Ambitwistor strings and the scattering equations at one loop

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[Cf. also Cachazo, He, Yuan arxiv:1306.2962, 1306.6575, 1307.2199, 1309.0885, 1412.3479]
Ambitwistor spaces: spaces of complex null geodesics.

- Extends Penrose/Ward’s gravity/Yang-Mills twistor constructions to non-self-dual fields.

Ambitwistor Strings:

- Tree S-Matrices in all dimensions for gravity, YM etc. [CHY]
- From strings in ambitwistor space [M. & Skinner 1311.2564]
- New models for Einstein-YM, DBI, BI, NLS, etc. [Casali, Geyer, M., Monteiro, Roehrig 1506.08771].
- Loop integrands from the Riemann sphere [Geyer, M., Monteiro, Tourkine, 1507.00321].

Provide string theories at $\alpha' = 0$ for field theory amplitudes.
Amplitudes from Feynman diagrams

Amplitudes are realized as sums of Feynman integrals.

Consider the five-gluon tree-level amplitude of QCD. Enters in calculation of multi-jet production at hadron colliders.

Described by following Feynman diagrams:

If you follow the textbooks you discover a disgusting mess.

Trees ↔ classical, loops ↔ quantum.
Need for new ideas

Result of a brute force calculation:

\[ k_1 \cdot k_4 \varepsilon_2 \cdot k_1 \varepsilon_1 \cdot \varepsilon_3 \varepsilon_4 \cdot \varepsilon_5 \]
The scattering equations

Take $n$ null momenta $k_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, $k_i^2 = 0$, $\sum_i k_i = 0$,

- define $P : \mathbb{C}P^1 \to \mathbb{C}^d$

\[
P(\sigma) := \sum_{i=1}^n \frac{k_i}{\sigma - \sigma_i}, \quad \sigma, \sigma_i \in \mathbb{C}P^1
\]

- Solve for $\sigma_i \in \mathbb{C}P^1$ with the $n$ scattering equations [Fairlie 197?]

\[
\text{Res}_{\sigma_i} \left( P^2 \right) = k_i \cdot P(\sigma_i) = \sum_{j=1}^n \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0.
\]

$\Rightarrow P^2 = 0 \ \forall \sigma$.

- For Mobius invariance $\Rightarrow P \in \mathbb{C}^d \otimes K$, $K = \Omega^{1,0} \mathbb{C}P^1$

- There are $(n-3)!$ solutions.

Amplitude formulae for massless theories.

Proposition (Cachazo, He, Yuan 2013,2014)

Tree-level massless amplitudes in $d$-dims are integrals/sums

$$\mathcal{M}_n = \delta^d \left( \sum_i k_i \right) \int_{(\mathbb{CP}^1)^n} \frac{l^l l^r \prod_i \delta(k_i \cdot P(\sigma_i))}{\text{Vol SL}(2, \mathbb{C}) \times \mathbb{C}^3}$$

where $l^l, l^r = l^l, l^r (\epsilon_i^l, k_i, \sigma_i)$ depend on the theory.

- polarizations $\epsilon_i^l$ for spin 1, $\epsilon_i^l \otimes \epsilon_i^r$ for spin-2 ($k_i \cdot \epsilon_i = 0 \ldots$).
- Introduce skew $2n \times 2n$ matrices $M = \begin{pmatrix} A & C \\ -C^t & B \end{pmatrix}$,

$$A_{ij} = \frac{k_i \cdot k_j}{\sigma_i - \sigma_j}, \quad B_{ij} = \frac{\epsilon_i \cdot \epsilon_j}{\sigma_i - \sigma_j}, \quad C_{ij} = \frac{k_i \cdot \epsilon_j}{\sigma_i - \sigma_j}, \quad \text{for } i \neq j$$

and $A_{ii} = B_{ii} = 0$, $C_{ii} = \epsilon_i \cdot P(\sigma_i)$.
- For YM, $l^l = Pf'(M)$, $l^r = \prod_i \frac{1}{\sigma_i - \sigma_{i-1}}$.
- For GR $l^l = Pf'(M^l)$, $l^r = Pf'(M^r)$. 


