Ambitwistor strings and the scattering equations at one loop

Lionel Mason

The Mathematical Institute, Oxford
lmason@maths.ox.ac.uk

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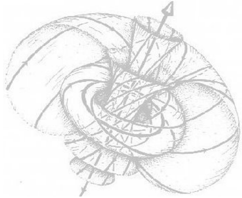
With David Skinner. arxiv:1311.2564, and collaborations with Tim Adamo, Eduardo Casali, Yvonne Geyer, Arthur Lipstein, Ricardo Monteiro, Kai Roehrig, & Piotr Tourkine, 1312.3828, 1404.6219, 1405.5122, 1406.1462, 1506.08771, 1507.00321.

[Cf. also Cachazo, He, Yuan arxiv:1306.2962, 1306.6575, 1307.2199, 1309.0885, 1412.3479]

Ambitwistor spaces: spaces of complex null geodesics.

- Extends Penrose/Ward's gravity/Yang-Mills twistor constructions to non-self-dual fields.
- Yang-Mills Witten and Isenberg, et. al. 1978, 1985.
- Conformal and Einstein gravity LeBrun [1983,1991] Baston & M. [1987] .



Ambitwistor Strings:

- Tree S-Matrices in all dimensions for gravity, YM etc. [CHY]
- From strings in ambitwistor space [M. & Skinner 1311.2564]
- New models for Einstein-YM, DBI, BI, NLS, etc. [Casali, Geyer, M., Monteiro, Roehrig 1506.08771].
- Loop integrands from the Riemann sphere [Geyer, M., Monteiro, Tourkine, 1507.00321].

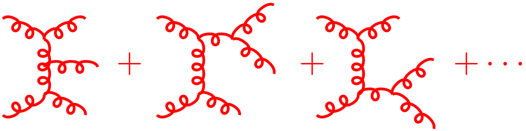
Provide string theories at $\alpha' = 0$ for field theory amplitudes.

Amplitudes from Feynman diagrams

Amplitudes are realized as sums of Feynman integrals.

Consider the five-gluon tree-level amplitude of QCD. Enters in calculation of multi-jet production at hadron colliders.

Described by following Feynman diagrams:

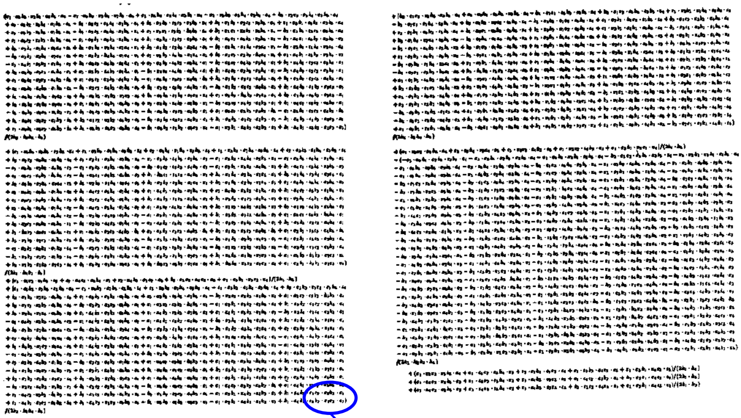


If you follow the textbooks you discover a disgusting mess.

Trees $\leftrightarrow$ classical, loops $\leftrightarrow$ quantum.

Need for new ideas

Result of a brute force calculation:



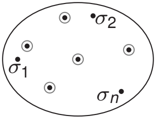
$k_1 \cdot k_4\, \varepsilon_2 \cdot k_1\, \varepsilon_1 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5$

The scattering equations

Take n null momenta $k_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, $k_i^2 = 0$, $\sum_i k_i = 0$,

- define $P : \mathbb{CP}^1 \to \mathbb{C}^d$

$$P(\sigma) := \sum_{i=1}^{n} \frac{k_i}{\sigma - \sigma_i}\,, \qquad \sigma, \sigma_i \in \mathbb{CP}^1$$



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- Solve for $\sigma_i \in \mathbb{CP}^1$ with the *n scattering equations* [Fairlie 197?]

$$\mathrm{Res}_{\sigma_i}\left(P^2\right) = k_i \cdot P(\sigma_i) = \sum_{j=1}^{n} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0\,.$$

$$\Rightarrow \quad P^2 = 0 \;\forall \sigma.$$

- For Mobius invariance $\Rightarrow P \in \mathbb{C}^d \otimes K$, $K = \Omega^{1,0}\mathbb{CP}^1$
- There are $(n-3)!$ solutions.

