Infinite discrete symmetries near singularities and modular forms

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Based on work with:

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[KNP = arXiv:1010.2212]

[FK, to be published]

Context and Plan

Hidden symmetries and cosmological billiards in supergravity [Damour, Henneaux 2000; Damour, Henneaux, Nicolai 2002]

Minisuperspace models for quantum gravity and quantum cosmology [DeWitt 1967; Misner 1969]

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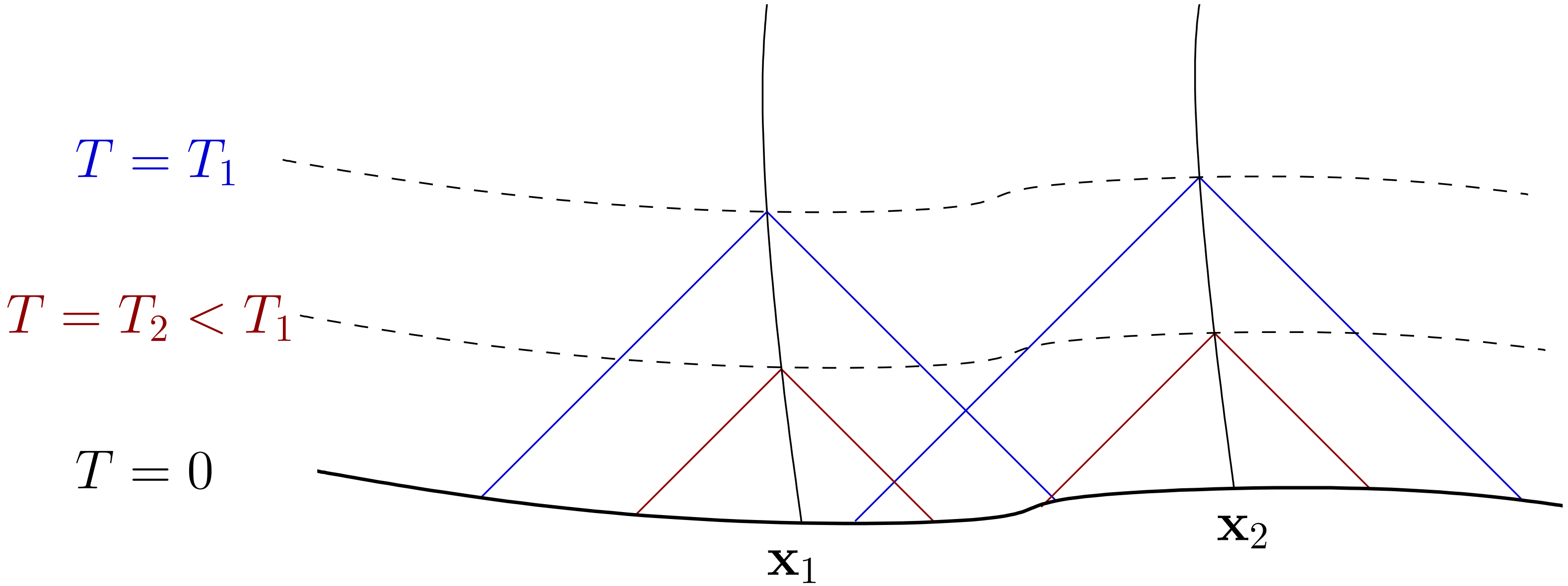
Plan

- Cosmological billiards and their symmetries
- Quantum cosmological billiards: arithmetic structure
- Modular forms for hyperbolic Weyl groups and infinite Chevalley groups
- Generalization and outlook

Cosmological billards: BKL

Supergravity dynamics near a space-like singularity simplify.

[Belinskii, Khalatnikov, Lifshitz 1970; Misner 1969; Chitre 1972]



Spatial points decouple $\overset{\text{(conj.)}}{\Rightarrow}$ dynamics becomes ultra-local.

Reduction of degress of freedom to spatial scale factors β^a

$$ds^2 = -N^2 dt^2 + \sum_{a=1}^{d} e^{-2\beta^a} dx_a^2 \qquad (t \sim -\log T)$$

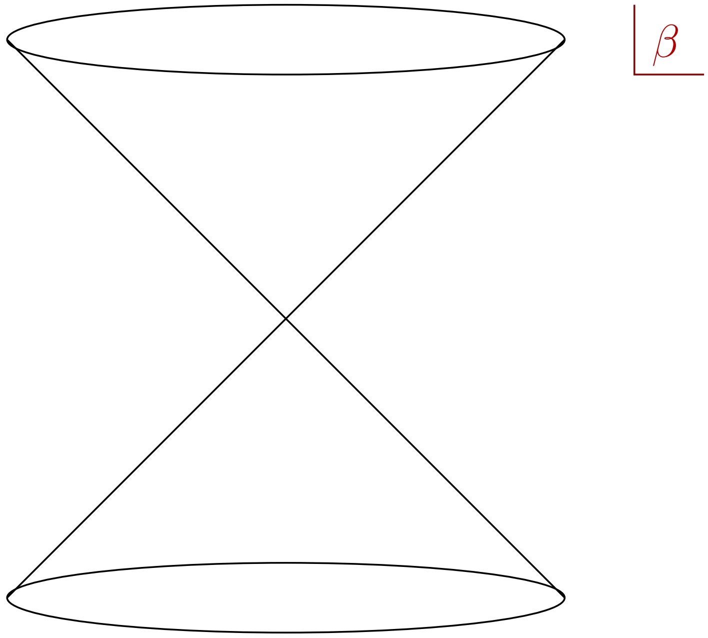
Cosmological billiards: Dynamics

Effective Lagrangian for $\beta^a(t)$ ($a = 1, \ldots, d$)