More CHY formulae:

Gravity

EM

EYM

$\phi^4$

BI

DBI

YM

YMS

NLSM

Figure: Theories studied by CHY and operations relating them.
Chiral bosonic strings at $\alpha' = 0$

Bosonic ambitwistor string action:

- $\Sigma$ Riemann surface, coordinate $\sigma \in \mathbb{C}$
- Complexify space-time $(M, g)$, coords $X \in \mathbb{C}^d$, $g$ hol.
- $(X, P) : \Sigma \to T^* M$, $P \in K$, holomorphic 1-forms on $\Sigma$.

$$S_B = \int_{\Sigma} P_\mu \bar{\partial} X^\mu - e P^2 / 2.$$

Underlying geometry:

- $e$ enforces $P^2 = 0$,
- $P^2$ generates gauge freedom: $\delta(X, P, e) = (\alpha P, 0, 2\bar{\partial}\alpha)$.

So target is

$$\mathbb{A} = T^* M|_{P^2 = 0}/\{\text{gauge}\}.$$

This is Ambitwistor space, space of complexified light rays.
Ambitwistor space $\mathbb{A}$ is space of complexified light rays.

- Light rays primary, events determined by lightcones $X \subset \mathbb{A}$ of light rays incident with $x$.
- Space-time $M = \text{space of such } X \subset \mathbb{A}$.

Space-time geometry is encoded in complex structure of $\mathbb{A}$.

**Theorem (LeBrun 1983 following Penrose 1976)**

*Complex structure of $\mathbb{A}$ determines $(M, [g])$. Correspondence stable under deformations of $P_{\mathbb{A}}$ that preserve $\theta = P_\mu dX^\mu$.***
Amplitudes from ambitwistor strings

Quantize bosonic ambitwistor string:

- \((X, P) : \Sigma \rightarrow T^* M, \)

\[
S_B = \int_{\Sigma} P_\mu (\bar{\partial} + \tilde{e} \partial) X^\mu - e P^2 / 2 .
\]

- Gauge fix \(\tilde{e} = e = 0, \sim \) ghosts & BRST \(Q\)
- Introduce vertex operators \(V_i \leftrightarrow \) field perturbations.

Amplitudes are computed as correlators of vertex ops

\[
\mathcal{M}_n = \langle V_1 \ldots V_n \rangle
\]

For gravity add type II worldsheet susy \(S_{\psi_1} + S_{\psi_2}\) where

\[
S_\psi = \int_{\Sigma} \psi_\mu \bar{\partial} \psi^\mu + \chi P \cdot \psi .
\]
Gravitons ↔ vertex operators $V_i$ = def’m of action $\delta S = \int_\Sigma \delta \theta$.

- $\theta$ determines complex structure on $PA$ via $\theta \wedge d\theta^{d-2}$. So:
- Deformations of complex structure $\leftrightarrow [\delta \theta] \in H^1_\partial(PA, L)$.

Proposition

For perturbation $\delta g_{\mu\nu} = e^{ik \cdot x} \epsilon_{\mu} \epsilon_{\nu}$ of flat space-time

$$\delta \theta = \bar{\delta}(k \cdot P)e^{ik \cdot X}(\epsilon \cdot P)^2$$

Proof: Penrose transform.

Ambitwistor repn $\Rightarrow \bar{\delta}(k \cdot P) \Rightarrow$ scattering equus.

Proposition

CHY formulae for massless tree amplitudes e.g. YM & gravity arise from appropriate choices of worldsheet matter.
• Take $e^{ik_i \cdot X(\sigma_i)}$ factors into action to give

$$S = \frac{1}{2\pi} \int \Sigma P \cdot \bar{\partial} X + 2\pi \sum_i ik \cdot X(\sigma_i).$$

• Gives field equations $\bar{\partial} X = 0$ and,

$$\bar{\partial} P = 2\pi \sum_i ik \delta^2(\sigma - \sigma_i).$$

• Solutions $X(\sigma) = X = \text{const.}$, $P(\sigma) = \sum_i \frac{k_i}{\sigma - \sigma_i} d\sigma$.

Thus path-integral reduces to

$$\mathcal{M}_n = \delta^d \left( \sum_i k_i \right) \int (\mathbb{C}P^1)^{n-3} \frac{\prod_i (\epsilon_i \cdot P(\sigma_i))^2 \delta(k_i \cdot P)}{\text{Vol } G}$$

We see $P(\sigma)$ appearing and scattering equations.

Unfortunately: amplitudes for $S \sim \int_M R + R^3$. 