Arise in large α' strings [Gross-Mende 1988] & twistor-strings [Roiban, Spradlin, Volovich, Witten 2004].

Amplitude formulae for massless theories.

Proposition (Cachazo, He, Yuan 2013,2014)

Tree-level massless amplitudes in d-dims are integrals/sums

$$\mathcal{M}_n = \delta^d\left(\sum_i k_i\right)\int_{(\mathbb{CP}^1)^n}\frac{I^l I^r \prod_i \bar\delta(k_i\cdot P(\sigma_i))}{\mathrm{Vol}\ \mathrm{SL}(2,\mathbb{C})\times\mathbb{C}^3}$$

where $I^{l/r} = I^{l/r}(\epsilon_i^{l/r}, k_i, \sigma_i)$ depend on the theory.

- polarizations ϵ_i^l for spin 1, $\epsilon_i^l\otimes\epsilon_i^r$ for spin-2 ($k_i\cdot\epsilon_i = 0\ldots$).
- Introduce skew $2n\times 2n$ matrices $M = \begin{pmatrix} A & C \\ -C^t & B\end{pmatrix}$,

$$A_{ij} = \frac{k_i\cdot k_j}{\sigma_i-\sigma_j},,\quad B_{ij} = \frac{\epsilon_i\cdot\epsilon_j}{\sigma_i-\sigma_j},\quad C_{ij} = \frac{k_i\cdot\epsilon_j}{\sigma_i-\sigma_j},\quad \text{for } i\neq j$$

  and $A_{ii} = B_{ii} = 0$, $C_{ii} = \epsilon_i\cdot P(\sigma_i)$.
- For YM, $I^l = Pf'(M)$, $I^r = \prod_i \frac{1}{\sigma_i-\sigma_{i-1}}$.
- For GR $I^l = Pf'(M^l)$, $I^r = Pf'(M^r)$.

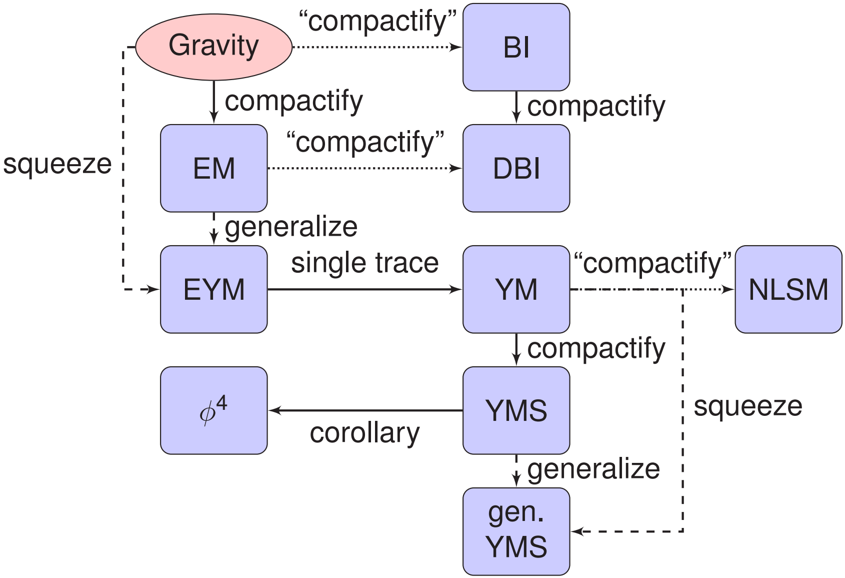


Figure: Theories studied by CHY and operations relating them.

Bosonic ambitwistor string action:

- Σ Riemann surface, coordinate $\sigma \in \mathbb{C}$
- Complexify space-time (M, g), coords $X \in \mathbb{C}^d$, g hol.
- $(X, P) : \Sigma \to T^*M, \quad P \in K$, holomorphic 1-forms on Σ.

$$S_B = \int_\Sigma P_\mu \bar{\partial} X^\mu - e\, P^2/2\,.$$

Underlying geometry:

- e enforces $P^2 = 0$,
- P^2 generates gauge freedom: $\delta(X, P, e) = (\alpha P, 0, 2\bar{\partial}\alpha)$.