$$\mathcal{L} = \frac{1}{2} \sum_{a,b=1}^{d} n^{-1} G_{ab} \dot{\beta}^a \dot{\beta}^b - V_{\text{eff}}(\beta)$$

[G_{ab}: DeWitt metric (Lorentzian signature)]

Close to the singularity V_{eff} consists of infinite potentials walls, obstructing free null motion of β^a.



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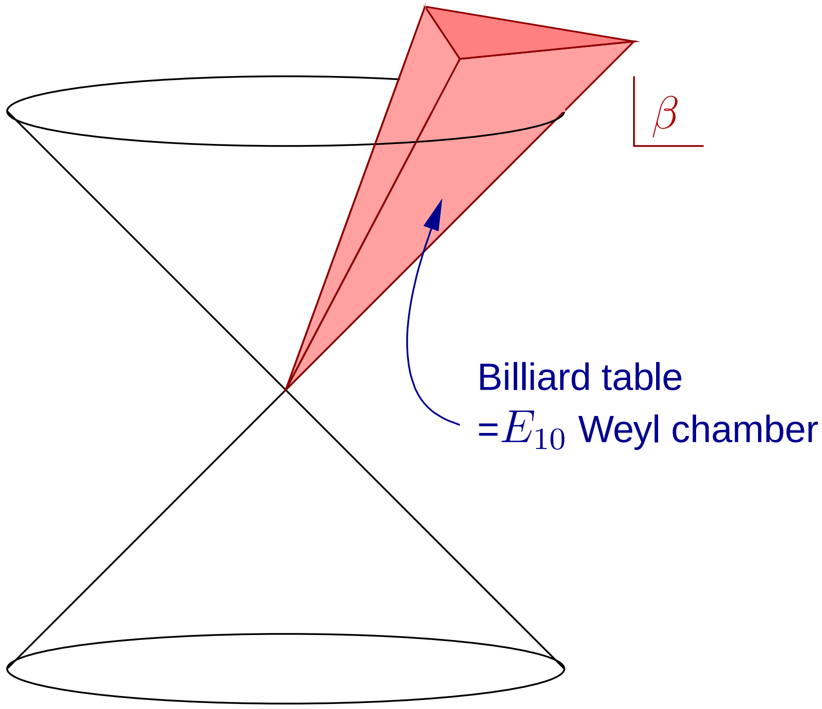
Resulting billiard geometry that of E_{10} Weyl chamber ($D = 11$, type (m)IIA and IIB).

[Damour, Henneaux 2000]



Cosmological billiards: Geometry

The sharp billiard walls come from

$$V_{\text{eff}}(\beta) = \sum_A c_A e^{-2w_A(\beta)}$$

with $w_A(\beta)$ a set of linear forms on β-space. For $G_{ab}\beta^a\beta^b \to -\infty$ (towards the singularity) the potential term becomes 0 or ∞, defining two sides of a wall.

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For the *dominant* terms $c_A \geq 0$ [Damour, Henneaux, Nicolai 2002]. Furthermore, the scalar product between the normals to those faces coincides with E_{10} Cartan matrix.

Associated E_{10} Weyl group $W(E_{10})$ are the symmetries of the unique even self-dual lattice $\mathrm{II}_{9,1} = \Lambda_{E_8} \oplus \mathrm{II}_{1,1}$.

Finite (hyperbolic) volume $\Rightarrow$ Chaos! [Damour, Henneaux 2000]

Quantum cosmological billiards

Setting $n = 1$ one has to quantize

$$\mathcal{L} = \frac{1}{2}\sum_{a,b=1}^{d} \dot{\beta}^a G_{ab} \dot{\beta}^b = \frac{1}{2}\left[\sum_{a=1}^{d} (\dot{\beta}^a)^2 - \left(\sum_{a=1}^{d} \dot{\beta}^a\right)^2\right]$$

with null constraint $\dot{\beta}^a G_{ab} \dot{\beta}^b = 0$ on billiard domain.

Canonical momenta: $\quad \pi_a = G_{ab}\dot{\beta}^b \quad \Rightarrow \quad \mathcal{H} = \frac{1}{2}\pi_a G^{ab} \pi_b$.