Evaluation of amplitude

- Take $e^{ik_i \cdot X(\sigma_i)}$ factors into action to give

$$S = \frac{1}{2\pi} \int_{\Sigma} P \cdot \bar{\partial} X + 2\pi \sum_i ik \cdot X(\sigma_i).$$

- Gives field equations $\bar{\partial} X = 0$ and,

$$\bar{\partial} P = 2\pi \sum_i ik \delta^2(\sigma - \sigma_i).$$

- Solutions $X(\sigma) = X = \text{const.}$, $P(\sigma) = \sum i \frac{k_i}{\sigma - \sigma_i} d\sigma$.

Thus path-integral reduces to

$$\mathcal{M}_n = \delta^d \left( \sum_i k_i \right) \int_{(\mathbb{CP}^1)^{n-3}} \frac{\prod_i (\epsilon_i \cdot P(\sigma_i))^2 \delta(k_i \cdot P)}{\text{Vol } G}$$

We see $P(\sigma)$ appearing and scattering equations.

**Unfortunately:** amplitudes for $S \sim \int_M R + R^3$. 
Worldsheet matter

- Decorate null geodesics with spin vectors, vectors for internal degrees of freedom & other holomorphic CFTs.
- Take
  \[ S = S_B + S^l + S^r \]
  where \( S^l, S^r \) are some worldsheet matter CFTs.
- Total vertex operators given by
  \[ \nu^l \nu^r \delta(k \cdot P) e^{i k \cdot X} \]
  with \( \nu^l, \nu^r \) worldsheet currents from \( S^l, S^r \) resp..
- Amplitudes become
  \[ \mathcal{M}_n = \delta^d \left( \sum_i k_i \right) \int_{(\mathbb{C}P^1)^n} \frac{\mathcal{I}^l \mathcal{I}^r \prod_i' \delta(k_i \cdot P)}{\text{Vol Gauge}} \]
  where \( \mathcal{I}^l, \mathcal{I}^r \) are worldsheet correlators of \( \nu^l \)s, \( \nu^r \)s resp..
- In good situations, \( Q \)-invariance and discrete symmetries (GSO) rule out unwanted vertex operators.
**Worldsheet matter models**

- **Worldsheet SUSY:** Let $\psi^\mu \in K^{1/2}$, spin 1/2 fermions on $\Sigma$,

  $$S_\psi = \int g_{\mu\nu} \psi^\mu \bar{\partial} \psi^\nu - \chi P_\mu \psi^\mu$$

  Replace $\nu = \epsilon \cdot P$ by $\nu = \epsilon \cdot P + \epsilon \cdot \psi k \cdot \psi$ (or $u = \delta(\gamma) \epsilon \cdot \psi$).

- **Worldsheet correlator**

  $$\mathcal{I}^{l/r} = \langle u_1 u_2 v_3 \ldots v_n \rangle = Pf'(M).$$

- **Free fermions and current algebras:** Free ‘real’

  Fermions $\rho^a \in \mathbb{C}^m \otimes K^{1/2}$

  $$S_\rho = \int_\Sigma \delta_{ab} \rho^a \bar{\partial} \rho^b, \quad a = 1, \ldots, m,$$

  With Lie alg structure const $f^{abc}$, set $\nu = t_a f^{abc} \rho^b \rho^c$.

  Correlators $\leadsto$ ‘Parke-Taylor’ + unwanted multi-trace terms

  $$\langle v_1 \ldots v_n \rangle = \frac{\text{tr}(t_1 \ldots t_n)}{\sigma_{12} \sigma_{23} \ldots \sigma_{n1}} + \ldots$$

  where $\sigma_{ij} = \sigma_i - \sigma_j$. 
Use fermions $\tilde{\rho}^a, \rho^a \in g \otimes K^{1/2}$, bosons $q^a, y^a \in g \otimes K^{1/2}$

$$S_{CS} = \int_{\Sigma} \tilde{\rho}^a \bar{\partial} \rho^a + q^a \bar{\partial} y^a + \chi \text{tr} \rho \left( \frac{[\tilde{\rho}, \rho]}{2} + [q, y] \right).$$