So target is

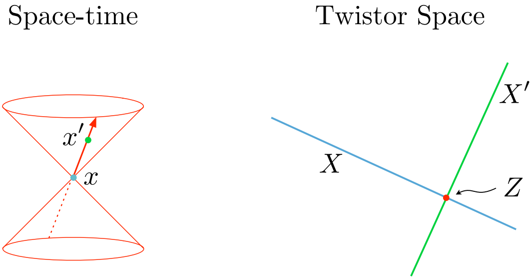
$$\mathbb{A} = T^*M|_{P^2=0}/\{\text{gauge}\}.$$

This is *Ambitwistor space*, space of complexified light rays.

The geometry of the space of light rays

Ambitwistor space $\mathbb{A}$ is space of complexified light rays.

- Light rays primary, events determined by lightcones $X \subset \mathbb{A}$ of light rays incident with x.
- Space-time $M =$ space of such $X \subset \mathbb{A}$.



Space-time geometry is encoded in complex structure of $\mathbb{A}$.

Theorem (LeBrun 1983 following Penrose 1976)

Complex structure of $\mathbb{A}$ *determines* $(M, [g])$. *Correspondence stable under deformations of* $P\mathbb{A}$ *that preserve* $\theta = P_\mu \mathrm{d}X^\mu$.

Quantize bosonic ambitwistor string:

- $(X, P) : \Sigma \to T^*M$,

$$S_B = \int_\Sigma P_\mu(\bar{\partial} + \tilde{e}\partial)X^\mu - e\,P^2/2\,.$$

- Gauge fix $\tilde{e} = e = 0$, $\rightsquigarrow$ ghosts & BRST Q
- Introduce vertex operators $V_i \leftrightarrow$ field perturbations.

Amplitudes are computed as correlators of vertex ops

$$\mathcal{M}_n = \langle V_1 \dots V_n \rangle$$

For gravity add type II worldsheet susy $S_{\Psi_1} + S_{\Psi_2}$ where

$$S_\Psi = \int_\Sigma \Psi_\mu \bar{\partial}\Psi^\mu + \chi P \cdot \Psi\,.$$

From deformations of $\mathbb{A}$ to the scattering equations

Gravitons $\leftrightarrow$ vertex operators V_i = def'm of action $\delta S = \int_\Sigma \delta\theta$.

- θ determines complex structure on $P\mathbb{A}$ via $\theta \wedge d\theta^{d-2}$. So:
- Deformations of complex structure $\leftrightarrow [\delta\theta] \in H^1_{\bar\partial}(P\mathbb{A}, L)$.

Proposition

For perturbation $\delta g_{\mu\nu} = \mathrm{e}^{ik\cdot x}\epsilon_\mu\epsilon_\nu$ of flat space-time

$$\delta\theta = \bar\delta(k \cdot P)\mathrm{e}^{ik\cdot X}(\epsilon \cdot P)^2$$

Proof: Penrose transform.

Ambitwistor repn $\Rightarrow \bar\delta(k \cdot P) \Rightarrow$ scattering equs.

Proposition

CHY formulae for massless tree amplitudes e.g. YM & gravity arise from appropriate choices of worldsheet matter.

- Take $e^{ik_i \cdot X(\sigma_i)}$ factors into action to give

$$S = \frac{1}{2\pi}\int_\Sigma P\cdot\bar\partial X + 2\pi\sum_i ik\cdot X(\sigma_i)\,.$$

- Gives field equations $\bar\partial X = 0$ and,

$$\bar\partial P = 2\pi\sum_i ik\delta^2(\sigma-\sigma_i)\,.$$

- Solutions $X(\sigma) = X = \text{const.}\,,\ P(\sigma) = \sum_i \frac{k_i}{\sigma-\sigma_i}d\sigma\,.$

Thus path-integral reduces to

$$\mathcal{M}_n = \delta^d\left(\sum_i k_i\right)\int_{(\mathbb{CP}^1)^{n-3}} \frac{\prod_i(\epsilon_i\cdot P(\sigma_i))^2\bar\delta(k_i\cdot P)}{\text{Vol } G}$$

We see $P(\sigma)$ appearing and scattering equations.

Unfortunately: amplitudes for $S\sim\int_M R + R^3$.

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We see $P(\sigma)$ appearing and scattering equations.

Unfortunately: amplitudes for $S \sim \int_M R + R^3$.

- Decorate null geodesics with spin vectors, vectors for internal degrees of freedom & other holmorphic CFTs.
- Take

$$S = S_B + S^l + S^r$$

where S^l, S^r are some worldsheet matter CFTs.