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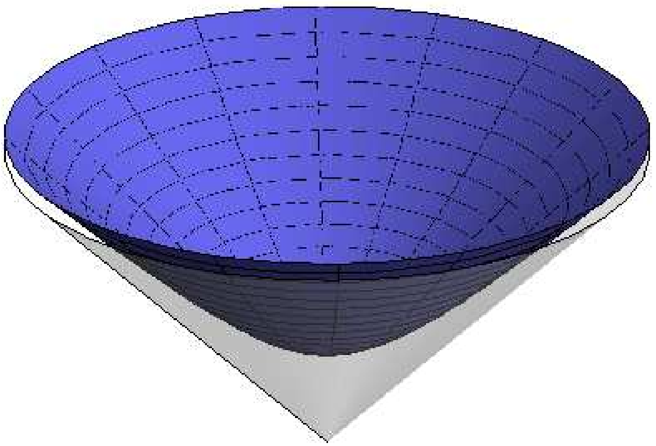
Wheeler–DeWitt (WDW) equation in canonical quantization

$$\mathcal{H}\Psi(\beta) = -\frac{1}{2}G^{ab}\partial_a\partial_b\Psi(\beta) = 0$$

Klein–Gordon ‘inner product’.

Quantum cosmological billiards (II)

Introduce new coordinates ρ and $\omega^a(z)$ from 'radius' and co-ordinates z on unit hyperboloid



$$\beta^a = \rho\omega^a\,, \quad \omega^a G_{ab}\omega^b = -1$$
$$\rho^2 = -\beta^a G_{ab}\beta^b$$

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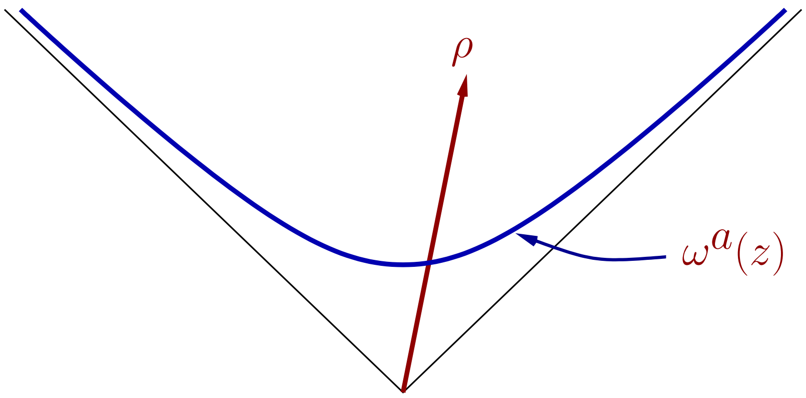
Singularity: $\rho \to \infty$



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Timeless WDW equation in these variables

$$\left[-\rho^{1-d}\frac{\partial}{\partial\rho}\left(\rho^{d-1}\frac{\partial}{\partial\rho}\right) + \rho^{-2}\Delta_{\mathsf{LB}}\right]\Psi(\rho, z) = 0$$

Laplace–Beltrami operator on unit hyperboloid

Solving the WDW equation

$$\left[-\rho^{1-d}\frac{\partial}{\partial\rho}\left(\rho^{d-1}\frac{\partial}{\partial\rho}\right)+\rho^{-2}\Delta_{\mathsf{LB}}\right]\Psi(\rho,z)=0$$

Separation of variables: $\Psi(\rho,z)=R(\rho)F(z)$

For

$$-\Delta_{\mathsf{LB}}F(z)=EF(z)$$

get

$$R_{\pm}(\rho)=\rho^{-\frac{d-2}{2}\pm i\sqrt{E-\left(\frac{d-2}{2}\right)^2}}$$

[Positive frequency coming out of singularity is $R_{-}(\rho)$.]

Left with spectral problem on hyperbolic space.

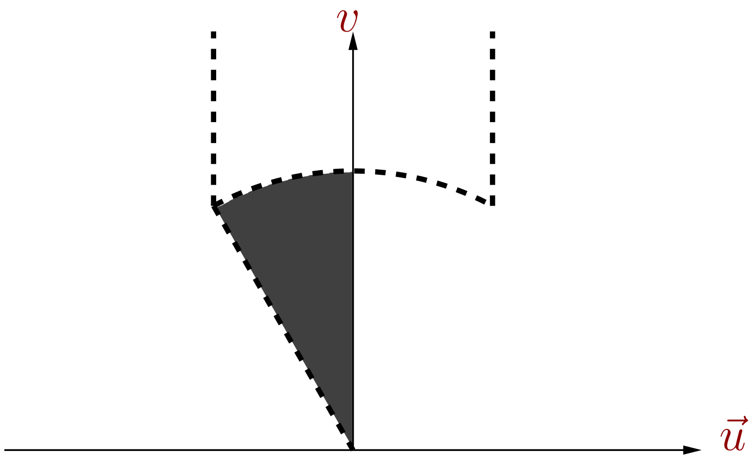
$\Delta_{\mathbf{LB}}$ and boundary conditions

The classical billiard ball is constrained to Weyl chamber with infinite potentials $\Rightarrow$ Dirichlet boundary conditions

