Spacetime approach to force-free magnetospheres

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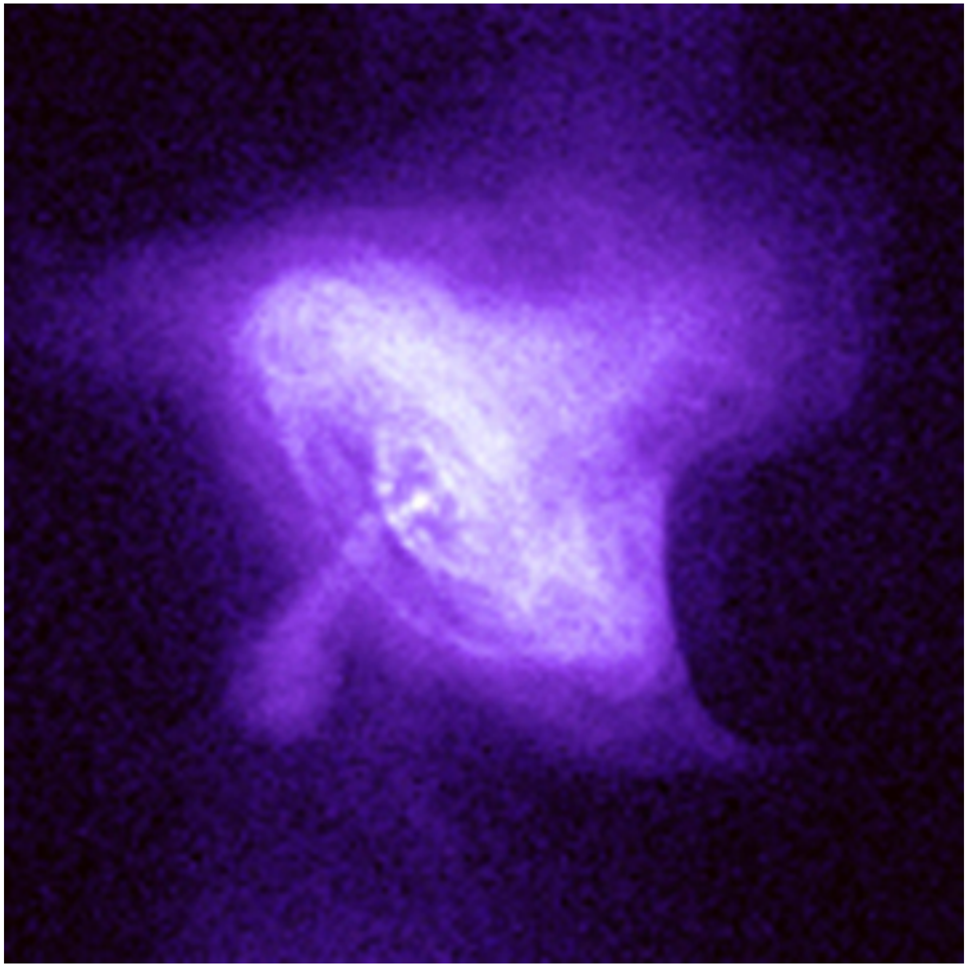
Based on arXiv:1305.6890 & (mainly) forthcoming

A peculiar field theory

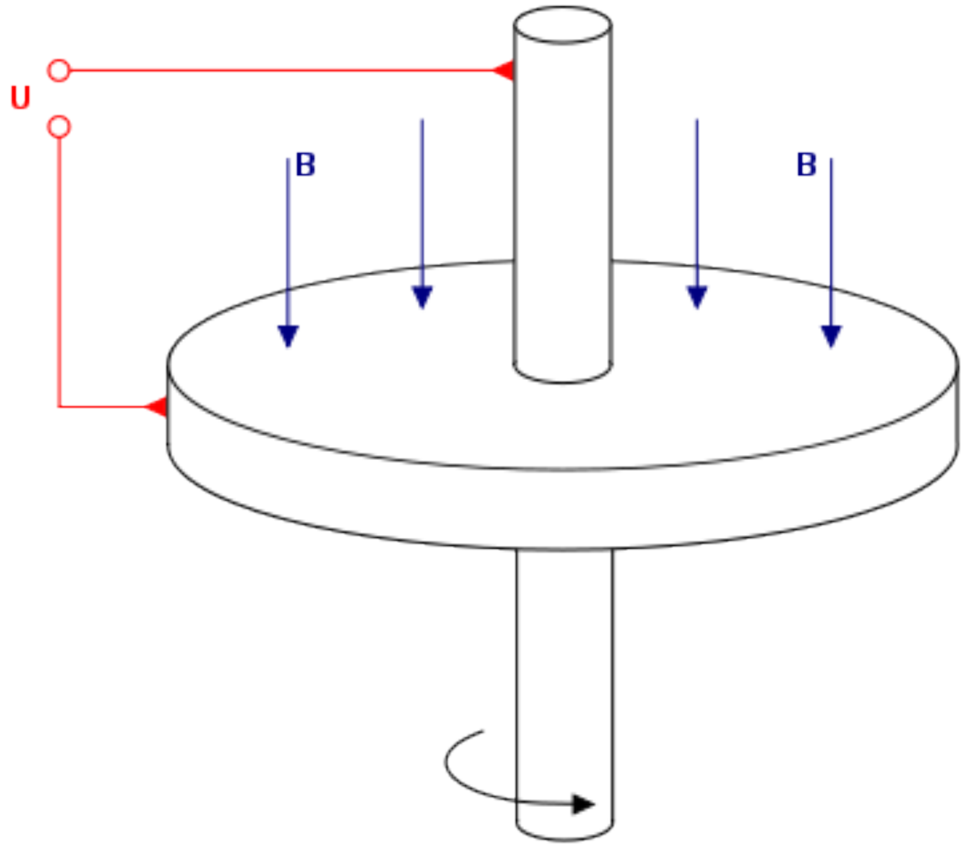
$$S[\phi_1, \phi_2] = \int d\phi_1 \wedge d\phi_2 \wedge *(d\phi_1 \wedge d\phi_2)$$