- Total vertex operators given by

$$v^l v^r \bar{\delta}(k \cdot P)\, \mathrm{e}^{ik \cdot X}$$

with v^l, v^r worldsheet currents from S^l, S^r resp..

- Amplitudes become

$$\mathcal{M}_n = \delta^d \left(\sum_i k_i \right) \int_{(\mathbb{CP}^1)^n} \frac{\mathcal{I}^l \mathcal{I}^r \prod_i{}' \bar{\delta}(k_i \cdot P)}{\text{Vol Gauge}}$$

where $\mathcal{I}^l$, $\mathcal{I}^r$ are worldsheet correlators of v^ls, v^rs resp..

- In good situations, Q-invariance and discrete symmetries (GSO) rule out unwanted vertex operators.

- **Worldsheet SUSY:** Let $\Psi^\mu \in K^{1/2}$, spin 1/2 fermions on Σ,

$$S_\Psi = \int g_{\mu\nu}\Psi^\mu \bar\partial \Psi^\nu - \chi P_\mu \Psi^\mu$$

Replace $v = \epsilon \cdot P$ by $v = \epsilon \cdot P + \epsilon \cdot \Psi k \cdot \Psi$ (or $u = \delta(\gamma)\epsilon \cdot \Psi$).
Worldsheet correlator

$$\mathcal{I}^{l/r} = \langle u_1 u_2 v_3 \dots v_n \rangle = Pf'(M)\,.$$

- **Free fermions and current algebras:** Free 'real' Fermions $\rho^a \in \mathbb{C}^m \otimes K^{1/2}$

$$S_\rho = \int_\Sigma \delta_{ab}\rho^a \bar\partial \rho^b\,, \qquad a = 1, \dots m,$$

With Lie alg structure const f^{abc}, set $v = t_a f^{abc}\rho^b\rho^c$.
Correlators $\rightsquigarrow$ 'Parke-Taylor' + unwanted multi-trace terms

$$\langle v_1 \dots v_n \rangle = \frac{\mathrm{tr}(t_1 \dots t_n)}{\sigma_{12}\sigma_{23}\cdots\sigma_{n1}} + \dots$$

where $\sigma_{ij} = \sigma_i - \sigma_j$.

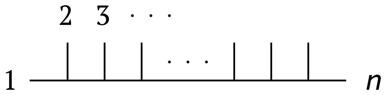
Comb system [Casali-Skinner]

Use fermions $\tilde{\rho}^a, \rho^a \in \mathfrak{g} \otimes K^{1/2}$, bosons $q^a, y^a \in \mathfrak{g} \otimes K^{1/2}$

$$S_{CS} = \int_\Sigma \tilde{\rho}_a \bar{\partial} \rho^a + q_a \bar{\partial} y^a + \chi\, \mathrm{tr} \rho \left(\frac{[\tilde{\rho}, \rho]}{2} + [q, y] \right) .$$

- Gauge fix $\chi = 0 \rightsquigarrow$ ghosts $(\beta, \gamma) \rightsquigarrow$ two fixed vertex operators to end chain of structure contants 'comb'.
- Vertex ops: $u = \delta(\gamma) t \cdot \rho\,, \quad \tilde{u} = \delta(\gamma) t \cdot \tilde{\rho}\,,$ (fixed)
  $v = t \cdot [\rho, \rho]\,, \quad \tilde{v} = t \cdot ([\tilde{\rho}, \rho] + [q, y])\,.$
- To be nontrivial, correlator must have just one untilded VO

$$\langle u_1 \tilde{u}_2 \tilde{v}_3 \ldots \tilde{v}_n \rangle = \mathcal{C}(1, \ldots, n) := \frac{\mathrm{tr}(t_1 [t_2, [t_3, \ldots [t_{n-1}, t_n] \ldots])}{\sigma_{12} \ldots \sigma_{n1}} .$$



The 2013 CHY formulae & ambitwistor models

Above lead essentially to original models & formulae:

- $(S^l, S^r) = (S_{\tilde{\Psi}}, S_\Psi) \rightsquigarrow$ type II gravity,
- $(S^l, S^r) = (S_{CS}, S_\Psi) \rightsquigarrow$ heterotic with YM,
- $(S^l, S^r) = (S_{CS}, S_{CS}) \rightsquigarrow$ bi-adjoint scalar.

