Hyperbolicity of constrained Hamiltonian systems

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Outline

Motivation

Model Hamiltonian

Application to GR

Conclusions

Why study formulations of GR?

Practical side: ‘good’ PDE systems for numerical relativity.

- Gravitational waves from compact objects.
- Critical phenomena.
- Examples: ADM, GHG, BSSNOK, Z4, FCF, EC, NOR, KST...

Principle questions: physical and mathematical properties.

- Does GR make sense dynamically? Gauge?
- Well-posedness of I(B)VP?
- Long-term existence?

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The Harmonic formulation

The vacuum field equations

$$R_{ab} = -\frac{1}{2}g^{cd}\partial_c\partial_d g_{ab} + \partial_{(a}\Gamma_{b)} - g^{cd}\Gamma_{cab}\Gamma_d + g^{cd}g^{ef}(\partial_e g_{ca}\partial_f g_{bd} - \Gamma_{ace}\Gamma_{bdf}) = 0,$$

with $\Gamma_a = g^{bc}\Gamma_{abc}$. Kill $\partial_{(a}\Gamma_{b)}$ term?

Choquet-Bruhat '53, Friedrich & Rendall '00, Garfinkle '02

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- Harmonic coordinates $\Box x^a = 0$. Constraints $C_a = \Gamma_a = 0$.

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- Harmonic coordinates $\Box x^a = 0$. Constraints $C_a = \Gamma_a = 0$.
- *Free-evolution point of view:* Constraint subsystem closed, work in expanded space. Other coordinates?

Choquet-Bruhat '53, Friedrich & Rendall '00, Garfinkle '02

Cute quotes

- Friedrich and Rendall, 2000: *"Ideally, one would like to exhibit a kind of 'hyperbolic skeleton' of the Einstein equations and a complete characterization of the freedom to fix the gauge from which all hyperbolic reductions should be derivable."*

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- Friedrich, 1996: *"To give a survey of hyperbolic reductions that is general enough to offer the optimal reduction in any given task appears a hopeless task."*

Gauge in General Relativity

We aim for a *complete* understanding of the local properties of different gauge choices in GR. Questions:

- Can the properties of gauge choices be characterized independently of the Einstein equations?
- If so can every hyperbolic gauge be coupled to the Einstein equations to form a hyperbolic PDE system?
- From the (free-evolution) PDEs point of view, how does GR compare to other physical theories with gauge freedom?

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A model constrained Hamiltonian

Consider constant coefficient Hamiltonian,

$$H = \frac{1}{2} \begin{pmatrix} \partial_i q \\ p \end{pmatrix}^\dagger \begin{pmatrix} V^{ij} & F^{\dagger\, i} \\ F^j & M^{-1} \end{pmatrix} \begin{pmatrix} \partial_j q \\ p \end{pmatrix} + g_q^\dagger C_{\mathcal{H}} V^{ij} \partial_i \partial_j q + g_p^\dagger C_{\mathcal{M}}{}^i M^{-1} \partial_i p \,,$$

- Positions and momenta (q, p).
- $F^i = \beta^i I$ for some shift vector β^i.

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Evolution equations

Variations in (q, p) and Hamilton's equations give,

$$\partial_t q = M^{-1} p + F^i \partial_i q - M^{-1} C_{\mathcal{M}}{}^{\dagger\, i} \partial_i g_p \,,$$
$$\partial_t p = V^{ij} \partial_i \partial_j q + F^i \partial_i p - V^{ij} C_{\mathcal{H}}{}^{\dagger} \partial_i \partial_j g_q \,.$$

- Gauge fields (g_q, g_p) not determined; their momenta are absent.

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The constraints

Variation with respect to gauge fields (g_q, g_p) reveals constraints

$$\mathcal{H} = C_{\mathcal{H}} V^{ij} \partial_i \partial_j q = 0 \,, \qquad \mathcal{M} = C_{\mathcal{M}}{}^{i} M^{-1} \partial_i p = 0 \,.$$

We insist that the constraint subsystem is closed, which gives,

$$C_{\mathcal{H}} V^{ij} = (A_{\mathcal{H}\mathcal{M}})^{(i} C_{\mathcal{M}}{}^{j)} \,, \qquad C^{(i}_{\mathcal{M}} M^{-1} V^{jk)} = (A_{\mathcal{M}\mathcal{H}})^{(i} C_{\mathcal{H}} V^{jk)} \,,$$
$$C^{(i}_{\mathcal{M}} M^{-1} V^{jk)} C^{\dagger}_{\mathcal{H}} = 0 \,.$$

- Dirac: "The constraints are *first class*."