Use upper half plane model

$$z = (\vec{u}, v)\,, \quad \vec{u} \in \mathbb{R}^{d-2}, v \in \mathbb{R}_{>0}$$

$$\Rightarrow \quad \Delta_{\mathsf{LB}} = v^{d-1}\partial_v(v^{3-d}\partial_v) + v^2\partial_{\vec{u}}^2$$



Δ_{LB} and boundary conditions

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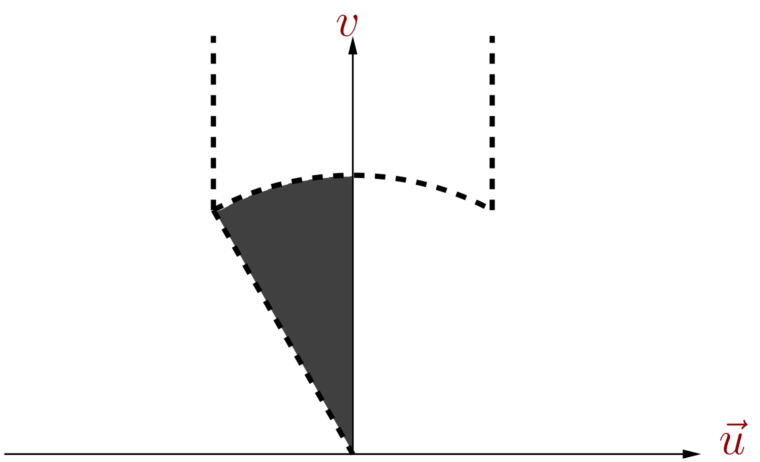


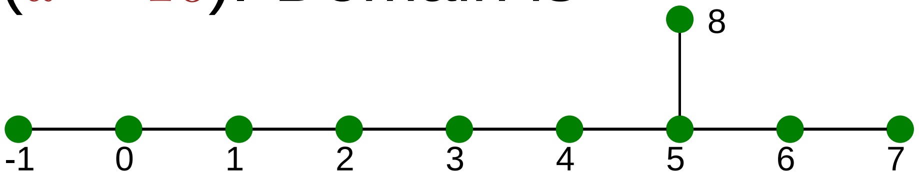
With *Dirichlet boundary conditions* ($d = 3$ in [Iwaniec])

$$-\Delta_{\mathsf{LB}}F(z) = EF(z) \quad \Rightarrow \quad E \geq \left(\frac{d-2}{2}\right)^2$$

Arithmetic structure (I)

Beyond general inequality details of spectrum depend on shape of domain. (‘Shape of the drum’ problem)

Focus on maximal supergravity ($d = 10$). Domain is determined by E_{10} Weyl group.



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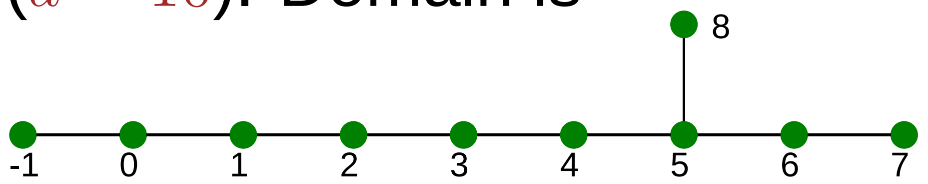

9-dimensional upper half plane with octonions: $u \equiv \vec{u} \in \mathbb{O}$

On $z = u + iv$ the ten fundamental Weyl reflections act by

$$w_{-1}(z) = \frac{1}{\bar{z}}\,, \; w_0(z) = -\bar{z} + 1\,, \; w_j(z) = -\varepsilon_j \bar{z} \varepsilon_j$$

ε_j simple E_8 rts. [Feingold, AK, Nicolai 2008]

Arithmetic structure (II)

Iterated action of

$$w_{-1}(z) = \frac{1}{\bar{z}}\,, \; w_0(z) = -\bar{z} + 1\,, \; w_j(z) = -\varepsilon_j \bar{z} \varepsilon_j$$

generates whole Weyl group $W(E_{10})$.

Even Weyl group $W^+(E_{10})$ gives 'holomorphic' maps

$$W^+(E_{10}) = PSL_2(\mathtt{O}).$$

Modular group over the integer 'octavians' $\mathtt{O}$.

[Example of family of isomorphisms between hyperbolic Weyl groups and modular groups over division algebras [Feingold, AK, Nicolai 2008].]

Modular wavefunctions (I)

Weyl reflections on wavefunction $\Psi(\rho, z)$

$$\Psi(\rho, w_I \cdot z) = \begin{cases} +\Psi(\rho, z) & \text{Neumann b.c.} \\ -\Psi(\rho, z) & \text{Dirichlet b.c.} \end{cases}$$

Use Weyl symmetry to *define* $\Psi(\rho, z)$ on the whole upper half plane, with Dirichlet boundary conditions $\Rightarrow$ $\Psi(\rho, z)$ is

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- Sum of eigenfunctions of Δ_{LB} on UHP
- Invariant under action of $W^+(E_{10}) = PSL_2(\mathsf{O})$. Anti-invariant under extension to $W(E_{10})$.