- 2nd order in time derivatives
- Lorentz and Weyl (in 4d) invariant
- global invariance under "area" preserving maps $(\phi_1, \phi_2) \rightarrow \Big(\phi'_1(\phi_1, \phi_2), \phi'_2(\phi_1, \phi_2)\Big)$
- Stationary at solutions to $d\Big(d\phi_{1,2} \wedge *(d\phi_1 \wedge d\phi_2)\Big) = 0$
- canonical momenta involve spatial derivatives of potentials

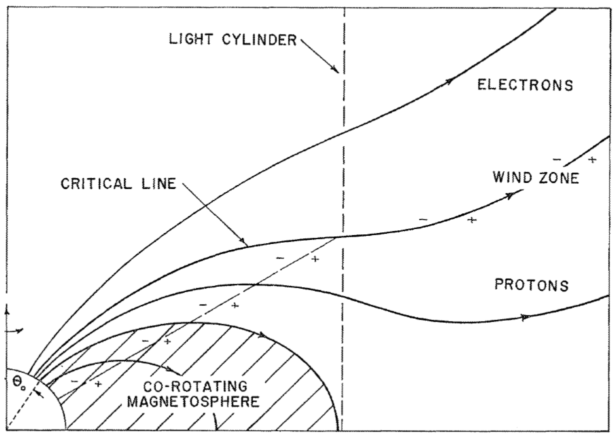


Crab pulsar (Chandra X-ray satellite image)