Gauge invariance I

The equations of motion are invariant under transformation

$$q \to \bar{q} = q - M^{-1} C_{\mathcal{M}}{}^{\dagger\, i} \partial_i \psi \,, \qquad p \to \bar{p} = p - V^{ij} C_{\mathcal{H}}{}^{\dagger} \partial_i \partial_j \theta \,,$$
$$g_q \to g_q + \bar{g}_q \,, \qquad g_p \to g_p + \bar{g}_p \,,$$

with θ and ψ satisfying

$$\partial_t \theta = \bar{g}_q + (A_{\mathcal{M}\mathcal{H}})^{\dagger\, i} \partial_i \psi + \beta^i \partial_i \theta \,,$$
$$\partial_t \psi = \bar{g}_p + (A_{\mathcal{H}\mathcal{M}})^{\dagger\, i} \partial_i \theta + \beta^i \partial_i \psi \,.$$

- Dirac: "constraints generate gauge transformation".

Gauge invariance II

Additionally require that the *field strength* $V^{ij}\partial_i\partial_j q$ is gauge invariant.

- Final conditions,

$$(A_{\mathcal{HM}})^{(i}C_{\mathcal{M}}{}^{j)} = C_{\mathcal{H}}V^{ij}\,, \qquad V^{(ij}M^{-1}C_{\mathcal{M}}{}^{\dagger\, k)} = 0\,,$$

- Can define electric and magnetic parts. Distinction if no $\mathcal{H}$'s.

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Gauge choice

Still missing equation of motion for the gauge fields. Choose,

$$\partial_t g_q = (A_{g_q g_q})^i \partial_i g_q + (A_{g_q g_p})^i \partial_i g_p + (A_{g_q p}) p \,,$$
$$\partial_t g_p = (A_{g_p g_q})^i \partial_i g_q + (A_{g_p g_p})^i \partial_i g_p + (A_{g_p q})^i \partial_i q \,.$$

- Assume $(A_{g_q p}) = A C_{\mathcal{H}}$ and $(A_{g_p q})^i = B {C_{\mathcal{M}}}^i + C^i C_{\mathcal{H}} M$ for some matrices A, B and C^i.
- This restriction can be dropped.
- Structure general enough for hyperbolic gauges.

Pure gauge subsystem

Evolve, but change ID by gauge transformation

$$q \to \bar{q} = q - M^{-1} C_{\mathcal{M}}{}^{\dagger\, i} \partial_i \bar{\psi}\,, \qquad p \to \bar{p} = p - V^{ij} C_{\mathcal{H}}{}^{\dagger} \partial_i \partial_j \bar{\theta}\,.$$

What happens?

- Difference evolves according to

$$\partial_t \bar{\theta} = \bar{g}_q + \beta^i \partial_i \bar{\theta}\,, \qquad \partial_t \bar{\psi} = \bar{g}_p + (A_{\mathcal{H}\mathcal{M}})^{\dagger\, i} \partial_i \bar{\theta} + \beta^i \partial_i \bar{\psi}\,,$$
$$\partial_t \bar{g}_q = (A_{g_q g_q})^i \partial_i \bar{g}_q + (A_{g_q g_p})^i \partial_i \bar{g}_p - (A_{g_q p}) V^{ij} C_{\mathcal{H}}{}^{\dagger} \partial_i \partial_j \bar{\theta}\,,$$
$$\partial_t \bar{g}_p = (A_{g_p g_q})^i \partial_i \bar{g}_q + (A_{g_p g_p})^i \partial_i \bar{g}_p - (A_{g_p q})^{(i} M^{-1} C_{\mathcal{M}}{}^{\dagger\, j)} \partial_i \partial_j \bar{\psi}\,.$$

- We call this the *pure gauge subsystem*. It is closed.