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$\Rightarrow$ Wavefunction is an odd Maass wave form of $PSL_2(\mathtt{O})$

[cf. [Forte 2008] for related ideas for Neumann conditions]

Modular wavefunctions (II)

The spectrum of odd Maass wave forms is (presumably) discrete but not known. For $PSL_2(\mathtt{O})$ the theory is not even developed (but see [Krieg]).

For lower dimensional cases like pure $(3+1)$-dimensional Einstein gravity with $PSL_2(\mathbb{Z})$ there are many numerical investigations. [Graham, Szépfalusy 1990; Steil 1994; Then 2003]

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Summary of analysis so far:

> Quantum billiard wavefunction $\Psi(\rho, z)$ is an odd Maass wave form (Dirichlet b.c.) for $PSL_2(\mathtt{O})$.

Interpretation (I)

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‘Wavefunction of the universe’ in this set-up formally

$$|\Psi_{\mathsf{full}}\rangle = \prod_{\mathbf{x}} |\Psi_{\mathbf{x}}\rangle$$

Product of quantum cosmological billiard wavefunctions, one for each spatial point (ultra-locality). [Also [Kirillov 1995]]

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Product of quantum cosmological billiard wavefunctions, one for each spatial point (ultra-locality). [Also [Kirillov 1995]]

Each factor contains a Maass wave form of the type $\Psi_{\mathbf{x}}(\rho, z) = \sum R_{\pm}(\rho) F(z)$ with

$$-\Delta_{\mathsf{LB}} F(z) = E F(z)\,, \quad R_{\pm}(\rho) = \rho^{-\frac{d-2}{2} \pm i \sqrt{E - \left(\frac{d-2}{2}\right)^2}}$$

Since $E \geq \left(\frac{d-2}{2}\right)^2$: $\Psi_{\mathbf{x}}(\rho, z) \to 0$ but cx. for $\rho \to \infty$

Interpretation (II)

- Absence of potential: $\exists$ a well-defined Hilbert space with positive definite metric.
- The wavefunction vanishes at the singularity. But it remains oscillating and complex. No bounce.
  $\Rightarrow$ Vanishing wavefunctions on singular geometries are one possible boundary condition. [DeWitt 1967]
- Complexity and notion of positive frequency
  $\Rightarrow$ Arrow of time? [Isham 1991; Barbour 1993]
- 'Semi-classical' states are expected to spread (quantum ergodicity). Numerical investigations, e.g. [Koehn 2011]

Generalization (I)

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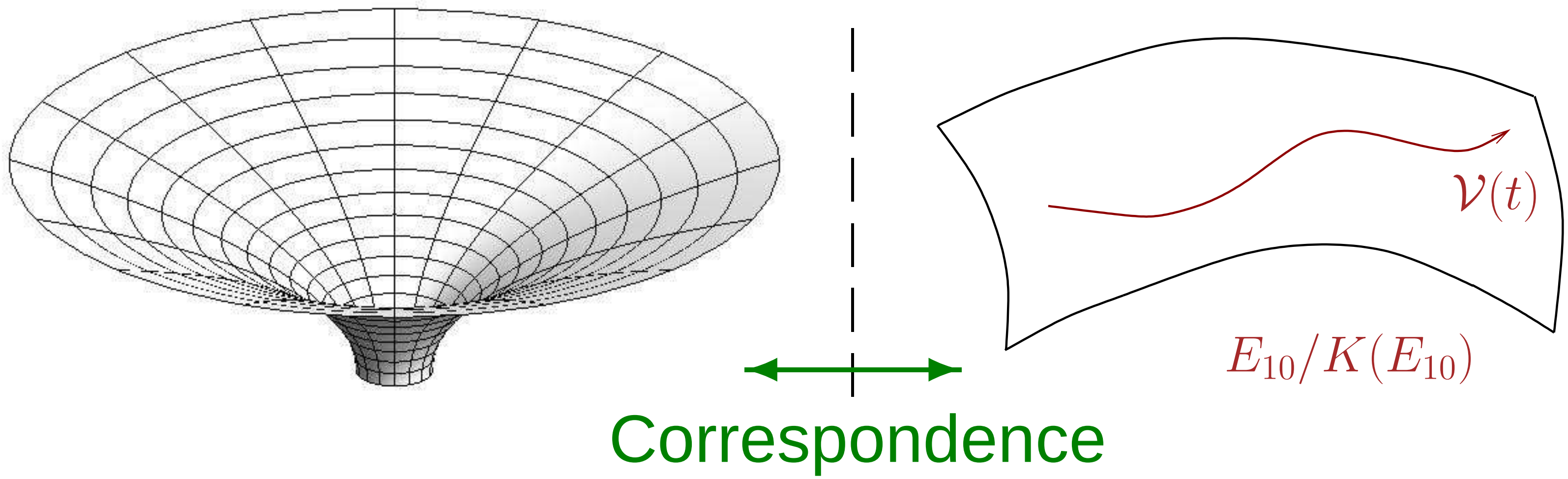
Classical cosmological billiards led to the E_{10} conjecture.