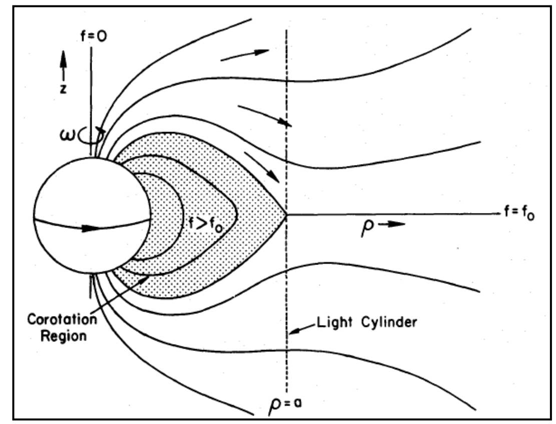
Faraday disk



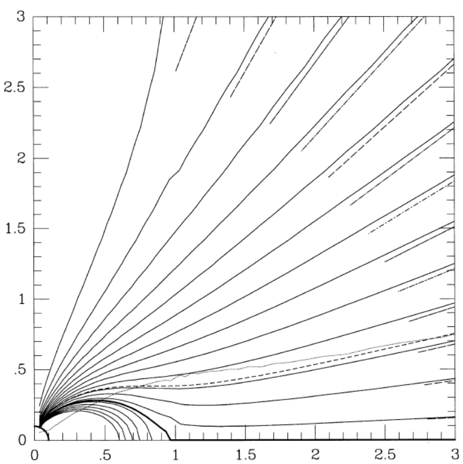
Dipole -> Monopole



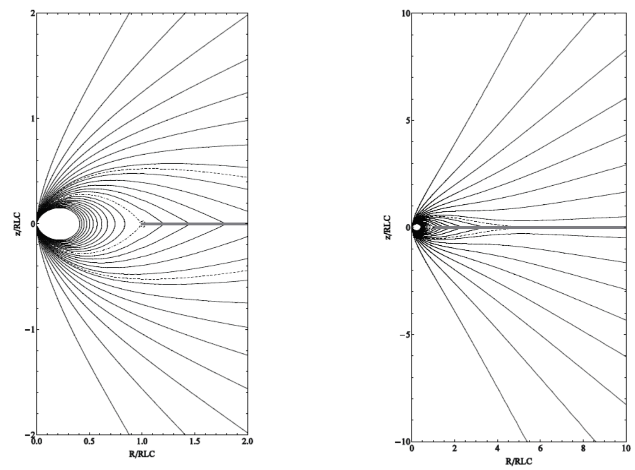
Goldreich and Julian 1969



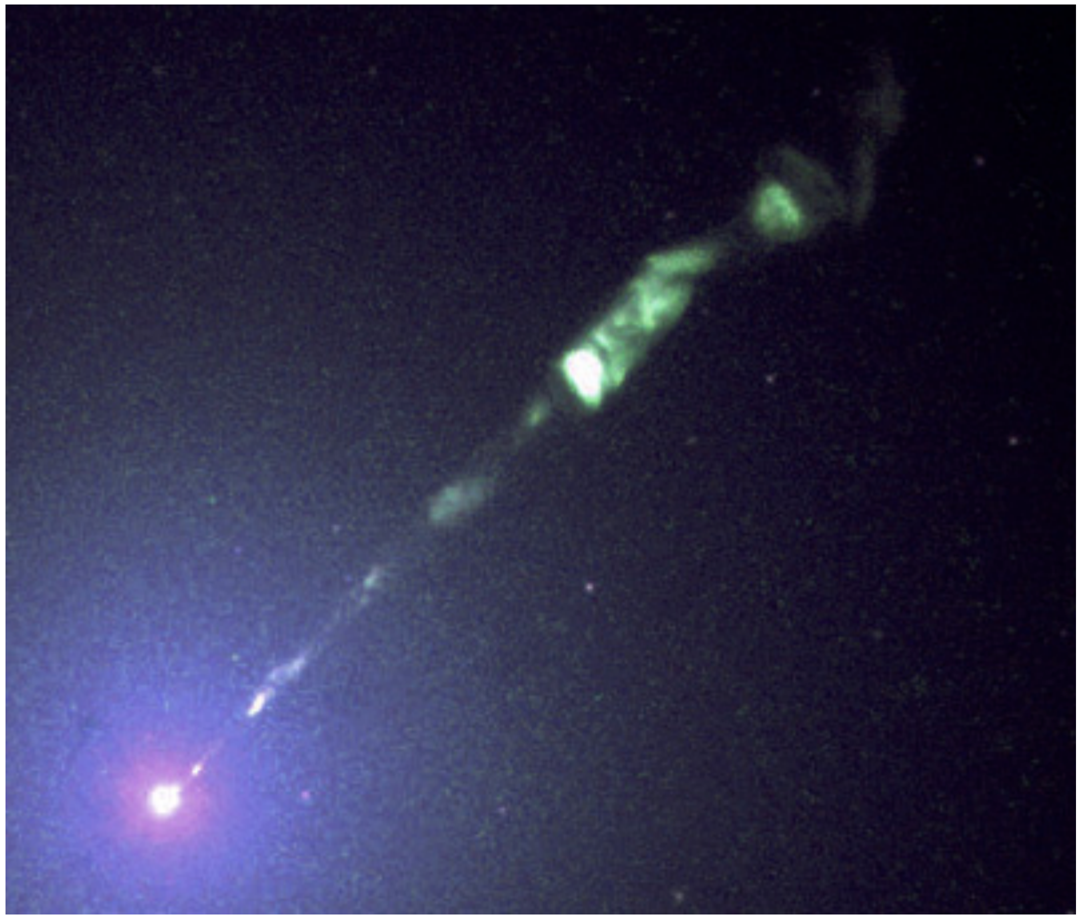
Michel 1974



Contopoulos, Kazanas, Fendt 1999



Contopoulos, Kalapotharakos, Kazanas 2013



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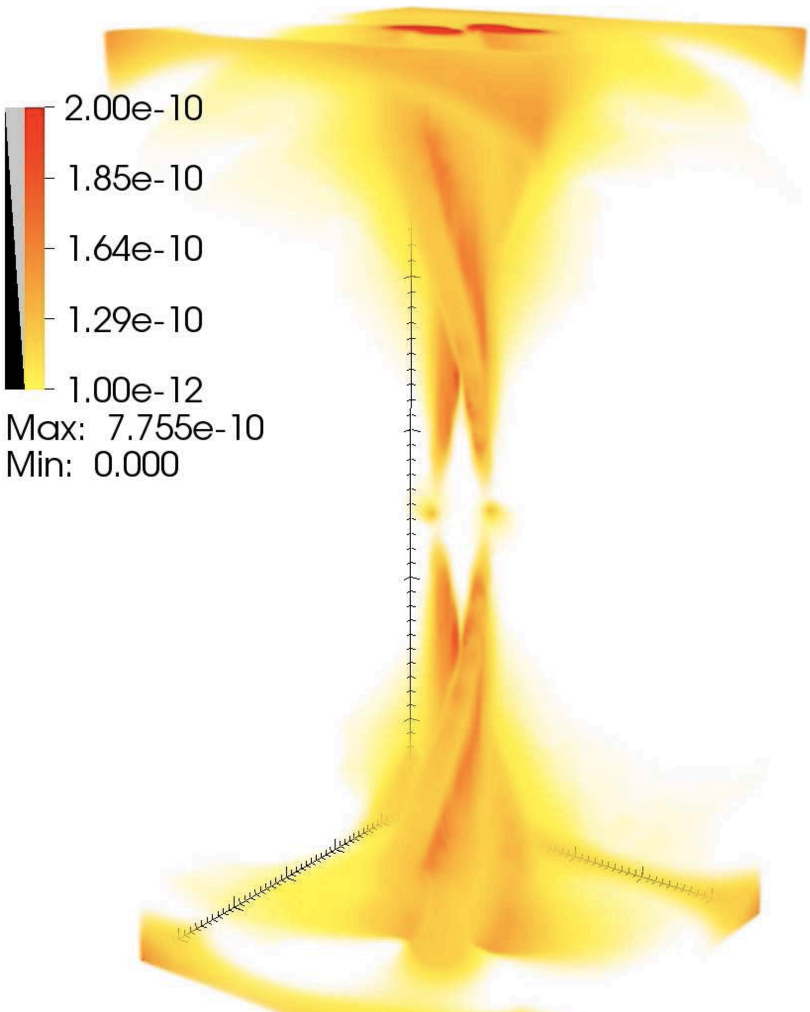
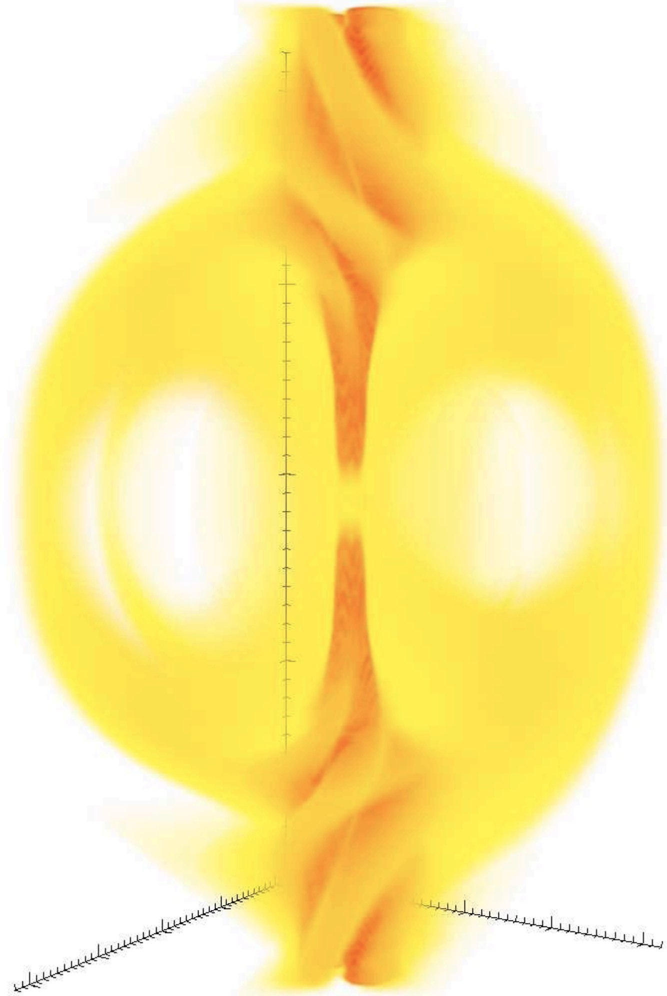
Jets from Boosted Black Holes





Magnetospheres of Black Hole Systems in Force-Free Plasma

Palenzuela, Garrett, Lehner, and Liebling, 2010

Force-free electrodynamics

If field energy >> matter energy
(because fields are very **strong** and/or matter is very **dilute**)
then energy transfer to matter is negligible, hence

$$\nabla^b T_{ab}^{\mathrm{EM}} = -F_{ab} J^b \approx 0$$

4-force vanishes,
EM energy and momentum conserved

Maxwell's equations $$\nabla_{[a} F_{bc]} = 0, \quad \nabla_b F^{ab} = J^a$$

Force-free electrodynamics $$\nabla_{[a} F_{bc]} = 0, \quad F_{ac} \nabla_b F^{ab} = 0$$

1) Non-linear
2) Deterministic??