The latter two come with unphysical gravity.

S_{CS} improves on current algebras in avoiding multi-trace terms and all models critical in 10d.

S_{Ψ_1,Ψ_2} $= S_{\Psi_1} + S_{\Psi_2}$ two worldsheet susy's for S^l or S^r. This is maximum. It gives VO currents

$$u = \delta(\gamma_1) k\cdot\Psi_2\,, \qquad v = k\cdot\Psi_1 k\cdot\Psi_2\,.$$

$S_{\Psi,\rho}$ $= S_\Psi + S_\rho$ combines 'real' Fermions with susy, $\rightsquigarrow$ VO currents as usual for S_Ψ and

$$u_t = \delta(\gamma) t\cdot\rho\,, \qquad v_t = k\cdot\Psi t\cdot\rho\,.$$

$S_{\Psi,CS}$ $= \int_\Sigma \Psi\cdot\bar\partial\Psi + \tilde\rho_a\bar\partial\rho^a + q_a\bar\partial y^a + \chi\left(P\cdot\Psi + \mathrm{tr}\rho\left(\frac{[\tilde\rho,\rho]}{2} + [q,y]\right)\right).$

With ghosts etc., the VO currents are those for S_Ψ and

$$\begin{aligned} \tilde u_t &= \delta(\gamma) t\cdot\tilde\rho\,, \qquad u_t = \delta(\gamma) t\cdot\rho\,, \\ \tilde v_t &= k\cdot\Psi t\cdot\tilde\rho + t\cdot([\tilde\rho,\rho] + [q,y])\,, \quad v_t = k\cdot\Psi t\cdot\rho + t\cdot[\rho,\rho]\,. \end{aligned}$$

GSO now reverses signs of all fields in matter system.

Ambitwistor strings with combinations of matter

CGMMRS 150?

S^l \ S^r	S_Ψ	S_{Ψ_1,Ψ_2}	$S^{(\tilde m)}_{\rho,\Psi}$	$S^{(\tilde N)}_{CS,\Psi}$	$S^{(\tilde N)}_{CS}$
S_Ψ	E				
S_{Ψ_1,Ψ_2}	BI	Galileon			
$S^{(m)}_{\rho,\Psi}$	*EM* $\mathrm{U}(1)^m$	DBI	*EMS* $\mathrm{U}(1)^m\times\mathrm{U}(1)^{\tilde m}$		
$S^{(N)}_{CS,\Psi}$	EYM	ext. DBI	*EYMS* $\mathrm{SU}(N)\times\mathrm{U}(1)^{\tilde m}$	*EYMS* $\mathrm{SU}(N)\times\mathrm{SU}(\tilde N)$	
$S^{(N)}_{CS}$	YM	Nonlinear σ	*EYMS* $\mathrm{SU}(N)\times\mathrm{U}(1)^{\tilde m}$	*gen. YMS* $\mathrm{SU}(N)\times\mathrm{SU}(\tilde N)$	*Biadjoint Scalar* $\mathrm{SU}(N)\times\mathrm{SU}(\tilde N)$

Table: Theories arising from the different choices of matter models.

Models from different geometric realizations of $\mathbb{A}$

Can start with other formulations of null superparticles

- Pure spinor version (Berkovits) $S = \int P \cdot \bar\partial X + p_\alpha \bar\partial \theta^\alpha + \dots$.
- In $d = 4$ have (super) Twistor space $\mathbb{T} := \mathbb{C}^{4|\mathcal{N}}$

$$\mathbb{A} = T^*\mathbb{PT} := \{(Z, W) \in \mathbb{T} \times \mathbb{T}^* | Z \cdot W = 0\} / \{Z \cdot \partial_Z - W \cdot \partial_W\}$$

$$S = \int_\Sigma W \cdot \bar\partial Z + a\, Z \cdot W$$

$\rightsquigarrow$ Twistor-strings [Witten, Berkovits & Skinner].

- In 4d have full ambitwistor representation [Geyer, Lipstein, M. 1404.6219]

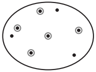
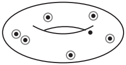
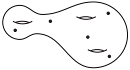
$$S = \int_\Sigma Z \cdot \bar\partial W - W \cdot \bar\partial Z + aZ \cdot W$$

Not twistor string: $(Z, W) \in K^{1/2}$ gives simpler 4d formulae with no moduli. Nonchiral, working with no supersymmetry.