$D = 11$ supergravity can be mapped to a constrained null geodesic motion on infinite-dimensional $E_{10}/K(E_{10})$ coset space. [Damour, Henneaux, Nicolai 2002]



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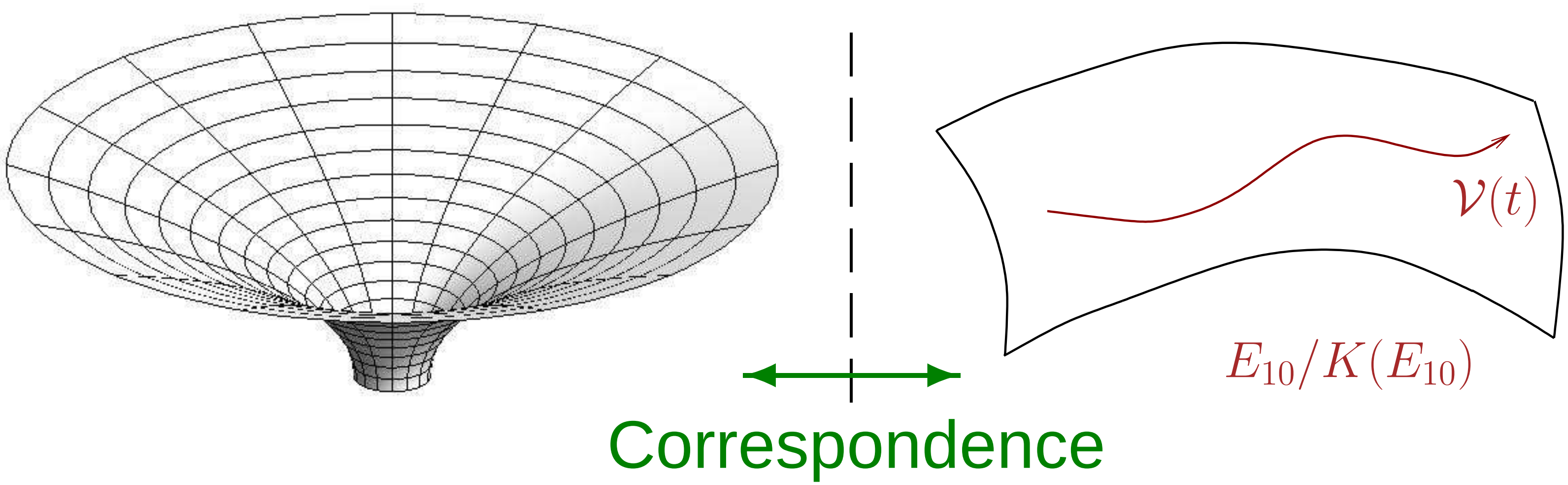

Symmetric space $E_{10}/K(E_{10})$ has $10 + \infty$ many directions.

Cartan subalgebra pos. step operators

Generalization (II)

Features of the conjectured E_{10} correspondence

- Billiard corresponds to 10 Cartan subalgebra generators
- ∞ many step operators correspond to remaining fields and spatial dependence. [Verified only at low 'levels' but for many different models]
- Space dependence introduced via *dual fields* (cf. Geroch group) — everything in terms of kinetic terms
- Space (de-)emergent via an algebraic mechanism
- Extension to E_{10} overcomes ultra-locality
- Appears that only supergravity captured; no higher spin fields [Henneaux, AK, Nicolai 2011]

Generalization (III)

$$\mathcal{H}_{\text{Bill}} \to \mathcal{H} \equiv \mathcal{H}_{\text{Bill}} + \sum_{\alpha \in \Delta_+(E_{10})} e^{-2\alpha(\beta)} \sum_{s=1}^{\text{mult}(\alpha)} \Pi^2_{\alpha,s}$$

is the unique quadratic E_{10} Casimir. Formally like free Klein–Gordon; positive norm could remain consistent?

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Full theory has more constraints than the Hamiltonian ($\mathcal{H}\Psi = 0$) constraint: diff, Gauss, etc.

- Global E_{10} symmetry provides ∞ conserved charges $\mathcal{J}$
- Evidence that constraints can be written as bilinears $\mathfrak{L} \sim \mathcal{J}\mathcal{J}$. [Damour, AK, Nicolai 2007; 2009]
- Analogy with affine Sugawara construction. Particularly useful for implementation as quantum constraints?

Aim: Quantize geodesic model!

Poincaré series for $PSL_2(\mathsf{O})$ (I)

Poincaré series for $W^+(E_{10}) = PSL_2(\mathsf{O})$ defined by

$$\mathcal{P}_s(z) = \sum_{\gamma \in W^+(E_9) \backslash W^+(E_{10})} I_s(\gamma(z))$$

with $z = u + iv$ and $I_s(z) = v^s$. $W^+(E_9)$ stabilises cusp at infinity. Converges for $\text{Re}(s) > 4$. $\mathcal{P}_s$ is eigenfunction of Δ_{LB}.

Cosets can be given an explicit octonionic description [KNP]. Result is

$$\mathcal{P}_s(z) = \frac{1}{240} \sum_{c,d \in \mathsf{O} \text{ left coprime}} \frac{v^s}{|cz+d|^{2s}}$$

'Left-coprimality' is defined via Euclidean algortihm [KNP].