3+1 split: determinism

$$F_{ab}J^b = 0 \;\leftrightarrow\; \vec{E}\cdot\vec{j} = 0 \quad \& \quad \rho\vec{E} + \vec{j}\times\vec{B} = 0$$

$$\Longrightarrow\; \vec{E}\cdot\vec{B} = 0$$

plus

$$\begin{aligned} \partial_t\vec{B} &= -\vec{\nabla}\times\vec{E}, \qquad \vec{\nabla}\cdot\vec{B} = 0 \\ \partial_t\vec{E} &= \vec{\nabla}\times\vec{B} - \vec{j}, \qquad \vec{\nabla}\cdot\vec{E} = \rho \end{aligned}$$

$$\Longrightarrow\; \vec{j} = \frac{1}{B^2}\left[(\vec{\nabla}\cdot\vec{E})\vec{E}\times\vec{B} + (\vec{B}\cdot\vec{\nabla}\times\vec{B} - \vec{E}\cdot\vec{\nabla}\times\vec{E})\vec{B}\right]$$

Equations hyperbolic, IVP well-posed if $B^2 > E^2$ ... [Komissarov '02 Pfeiffer & MacFayden '13]
but this condition is *not* automatically preserved. When violated, need more physics; in practice, e.g., just prune E to impose $B^2 = E^2$ and continue (numerical) evolution. This *may* model physics...

Covariant structure of force-free fields

B. Carter, T. Uchida

$$F_{ab}J^b = 0 \implies F_{[ab}F_{cd]}J^d = 0 \implies F_{[ab}F_{cd]} = 0$$

$$(\vec{E} \cdot \vec{B} = 0)$$

So F is a *simple* 2-form, aka *degenerate*:

$$F \wedge F = 0 \implies F = \alpha \wedge \beta, \quad \text{for some } \alpha \text{ and } \beta$$

and integrable:

$$dF = 0 \text{ and } F = \alpha \wedge \beta \implies F = d\phi_1 \wedge d\phi_2$$

So F(v,.) = 0 for v tangent to a foliation by 2d submanifolds, the level sets of the *Euler potentials.*

In *magnetic* case, $F_{ab}F^{ab} > 0$, these surfaces are swept by magnetic field lines along zero E congruences. They were called “flux surfaces” by Carter, and Uchida, but we call them “field sheets”.

Frozen-in theorem & field sheets

Zero resistance along direction of magnetic field implies $\vec{E} \cdot \vec{B} = 0$

For magnetically dominated degenerate fields, there is a 1-parameter family of frames U in which the electric field vanishes:

$$F_{ab}U^b = 0$$

In “ideal MHD” there is a preferred U, the rest frame of the plasma (ions), but there need not be.

implies $$\mathcal{L}_U\, F = U \cdot dF + d(U \cdot F) = 0$$

FROZEN-IN THEOREM

Aside: conservation of magnetic helicity

$$d(A \wedge F) = F \wedge F = 0$$

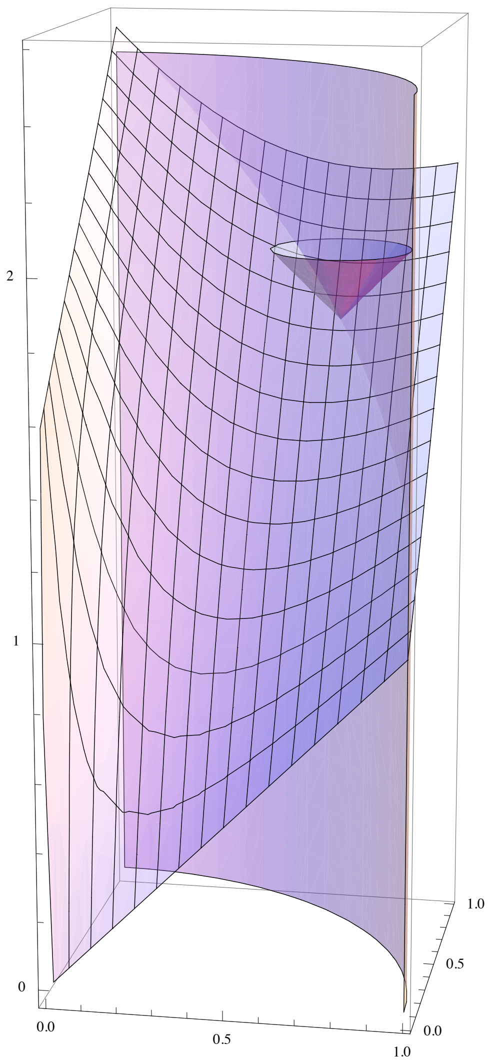
helicity current

Minimum energy magnetic field at fixed helicity is force-free.

Helicity is approximately conserved even when energy is not.

Helicity is injected into the solar corona from the interior, and blown off by wind and coronal mass ejections.

Must use ``relative helicity” to describe helicity injection from a boundary.



Field sheet metric

The induced metric on the field sheet governs the propagation of charged particles and Alfvén waves.

Collisionless charges stuck to the field sheet,

action = proper time + radiative em coupling -> geodesic + curvature radiation

Alfven waves: transverse oscillations of field lines.

Dispersion relation depends on pullback of wave 4-vector to field sheet ➔ group 4-velocity tangent to field sheet.

Force-free limit: wavevector null wrt field sheet metric, group velocity along field sheet lightrays.