Free evolution on the expanded phase space I

Define constraints (Θ, Z). Choose:

$$\partial_t\Theta = \beta^i\partial_i\Theta + (A_{\Theta Z})^i\partial_i Z + (A_{\Theta\mathcal{H}})\mathcal{H}\,,$$
$$\partial_t Z = (A_{Z\Theta})^i\partial_i\Theta + \beta^i\partial_i Z + (A_{Z\mathcal{M}})\mathcal{M}\,.$$

- Remember: can modify dynamics away from the constraint satisfying hypersurface in phase space if constraint subsystem remains closed.
- If we insisted on the whole system having Poisson-bracket structure (Θ, Z) would be canonical momenta of gauge fields.

Free evolution on the expanded phase space II

Couple new constraints to gauge conditions

$$\partial_t g_q = (A_{g_q g_q})^i \partial_i g_q + (A_{g_q g_p})^i \partial_i g_p + (A_{g_q p}) p + (A_{g_q \Theta}) \Theta \,,$$
$$\partial_t g_p = (A_{g_p g_q})^i \partial_i g_q + (A_{g_p g_p})^i \partial_i g_p + (A_{g_p q})^i \partial_i q + (A_{g_p Z}) Z \,.$$

and the rest of the system

$$\partial_t q = M^{-1} p + F^i \partial_i q - M^{-1} {C_{\mathcal{M}}}^{\dagger\, i} \partial_i g_p + (A_{q\Theta}) \Theta \,,$$
$$\partial_t p = V^{ij} \partial_i \partial_j q + F^i \partial_i p - V^{ij} {C_{\mathcal{H}}}^{\dagger} \partial_i \partial_j g_q + (A_{pZ})^i \partial_i Z + (A_{p\mathcal{H}}) \mathcal{H} \,.$$

Constraint subsystem

The full constraint subsystem is closed by

$$\partial_t \mathcal{H} = (A_{\mathcal{H}\Theta})^{ij} \partial_i \partial_j \Theta + \beta^i \partial_i \mathcal{H} + (A_{\mathcal{H}\mathcal{M}})^i \partial_i \mathcal{M}\,,$$
$$\partial_t \mathcal{M} = (A_{\mathcal{M}Z})^{ij} \partial_i \partial_j Z + (A_{\mathcal{M}\mathcal{H}})^i \partial_i \mathcal{H} + \beta^i \partial_i \mathcal{M}\,,$$

with matrices

$$(A_{\mathcal{H}\Theta})^{ij} = C_{\mathcal{H}} V^{ij} (A_{q\Theta})\,, \qquad (A_{\mathcal{M}Z})^{ij} = C_{\mathcal{M}}{}^{(i} M^{-1} (A_{pZ})^{j)}\,,$$
$$(A_{\mathcal{M}\mathcal{H}})^i = C_{\mathcal{M}}{}^i M^{-1} (A_{p\mathcal{H}})\,.$$

- Two closed subsystems. PDE properties inherited by the full formulation?

Natural choice of variables I

Final assumption. For every unit spatial vector s^i

- Rows of $C_{\mathcal{H}}$ and $C_{\mathcal{M}}{}^s \equiv C_{\mathcal{M}}{}^i s_i$ are contained in the span of the union of the rows of $V = C_{\mathcal{H}} V^{ss}$ and $W = C_{\mathcal{M}}{}^s M^{-1}$.
- Rows of V and W are independent.
- Furthermore the contractions $X = V C_{\mathcal{H}}{}^\dagger$ and $Y = W C_{\mathcal{M}}{}^{\dagger s}$ are invertible.

This really is key, as it ensures that variables decompose nicely. Notice assumptions not placed on gauge choice.