Poincaré series for $PSL_2(\mathsf{O})$ (II)

In terms of unrestricted sum

$$\sum_{(c,d)\in\mathsf{O}^2\backslash\{(0,0)\}} \frac{v^s}{|cz+d|^{2s}} = \zeta_{\mathsf{O}}(s)\frac{1}{240} \sum_{c,d\in\mathsf{O}\text{ left coprime}} \frac{v^s}{|cz+d|^{2s}}$$

Dedekind Zeta, related to E_8 Theta

Fourier expansion

$$\mathcal{P}_s(z) = v^s + a(s)v^{8-s} + v^4 \sum_{\mu\in\mathsf{O}^*\backslash\{0\}} a_\mu K_{s-4}(2\pi|\mu|v)e^{2\pi i\mu(u)}$$

- Only abelian Fourier modes, only two constant terms
- Functional relation (?): $\xi_{\mathsf{O}}(s)\mathcal{P}_s(z) = \xi_{\mathsf{O}}(8-s)\mathcal{P}_{8-s}(z)$
- Neumann boundary conditions

Eisenstein series for $E_9(\mathbb{Z})$ and $E_{10}(\mathbb{Z})$ (I)

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[work in progress... [FK]]

String theory *seems* to require $E_{10}(\mathbb{Z}) \supset W(E_{10})$ [Hull, Townsend 1995; Ganor 1999].
For smaller rank [Green, Gutperle 1997; Obers, Pioline 1998; Green, Miller, Russo, Vanhove 2010].

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Eisenstein series for the Chevalley groups $E_n(\mathbb{Z})$, $n > 8$?

Very little literature on the subject... But [Garland 2001].
Affine case $G = E_9$:

$$E_\lambda^G(g, r) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda+\rho, H(\gamma g e^{rD})\rangle}$$

g does not include derivation D.

Eisenstein series for $E_9(\mathbb{Z})$ and $E_{10}(\mathbb{Z})$ (II)

Constant term (in minimal parabolic) [Langlands; Garland]

$$\sum_{w \in W(E_9)} e^{\langle w\lambda+\rho, H(ge^{rD})\rangle} M(w,\lambda)$$

$$\uparrow$$

$$= \prod_{\alpha>0 : w\alpha<0} \frac{\xi(\langle\lambda,\alpha\rangle)}{\xi(\langle\lambda,\alpha\rangle+1)}$$

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Affine Weyl group is infinite but for special values of λ, the infinite sum collapses since $M(w,\lambda)=0$. For $\lambda = 2s\Lambda_i - \rho$ this can only happen for $2s\in\mathbb{Z}$.

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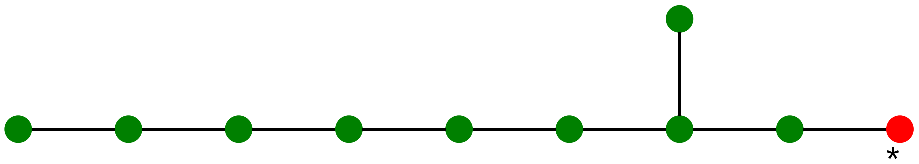
Assume same formal expression for $E_{10}(\mathbb{Z})$...

Constant terms for $E_9(\mathbb{Z})$ and $E_{10}(\mathbb{Z})$

Example: $\Lambda_i = \Lambda_*$



	$s=1/2$	$s=1$	$s=3/2$	$s=2$	$s=5/2$	$s=3$
E_7	2	126	8	14	35	56
E_8	2	2160	9	16	44	72
E_9	2	∞	10	18	54	90
E_{10}	2	∞	11	20	65	110

Constant terms in maximal parabolic can also be evaluated.

Full Fourier decomposition (constant + abelian + non-abelian)?

Summary and outlook

Done:

- Quantum cosmological billiards wavefunctions involve automorphic forms of $PSL_2(\mathsf{O})$
- Extendable to supersymmetric case
- Studied parts of modular forms for $W^+(E_{10})$ and $E_{10}(\mathbb{Z})$

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To do:

- Construct wavefunctions (with Dirichlet boundary conditions)?
- Include more variables $\Rightarrow$ E_{10} coset model? Constraints? Observables?
- Understand $E_9(\mathbb{Z})$ and $E_{10}(\mathbb{Z})$ modular forms better and relation to string scattering

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Thank you for your attention!