Action for Euler potentials

Uchida, 1997

$$S[\phi_1, \phi_2] = \int d\phi_1 \wedge d\phi_2 \wedge *(d\phi_1 \wedge d\phi_2)$$

Euler-Lagrange equation: $d\Big(d\phi_{1,2} \wedge *(d\phi_1 \wedge d\phi_2)\Big) = 0$

$$d\phi_{1,2} \wedge d * (d\phi_1 \wedge d\phi_2) = 0$$

Force-free field equations

Outgoing Poynting flux in Schwarzschild spacetime

$$ds^2 = G(r)\, du^2 + 2du dr - r^2 d\Omega^2$$

u = t – r*, retarded time.
du is *null*

$$\begin{aligned} A &= f(u,\theta,\phi)\, du \\ F &= df \wedge du = (f_\theta\, d\theta + f_\phi\, d\phi) \wedge du \\ *F &= (A\, d\theta + B\, d\phi) \wedge du \\ d*F &= C\, d\theta \wedge d\phi \wedge du \\ df \wedge d*F &= 0 \\ du \wedge d*F &= 0 \end{aligned}$$

null field

null radial current

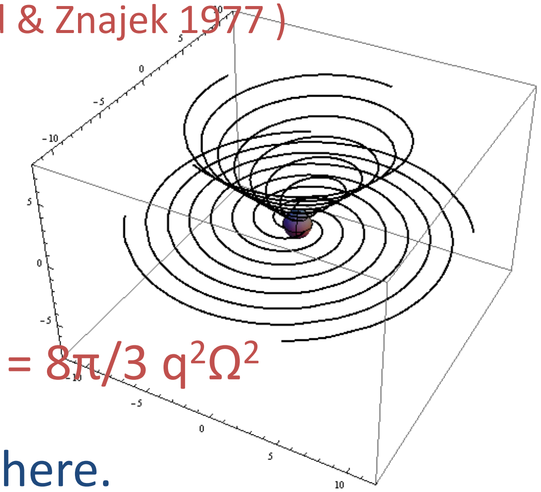
FF eqns satisfied

- Arbitrary (u, θ, Φ) dependence: new solutions! (Brennan, Gralla, TJ, 2013)
- Generalizes to Kerr (stat-axi-symm: Menon & Dermer, genl: BGJ)
- No scattering!? Robinson solutions (1961) with a current.

The “Michel rotating monopole”:

(Michel 1973; Schwarzschild version Blandford & Znajek 1977)

$$F^{\mathrm{Michel}} = q\, d(\cos\theta) \wedge (d\phi - \Omega\, du)$$



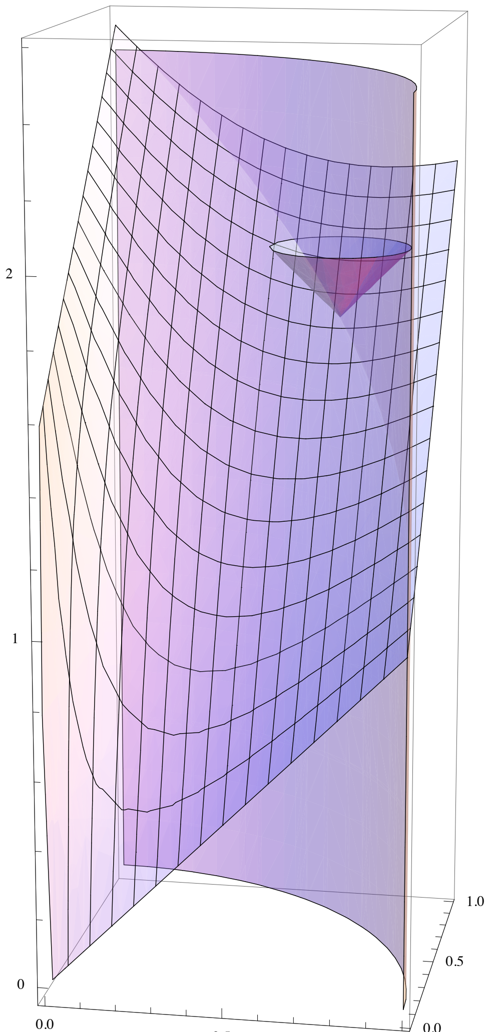
Co-rotates with conducting star, extracts “Poynting flux” : L = $8\pi/3\ q^2\Omega^2$

Current flows in the northern and out the southern hemisphere.

Whirling monopole: (fixed axis case, Lyutikov, 2011; general case: Gralla, TJ 2013)

$$F^{\mathrm{whirling}} = q\, \hat{\Omega}(u) \cdot d\hat{n}(\theta,\phi) \wedge \left(d\phi_{\Omega(u)} - \Omega(u)\, du \right)$$

Field sheet geometry of the rotating monopole

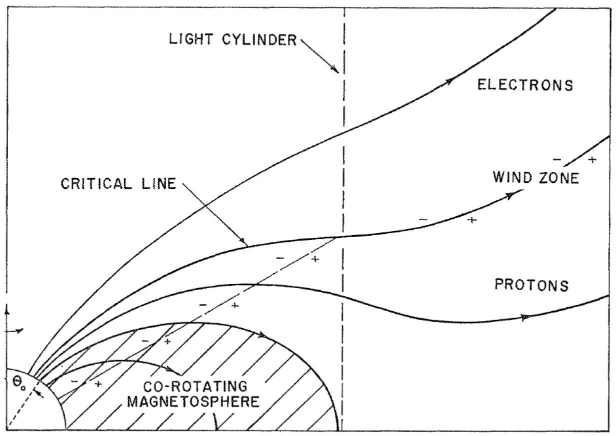


Equatorial field sheet of
the Michel rotating monopole.