Natural choice of variables II

Define s^i dependent projection operators

$$C_\theta = -X^{-1} C_{\mathcal{H}}\,,$$
$$C_\psi = -Y^{-1} C_{\mathcal{M}}{}^s + (A_{\mathcal{H}\mathcal{M}})^{\dagger s} C_\theta [M - C_{\mathcal{M}}{}^{\dagger s} Y^{-1} C_{\mathcal{M}}{}^s]\,,$$
$$\perp = I - V^\dagger [V V^\dagger]^{-1} V - W^\dagger [W W^\dagger]^{-1} W\,.$$

Decomposition into

$$[\partial_s^2 \theta] = C_\theta p + (A_{\theta\Theta})\Theta\,, \qquad [\partial_s^2 \psi] = C_\psi \partial_s q + (A_{\psi Z}) Z\,,$$
$$[\mathcal{H}] = V \partial_s q\,, \qquad [\mathcal{M}] = W p\,,$$
$$[\partial_s P_q] = \perp \partial_s q\,, \qquad [P_p] = \perp p\,,$$

is invertible. Names foreshadow conclusion.

Principal symbol

The principal symbol of the formulation in the s^i direction is,

$$P^s = \begin{pmatrix} P^s_{\mathcal{G}} & P^s_{\mathcal{GC}} & 0 \\ 0 & P^s_{\mathcal{C}} & 0 \\ 0 & 0 & P^s_{\mathcal{P}} \end{pmatrix} .$$

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Principal symbol

The upper left block,

$$P^s_{\mathcal{G}} = \begin{pmatrix} \beta^s & 0 & I & 0 \\ (A_{\mathcal{HM}})^{\dagger s} & \beta^s & 0 & I \\ -(A_{g_q p})V^\dagger & 0 & (A_{g_q g_q})^s & (A_{g_q g_p})^s \\ 0 & -(A_{g_p q})^s W^\dagger & (A_{g_p g_q})^s & (A_{g_p g_p})^s \end{pmatrix},$$

is identical to that of the pure gauge subsystem!

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Principal symbol

The central block,

$$P^s_{\mathcal{C}} = \begin{pmatrix} \beta^s & (A_{\Theta Z})^s & (A_{\Theta\mathcal{H}}) & 0 \\ (A_{Z\Theta})^s & \beta^s & 0 & (A_{Z\mathcal{M}}) \\ (A_{\mathcal{H}\Theta})^{ss} & 0 & \beta^s & (A_{\mathcal{H}\mathcal{M}})^s \\ 0 & (A_{\mathcal{M}Z})^{ss} & (A_{\mathcal{M}\mathcal{H}})^s & \beta^s \end{pmatrix},$$

Is identical to that of the constraint subsystem!

Principal symbol

The off-diagonal block,

$$P^s_{\mathcal{GC}} = \begin{pmatrix} 0 & (A_{\theta Z}) & (A_{\theta\mathcal{H}}) & 0 \\ (A_{\psi\Theta}) & 0 & 0 & (A_{\psi\mathcal{M}}) \\ (A_{\Theta}) & 0 & 0 & 0 \\ 0 & (A_Z) & 0 & 0 \end{pmatrix},$$

with sub-matrices,

$$\begin{aligned} (A_{\theta Z}) &= (A_{\theta\Theta})(A_{\Theta Z}) + C_\theta (A_{pZ})^s\,, \\ (A_{\theta\mathcal{H}}) &= (A_{\theta\Theta})(A_{\Theta\mathcal{H}}) - X^{-1} + C_\theta (A_{p\mathcal{H}})\,, \\ &\dots \end{aligned}$$

parametrizes the coupling of the gauge fields to the constraints.

Principal symbol

The lower right block,

$$P^s_{\mathcal{P}} = \begin{pmatrix} \beta^s & \perp M^{-1} \\ \perp V^{ss} & \beta^s \end{pmatrix},$$

contains neither constraint addition or gauge parameters.
Why look at the principal symbol?

- *Strong hyperbolicity:* A necessary condition for SH is that P^s has real eigenvalues and a complete set of eigenvectors for every s^i.
- SH $\iff$ Well-posed initial value problem.

Basic properties

The principal symbol

$$P^s = \begin{pmatrix} P^s_{\mathcal{G}} & P^s_{\mathcal{GC}} & 0 \\ 0 & P^s_{\mathcal{C}} & 0 \\ 0 & 0 & P^s_{\mathcal{P}} \end{pmatrix}.$$

- No formulation is SH if the physical sub-block is not.
- Necessary condition for SH of a formulation is that the pure gauge and constraint violating subsystems are SH. Follows from lemma on upper block diagonal matrices.