Can adapt also to geometry of null infinity, $\mathbb{A} = T^*\mathscr{I}$ and connect to BMS symmetries & conformal scattering theory.

Loops

The string paradigm gives

$\mathcal{M}_n =$  $+$  $+ \ldots +$  $+ \ldots$

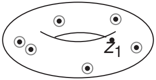
Can we make sense of this at 1 loop, i.e., on a torus?

Need critical model with all anomalies cancelling, i.e., type II super-gravity.

1-loop: the scattering equations on a torus

[Adamo, Casali, Skinner 2013, Casali Tourkine 2014 Geyer,M., Monteiro, Tourkine 2015

On torus $\Sigma_q = \mathbb{C}/\{\mathbb{Z}\oplus\mathbb{Z}\tau\}$, $q = e^{2\pi i\tau}$, solve



$$\bar\partial P = 2\pi i \sum_i k_i \bar\delta(z - z_i) dz \qquad \text{with}$$

$$P = 2\pi i\, \ell dz + \sum_i k_i \left(\frac{\theta_1'(z - z_i)}{\theta_1(z - z_i)} + \sum_{j\neq i} \frac{\theta_1'(z_{ij})}{n\,\theta_1(z_{ij})} \right) dz\,.$$

zero-modes $\ell \in \mathbb{R}^d \leftrightarrow$ loop momenta.

Scattering eqs:

$$\mathrm{Res}_{z_i} P^2 := k_i \cdot P(z_i) = 0, \quad i = 2, \ldots, n, \qquad P(z_0)^2 = 0\,.$$

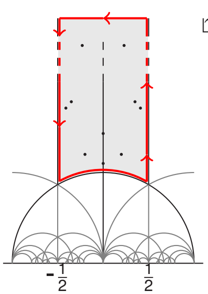
Gives amplitude formula

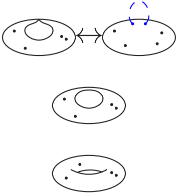
$$\mathcal{M}^{(1)}_{\mathrm{SG}} = \int \mathcal{I}_q\, d^d\ell\, d\tau\, \bar\delta(P^2(z_0)) \prod_{i=2}^{n} \bar\delta(k_i \cdot P(z_i)) dz_i\,.$$

Localizes on discrete set of solutions to scattering eqs.
With $\mathcal{I}_q = 1$, conjectured to be permutations sum of n-gons.

From the elliptic curve to the Riemann sphere

[Geyer, M., Monteiro, Tourkine 1507.00321]





$\sum$ {residues at $P^2(z_0) = 0$} = {residue at $q = 0$} so

$$
\begin{aligned}
\mathcal{M}_n^{(1)} &= \int \mathcal{I}_q \, d^d\ell \, \frac{dq}{q} \, \bar{\partial} \frac{1}{P^2(z_0)} \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(z_i)) dz_i \, , \\
&= -\int \mathcal{I}_0 \, d^d\ell \, \frac{1}{\ell^2} \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i^2} \, ,
\end{aligned}
$$

Off-shell scattering eqs and n-gon conjecture

At $q = 0$

$$P(z) = P(\sigma) = \ell \, \frac{d\sigma}{\sigma} + \sum_{i=1}^{n} \frac{k_i \, d\sigma}{\sigma - \sigma_i} \, .$$

Set $S = P^2 - \ell^2 \, \mathrm{d}\sigma^2/\sigma^2$, gives *off-shell scattering equations*:

$$0 = \mathrm{Res}_{\sigma_i} S = k_i \cdot P(\sigma_i) = \frac{k_i \cdot \ell}{\sigma_i} + \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} \, .$$

The n-gon conjecture becomes

$$\mathcal{M}^{(1)}_{n-\mathrm{gon}} = - \int d^{2n+2}\ell \, \frac{1}{\ell^2} \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i^2} \, ,$$

which yields

$$\mathcal{M}^{(1)}_{n} = \frac{(-1)^n}{\ell^2} \sum_{\sigma \in S_n} \prod_{i=1}^{n-1} \frac{1}{\ell \cdot K_{\sigma_i} + \frac{1}{2} K_{\sigma_i}^2} \, , \qquad K_{\sigma_i} = \sum_{j=1}^{i} k_{\sigma_i(j)}$$

Partial fractions + shifts in ℓ gives permutation sum of n-gons.