More on hyperbolic Weyl groups (I)

Consider only over-extended hyperbolic algebras $\mathfrak{g}^{++}$ (rank$(\mathfrak{g}) \equiv \ell = 1, 2, 4, 8$). Their root lattices can be realized in $R^{1,1+\ell} \cong H_2(\mathbb{K})$ for a normed division algebra $\mathbb{K}$

$$(\mathtt{X}_1|\mathtt{X}_2) = -\det(\mathtt{X}_1 + \mathtt{X}_2) + \det(\mathtt{X}_1) + \det(\mathtt{X}_2)\,, \qquad \mathtt{X}_i \in H_2(\mathbb{K})$$

More on hyperbolic Weyl groups (I)

Consider only over-extended hyperbolic algebras $\mathfrak{g}^{++}$ ($\text{rank}(\mathfrak{g}) \equiv \ell = 1,2,4,8$). Their root lattices can be realized in $R^{1,1+\ell} \cong H_2(\mathbb{K})$ for a normed division algebra $\mathbb{K}$

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Choose a_i ($i = 1,\ldots,\ell$) such that

$$a_i\bar{a}_j + a_j\bar{a}_i = \text{Cartan matrix of } \mathfrak{g}$$

Prop 1. $\mathfrak{g}^{++}$ *Cartan matrix from simple roots*

$$\alpha_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_0 = \begin{pmatrix} -1 & -\theta \\ -\bar{\theta} & 0 \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} 0 & a_i \\ \bar{a}_i & 0 \end{pmatrix}$$

More on hyperbolic Weyl groups (II)

Thm 1. *Fundamental Weyl reflections of* $W \equiv W(\mathfrak{g}^{++})$ *are*

$$w_I(\mathtt{X}) = M_I \bar{\mathtt{X}} M_I^\dagger \quad , \; I = -1, 0, 1, \ldots, \ell$$

with unit versions of $\mathfrak{g}$ *simple roots* $\varepsilon_i = a_i / \sqrt{N(a_i)}$ *and*

$$M_{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_0 = \begin{pmatrix} -\theta & 1 \\ 0 & \bar{\theta} \end{pmatrix}, \quad M_i = \begin{pmatrix} \varepsilon_i & 0 \\ 0 & -\bar{\varepsilon}_i \end{pmatrix}$$

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Remarks

- Formula well-defined for all $\mathbb{K}$, including octonions
- Involves complex conjugation of $\mathtt{X}$
- $\varepsilon_i \neq a_i$ only if $\mathfrak{g}$ not simply laced

More on hyperbolic Weyl groups (III)

For generalizations of modular group $PSL_2(\mathbb{Z})$ need

Thm 2. *Even Weyl group* $W^+ \equiv W^+(\mathfrak{g}^{++})$ *generated by*

$$(w_{-1}w_i)(\mathtt{X}) = S_i \mathtt{X} S_i^\dagger \quad , \; i = 0, 1, \ldots, \ell$$

with

$$S_0 = \begin{pmatrix} 0 & \theta \\ -\bar{\theta} & 1 \end{pmatrix} , \quad S_i = \begin{pmatrix} 0 & -\varepsilon_i \\ \bar{\varepsilon}_i & 0 \end{pmatrix}$$

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Remarks

- Formula well-defined for all $\mathbb{K}$, including octonions
- If det. *were* defined: $\det S = 1$, cf. $W^+ \subset SO(1, \ell+1; \mathbb{R})$
- Does not involve complex conjugation of $\mathtt{X}$
  $\Longrightarrow$ matrix subgroups of $PSL_2(\mathbb{K})$ in associative cases!

List of hyperbolic Weyl groups

$\mathbb{K}$	$\mathfrak{g}$	'Ring'	$W(\mathfrak{g})$	$W^+(\mathfrak{g}^{++})$
$\mathbb{R}$	A_1	$\mathbb{Z}$	$2 \equiv \mathbb{Z}_2$	$PSL_2(\mathbb{Z})$
$\mathbb{C}$	A_2	Eisenstein $\mathtt{E}$	$\mathbb{Z}_3 \rtimes 2$	$PSL_2(\mathtt{E})$
$\mathbb{C}$	$B_2 \equiv C_2$	Gaussian $\mathtt{G}$	$\mathbb{Z}_4 \rtimes 2$	$PSL_2(\mathtt{G}) \rtimes 2$
$\mathbb{C}$	G_2	Eisenstein $\mathtt{E}$	$\mathbb{Z}_6 \rtimes 2$	$PSL_2(\mathtt{E}) \rtimes 2$
$\mathbb{H}$	A_4	Icosians $\mathtt{I}$	$\mathfrak{S}_5$	$PSL_2^{(0)}(\mathtt{I})$
$\mathbb{H}$	B_4	Octahedral $\mathtt{R}$	$2^4 \rtimes \mathfrak{S}_4$	$PSL_2^{(0)}(\mathtt{H}) \rtimes 2$
$\mathbb{H}$	C_4	Octahedral $\mathtt{R}$	$2^4 \rtimes \mathfrak{S}_4$	$\widetilde{PSL}_2^{(0)}(\mathtt{H}) \rtimes 2$
$\mathbb{H}$	D_4	Hurwitz $\mathtt{H}$	$2^3 \rtimes \mathfrak{S}_4$	$PSL_2^{(0)}(\mathtt{H})$
$\mathbb{H}$	F_4	Octahedral $\mathtt{R}$	$2^5 \rtimes (\mathfrak{S}_3 \times \mathfrak{S}_3)$	$PSL_2(\mathtt{H}) \rtimes 2$
$\mathbb{O}$	E_8	Octavians $\mathtt{O}$	$2.O_8^+(2).2$	$PSL_2(\mathtt{O})$