(\ell e_++m_-e_-)\\&\quad-m_-(\ell e_-+m_-e_+)\Big]+\ell(e_+-e_-)S_2\end{aligned}$$

1) ***Not available for analysis – see the next page***

Equilibrium configurations of two Reissner - Nordstrom sources[1)]

$$g_{tt} = \frac{[(r_1 - m_1)^2 - \sigma_1^2 + \gamma^2 \sin^2\theta_2][(r_2 - m_2)^2 - \sigma_2^2 + \gamma^2 \sin^2\theta_1]}{[r_1 r_2 - (q_1 - \gamma\cos\theta_2)(q_2 - \gamma\cos\theta_1)]^2},$$

$$A_t = \frac{q_1(r_2 - m_2) + q_2(r_1 - m_1) + \gamma(m_1\cos\theta_1 + m_2\cos\theta_2)}{r_1 r_2 - (q_1 - \gamma\cos\theta_2)(q_2 - \gamma\cos\theta_1)},$$

$$f = \frac{[r_1 r_2 - (q_1 - \gamma\cos\theta_2)(q_2 - \gamma\cos\theta_1)]^2}{(r_1 - m_1)^2 - \sigma_1^2\cos^2\theta_1][(r_2 - m_2)^2 - \sigma_2^2\cos^2\theta_2]}$$

In equilibrium

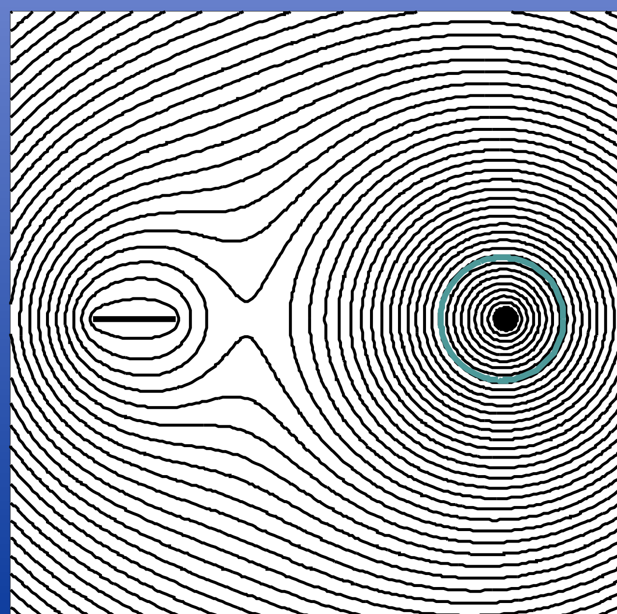
$$m_1 m_2 = q_1 q_2$$

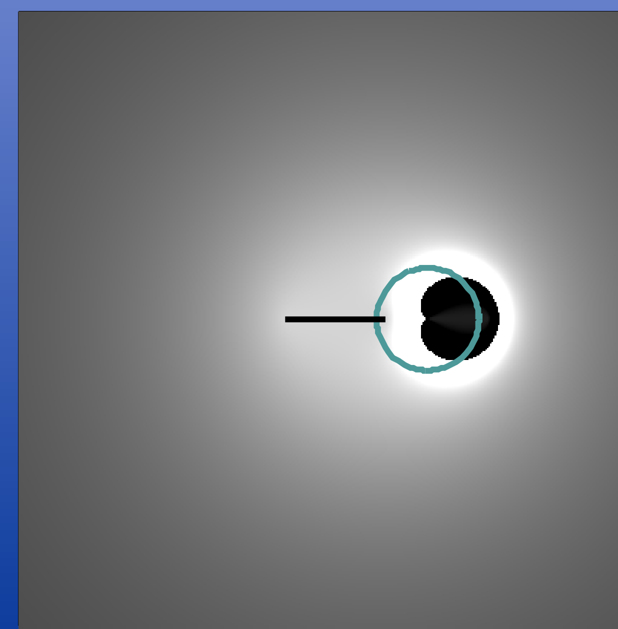
$$\ell = \frac{m_2 q_1 - m_1 q_2}{\gamma}$$