Supergravity 1-loop integrand

For supergravity $\mathcal{I}_q = \mathcal{I}_q^L \mathcal{I}_q^R$ with $\mathcal{I}^{L/R} \equiv \mathcal{I}^{L/R}(k_i, \epsilon_i^{L/R}, z_i|q)$.
At $q = 0$

$$\mathcal{I}_0^{L/R} = 16 \left(\mathrm{Pf}(M_2^{L/R}) - \mathrm{Pf}(M_3^{L/R}) \right) - 2\, \partial_{q^{1/2}} \mathrm{Pf}(M_3^{L/R}) \,,$$

So the 1-loop supergravity integrand is

$$\mathcal{M}_n^{(1)} = - \int \mathcal{I}_0^L \mathcal{I}_0^R \, \frac{1}{\ell^2} \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i^2} \,.$$

Checked at 4 points algebraically and 5 points numerically.

Super Yang-Mills 1-loop integrand

This leads to conjecture for super Yang-Mills at 1 loop;

$$\mathcal{M}_n^{(1)} = \int \mathcal{I}_0^L \, PT_n \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i} \, .$$

Here, $\mathcal{I}_0^R$ factor is replaced by cyclic sum of Parke-Taylors running through loop,

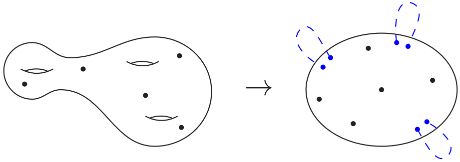
$$PT_n = \sum_{i=1, i \bmod n}^{n} \frac{\sigma_{0\,\infty}}{\sigma_{0\,i}\sigma_{i\,i+1}\sigma_{i+1\,i+2}\cdots\sigma_{i+n\,\infty}} \, .$$

Checked at 4 and 5 points.

PT_n^2 integrand also work for bi-adjoint scalar [Bjerrum-Bohr, Bourjailly, Damsgaard] & [He & Yuan]. $\rightsquigarrow$ KLT at 1-loop.

Outlook: All-loop Scattering equations on $\mathbb{CP}^1$

Use residue thms to localize genus g moduli integrals to bdy cpt with g a-cycles contracted $\rightsquigarrow \mathbb{CP}^1$ with g nodes.



Fixes g moduli, remaining $2g-3 \leftrightarrow 2g$ new marked points. 1-form P becomes

$$P = \sum_{r=1}^{g} \ell_r \omega_r + \sum_i k_i \frac{d\sigma}{\sigma - \sigma_i} \,,$$

here ω_r is basis of g global holomorphic 1-forms on nodal $\mathbb{CP}^1$. Set $S(\sigma) := P^2 - \sum_{r=1}^{g} \ell_r^2 \omega_r^2$, off-shell scattering equations are

$$\mathrm{Res}_{\sigma_i} S = 0 \,, \quad i = 1, \ldots, n+2g, .$$

Leads to proposal for all-loop integrand;

$$\mathcal{M}_n^{(g)} = \int_{(\mathbb{CP}^1)^{n+2g}} d^{dg}\ell \, \frac{\mathcal{I}_0^L \mathcal{I}_0^R}{\mathrm{Vol}\, G} \prod_{r=1}^{g} \frac{1}{\ell_r^2} \prod_{i=1}^{n+2g} \bar{\delta}(\mathrm{Res}_{\sigma_i} S(\sigma_i)) \, ,$$

$$\text{where } \mathcal{I}_0 = \begin{cases} \mathcal{I}_0^L \mathcal{I}_0^R, & \text{gravity} \\ \mathcal{I}_0^L PT_n, & \text{Yang-Mills} \\ PT_n PT_n' & \text{biadjoint scalar} \end{cases} .$$

Suggests n-point g-loop integrands have similar complexity to $n+2g$-point tree amplitudes.

Chiral $\alpha' = 0$ ambitwistor strings use LeBrun's correspondence to give theories generalizing twistor-strings to CHY formulae.

- Incorporates colour/kinematics Yang-Mills/gravity ideas.
- Extends to many theories from DBI to Nonlinear Sigma models.
- Critical models extend to loops on a Riemann surface.
- Higher genus Riemann surface formulae reduce to simpler formulae on $\mathbb{CP}^1$.
- *Off-shell scattering equations* on $\mathbb{CP}^1$ can be used to find loop integrands for non-critical models.
- Gives canonical choice of loop momenta.

Can we do optimal powercounting for $\mathcal{N} = 8$ SUGRA?

Thank You