The "light cylinder" is where the "field line rotates at the speed of light", i.e. the intersection of the field sheet with the cylinder is a null curve, which is a Killing horizon for the helical Killing vector $\partial_t + \Omega\, \partial_\varphi$ on the field sheet.

For the Michel monopole, the field sheet geometry is de Sitter space, and the Hubble constant/surface gravity of the horizon is $\Omega \sin\theta$.

Dipole -> Monopole



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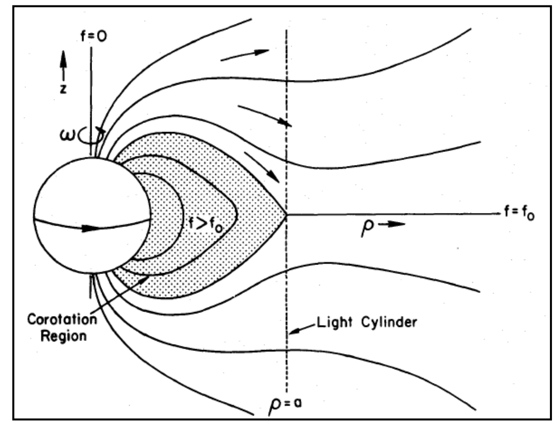

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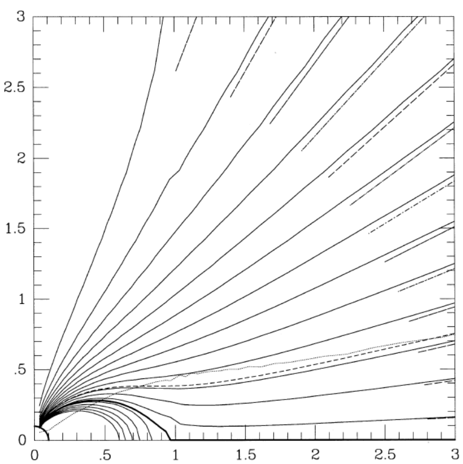

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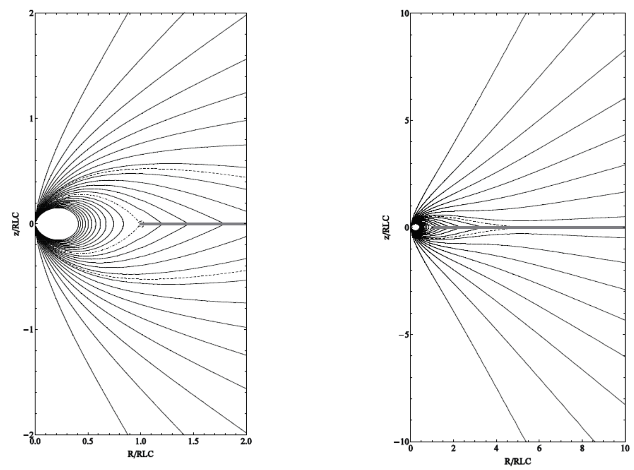
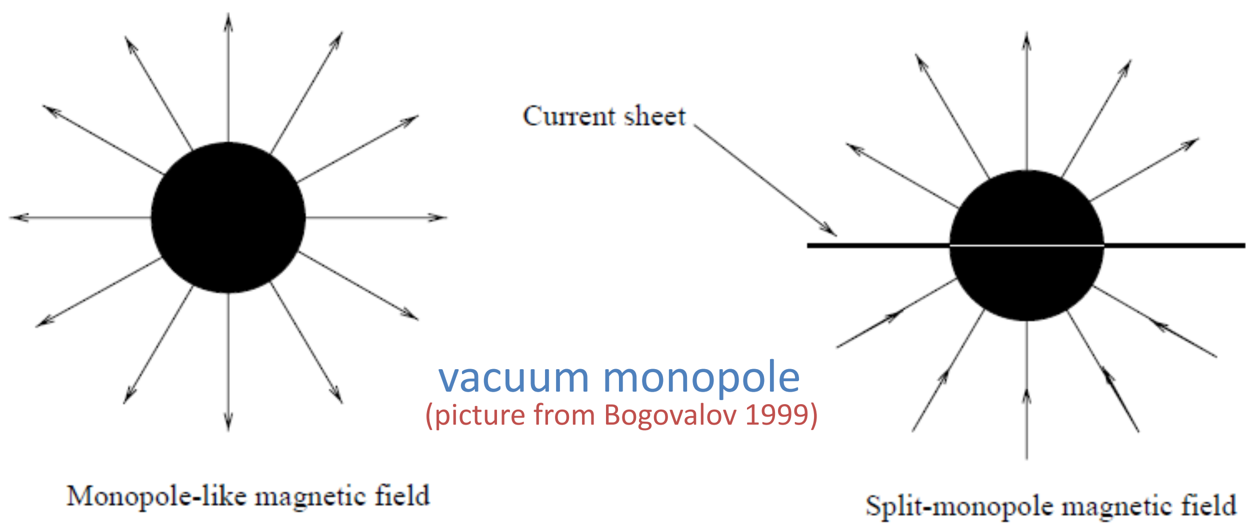