$$q_1 = e_1 - \gamma$$

$$q_2 = e_2 + \gamma$$

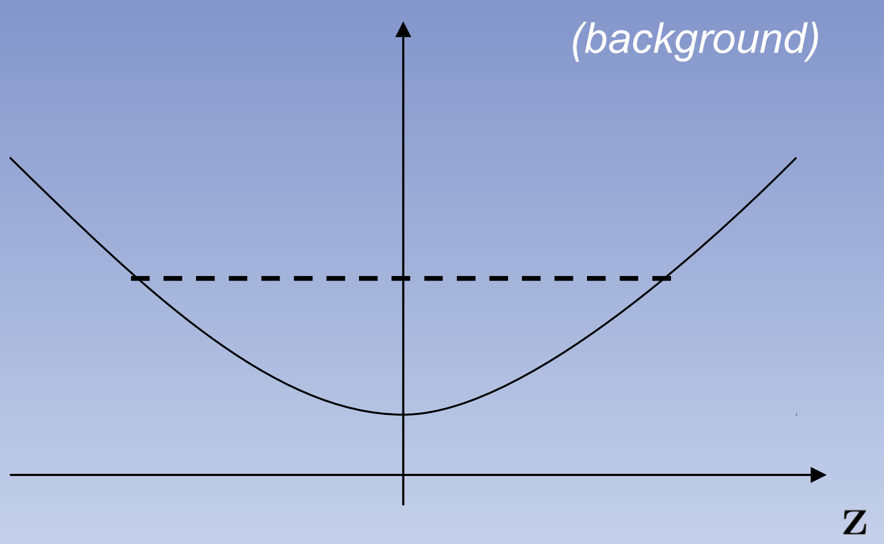
$$\{m_1, m_2, q_1, q_2, \gamma\}$$





1) *GA and V.Belinski Phys.Rev. D (2007)*

Charged black hole accelerated by homogeneous electric field



Reissner - Nordstrom black hole in a homogeneous electric field

Formal solution for metric and electromagnetic potential:

$$H = \frac{(x_1^2 + b^2)(x_2^2 - \sigma^2)\mathbf{A}(x_1, x_2)}{(b^2 x_2^2 + \sigma^2 x_1^2)^2 \mathbf{p}^2(x_1, x_2)\mathbf{q}^2(x_1, x_2)}$$

$$\Phi = \frac{\mathbf{B}(x_1, x_2)}{(b^2 x_2^2 + \sigma^2 x_1^2)\mathbf{p}(x_1, x_2)\mathbf{q}(x_1, x_2)}$$

Auxiliary polynomials:

$$\mathbf{p}(x_1, x_2) = e^2(x_1^2 + b^2) - [(\ell + m)^2 + b^2](x_2^2 - \sigma^2)$$