Too many refs. Review: Sarbach Tiglio '12. For lemma, see Gundlach M-García '06

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Gauge conditions

Choose gauge conditions,

$$\partial_t\alpha = -g_1\alpha^2 K + g_2\alpha\partial_i\beta^i + \beta^i\partial_i\alpha\,,$$
$$\partial_t\beta^i = \alpha^2[g_3\gamma^{kl}\gamma^{ij} + g_4\gamma^{il}\gamma^{jk}]\partial_l\gamma_{jk} - g_5\alpha\partial^i\alpha + \beta^j\partial_j\beta^i\,.$$

with $g_1 > 0$ and $\bar{g}_3 = 2(g_3 + g_4) > 0$.

- Harmonic gauge is $g_1 = g_3 = -2g_4 = g_5 = 1$ and $g_2 = 0$.
- Moving puncture gauge also included in this family.

Linearized pure gauge subsystem I

Upon linearizing the ADM Hamiltonian of GR takes the form that we have been considering.

- First part of pure gauge equations of motion are

$$\partial_t\theta = U - \psi_i D^i\alpha + \beta^i\partial_i\theta,$$
$$\partial_t\psi^i = V^i + \alpha D^i\theta - \theta D^i\alpha + \mathcal{L}_\beta\psi^i,$$

  where $\theta = -n_a\Delta[x^a]$, $\psi^i = -\perp^i_a\Delta[x^a]$, $U = \Delta[\alpha]$ and $V^i = \Delta[\beta^i]$.

- Effect of infinitsimal gauge change on spatial metric and extrinsic curvature gives ADM equations with $\alpha \to \theta, \beta^i \to \psi^i$.

Linearization: Moncrief '74. Pure gauge: Khoklov Novikov '02

Linearized pure gauge subsystem II

The principal part of the pure gauge system is thus

$$\partial_t\theta \simeq U + \beta^i\partial_i\theta, \quad \partial_t\psi^i \simeq V^i + \alpha\partial^i\theta + \beta^j\partial_j\psi^i,$$
$$\partial_t U \simeq g_1\alpha^2\Delta\theta + g_2\partial_i V^i + \beta^i\partial_i U,$$
$$\partial_t V^i \simeq g_3\alpha^2\Delta\psi^i + (g_3 + 2g_4)\alpha^2\partial^i\partial_j\psi^j - \alpha g_5\partial^i U + \beta^j\partial_j V^i,$$

Simply read off P^s. Eigenvalues $\pm\sqrt{g_3}\,, \pm v_\pm$ with

$$2\,v_\pm^2 = g_1 + \bar{g}_3 - g_2 g_5 \pm \sqrt{(g_1 + \bar{g}_3 - g_2 g_5)^2 - 4(g_1 - g_2)\bar{g}_3}\,.$$

Hyperbolicity of pure gauge

- Pure gauge system is SH if $g_3 > 0$ and either
  i). $0 \neq g_2 < g_1$ and $g_2 g_5 < g_1 - 2\sqrt{g_1 - g_2}\sqrt{\bar{g}_3} + \bar{g}_3$,
  ii). $g_2 = 0$ and $\bar{g}_3 \neq g_1$ or $g_2 = 0, \bar{g}_3 = g_1$ and $g_5 = 1$.
- In the paper we present a choice for the constraints that sets off-diagonal block in P^s to vanish.
- We furthermore use the remaining freedom to make the constraint subsystem SH.

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This gives a SH five parameter family generalization of the Harmonic gauge formulation.

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Summary

We have

- Studied a model constrained Hamiltonian system.
- Identified the pure gauge subsystem.
- Demonstrated that a necessary condition for SH is SH of the pure gauge and constraint subsystems.
- Constructed a SH five parameter family of gauges
- Coupled gauges GR; natural generalization of Harmonic formulation.

Future work

Outstanding issues include

- Gauge freedom and the initial boundary value problem.
- Elliptic and parabolic pure gauge conditions.
- Long-term existence in GR with different gauges.