Contopoulos, Kalapotharakos, Kazanas 2013

Split Monopole: crude model of a dipole with current sheet

+q in northern, -q in southern hemisphere



Current flows out *both* hemispheres, in along equatorial current sheet.

Rotating black hole (Blandford & Znajek, 1977)

Imagine 1-parameter family of solutions F^a, with F^0 = (split) monopole, and a = BH spin parameter. Linearization wrt a implies

$$d\,{}^0\phi_{1,2} \wedge [d\,{}^1{*}\,{}^0F + d\,{}^0{*}\,{}^1F] = 0$$

The Michel solution, exported to Kerr in Boyer-Lindquist coordinates, satisfies this:

$$F^{\text{Michel}} = q\,d(\cos\theta) \wedge (d\phi - \Omega\,du)$$

$$
\begin{aligned}
d\,{}^{1}\!\ast\,(d\cos\theta\wedge d\phi) &\quad\sim\quad d\theta\wedge d\phi\wedge dr,\\
d\,{}^{0}\!\ast\,(d\cos\theta\wedge du) &\quad\sim\quad d\theta\wedge d\phi\wedge du
\end{aligned}
$$

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$$F^{\mathrm{Michel}} = q\, d(\cos\theta) \wedge (d\phi - \Omega\, du)$$

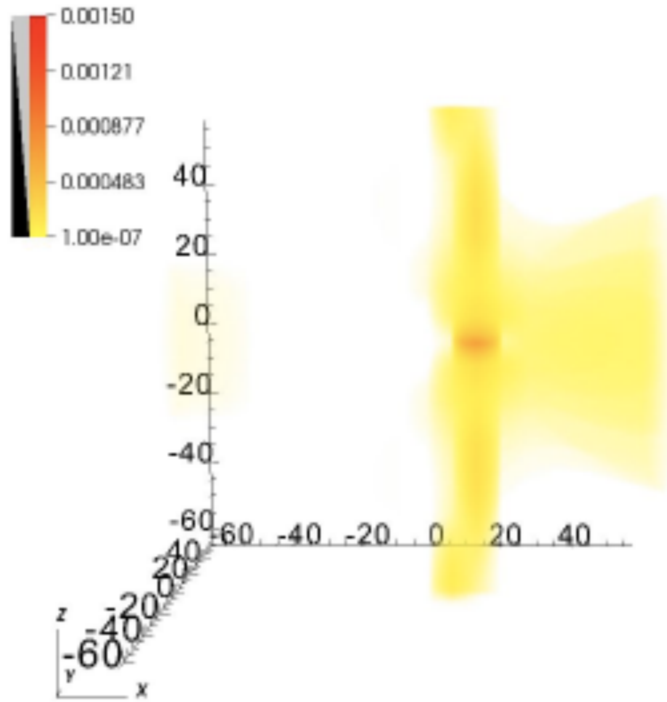
Regularity on the future horizon requires*: $\Omega = \frac{\Omega_H}{2} = \frac{a/2}{r_H^2 + a^2}$

Then Poynting flux same as rotating conducting star (to $O(a^2)$): ***negative energy flux into BH***

$$\begin{aligned} d\phi &= d\psi - \frac{a}{\Delta} dr \\ du &= dv - 2\frac{r^2 + a^2}{\Delta} dr \end{aligned}$$

*Psi and v are regular on the future Horizon where Delta = 0

Energy conservation in boosted black hole jet?



Neilsen, Lehner, Palenzuela, Hirschmann, Liebling, Motl, Garrett (2010)

Stationary axisymmetric magnetospheres …

We don't have time to discuss it now but there's a classic theory of this, which is beautiful and simple using exterior calculus…

… and raises an interesting mathematical question about uniqueness of solutions to the "stream equation" with given boundary conditions and regular singular points …

Conclusion

Force-free electrodynamics is remarkably simple and elegant when treated relativistically with differential forms.

We have recovered, systematized, and generalized a menagerie of known solutions, including astrophysically relevant ones.

In and upcoming paper, we also provide an introduction and derivation of the general properties of stationary axisymmetric force-free solutions, and apply this to derive the general structure of neutron star and black hole magnetospheres.

Perhaps these methods can lead to more exact, or approximate solutions that could help us to understand the physics of jets and other plasma phenomena.