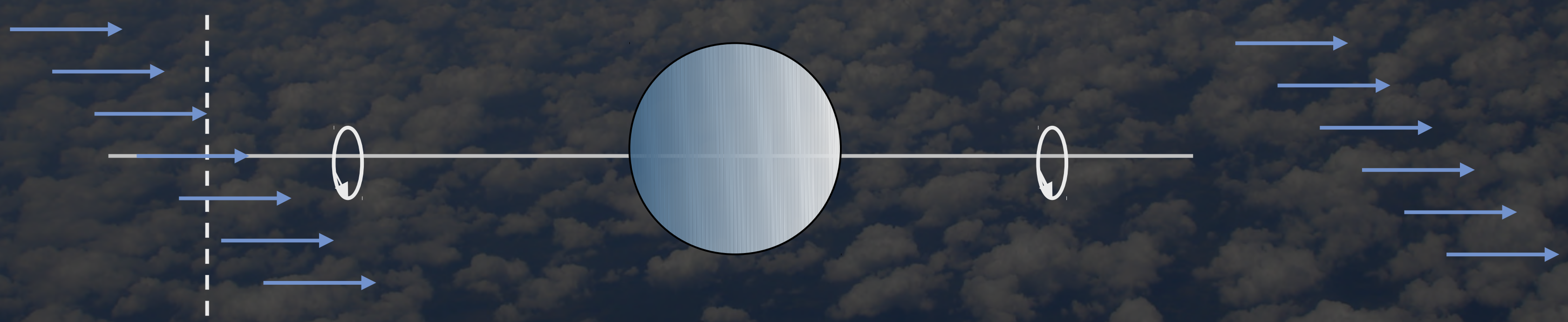
$$\begin{aligned}\mathbf{q}(x_1, x_2) = [\ell^2 - (x_1 - x_2)^2[m\sqrt{(\ell + m)^2 + b^2} + e(\ell + m)]^2 + \\ + \left[x_1^2\sigma^2 - b^2 x_2^2 + (\ell^2 + b^2 - \sigma^2)x_1 x_2\right] \times \\ \times \left[\sqrt{(\ell + m)^2 + b^2} + e\right]^2 - x_1 x_2[(\ell^2 + b^2 - \sigma^2)^2 + 4b^2\sigma^2],\end{aligned}$$

$$\sigma^2 = m^2 - e^2$$

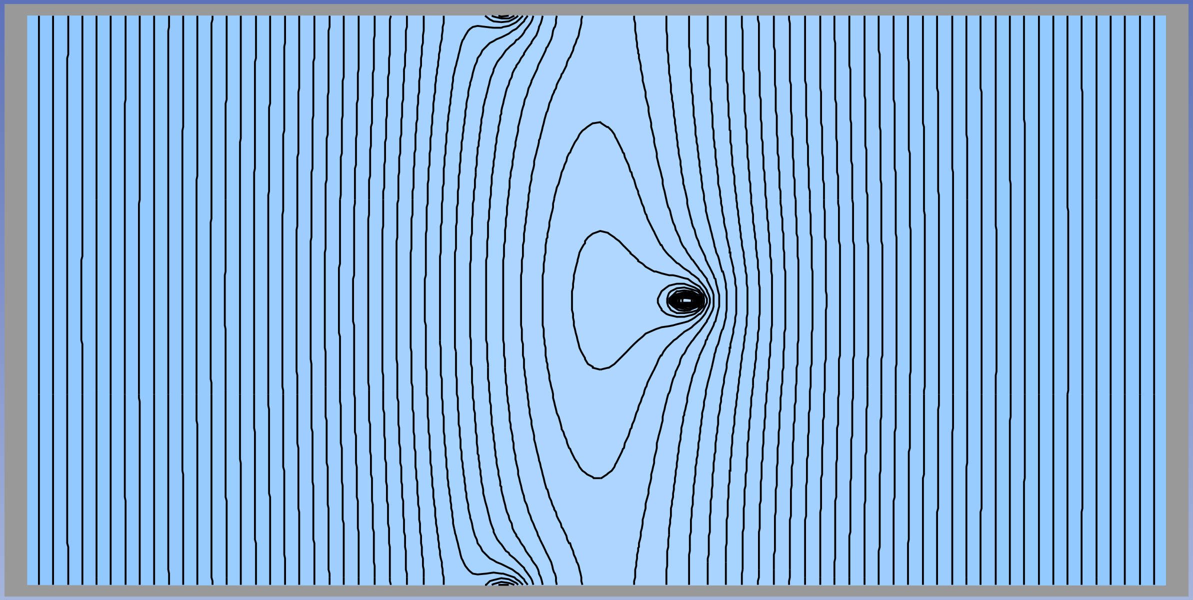
Black hole in external electromagnetic field

Balance of forces
In Newtonian mechanics

Regularity of space-time geometry
in General Relativity



Charged black hole accelerated by the external homogeneous electric field




Thank you
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