- Gauge fix $\chi = 0 \leadsto$ ghosts $(\beta, \gamma) \leadsto$ two fixed vertex operators to end chain of structure contants ‘comb’.
- Vertex ops: $u = \delta(\gamma) t \cdot \rho$, $\tilde{u} = \delta(\gamma) t \cdot \tilde{\rho}$, (fixed)
  $v = t \cdot [\rho, \rho]$, $\tilde{v} = t \cdot ([\tilde{\rho}, \rho] + [q, y])$.
- To be nontrivial, correlator must have just one untilded VO

$$\langle u_1 \tilde{u}_2 \tilde{v}_3 \ldots \tilde{v}_n \rangle = C(1, \ldots, n) := \frac{\text{tr}(t_1 [t_2, [t_3, \ldots [t_{n-1}, t_n] \ldots])}{\sigma_1 \sigma_2 \ldots \sigma_{n-1}}.$$

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Above lead essentially to original models & formulae:

- \((S^l, S^r) = (S_{\bar{\Psi}}, S_{\Psi}) \leadsto \text{type II gravity,}\)
- \((S^l, S^r) = (S_{CS}, S_{\Psi}) \leadsto \text{heterotic with YM,}\)
- \((S^l, S^r) = (S_{CS}, S_{CS}) \leadsto \text{bi-adjoint scalar.}\)

The latter two come with unphysical gravity.

\(S_{CS}\) improves on current algebras in avoiding multi-trace terms and all models critical in 10d.
Combined matter systems

\[ S_{\psi_1,\psi_2} = S_{\psi_1} + S_{\psi_2} \] two worldsheet susy’s for \( S' \) or \( S'' \). This is maximum. It gives VO currents

\[ u = \delta(\gamma_1) k \cdot \psi_2, \quad v = k \cdot \psi_1 k \cdot \psi_2. \]

\[ S_{\psi,\rho} = S_{\psi} + S_{\rho} \] combines ‘real’ Fermions with susy, \( \sim \) VO currents as usual for \( S_{\psi} \) and

\[ u_t = \delta(\gamma) t \cdot \rho, \quad v_t = k \cdot \psi t \cdot \rho. \]

\[ S_{\psi,CS} = \int_\Sigma \psi \cdot \bar{\partial} \psi + \tilde{\rho}_a \bar{\partial} \rho^a + q_a \bar{\partial} y^a + \chi \left( P \cdot \psi + \text{tr} \rho \left( \frac{[\tilde{\rho},\rho]}{2} + [q,y] \right) \right). \]

With ghosts etc., the VO currents are those for \( S_{\psi} \) and

\[ \tilde{u}_t = \delta(\gamma) t \cdot \tilde{\rho}, \quad u_t = \delta(\gamma) t \cdot \rho, \]

\[ \tilde{v}_t = k \cdot \psi t \cdot \tilde{\rho} + t \cdot ([\tilde{\rho},\rho] + [q,y]), \quad v_t = k \cdot \psi t \cdot \rho + t \cdot [\rho,\rho]. \]

GSO now reverses signs of all fields in matter system.
Ambitwistor strings with combinations of matter

<table>
<thead>
<tr>
<th>$S^{'}$</th>
<th>$S^{'}$</th>
<th>$S_{\psi}$</th>
<th>$S_{\psi_1,\psi_2}$</th>
<th>$S^{(\tilde{m})}_{\rho,\psi}$</th>
<th>$S^{(\tilde{N})}_{CS,\psi}$</th>
<th>$S^{(\tilde{N})}_{CS}$</th>
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<tr>
<td>$S_{\psi}$</td>
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<td>$S_{\psi_1,\psi_2}$</td>
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<td>Galileon</td>
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<td>$S^{(m)}_{\rho,\psi}$</td>
<td>$EM$</td>
<td>$DBI$</td>
<td>$EMS$</td>
<td>$EMS$</td>
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<td>$U(1)^m \times U(1)^{\tilde{m}}$</td>
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<tr>
<td>$S^{(N)}_{CS,\psi}$</td>
<td>EYM</td>
<td>ext. DBI</td>
<td>$EYMS$</td>
<td>$EYMS$</td>
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<td>$SU(N) \times SU(\tilde{N})$</td>
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<tr>
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<td>YM</td>
<td>Nonlinear $\sigma$</td>
<td>$EYMS$</td>
<td>$EYMS$</td>
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<td>$\text{gen. YMS}$</td>
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<td>$\text{Biadjoint Scalar}$</td>
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**Table:** Theories arising from the different choices of matter models.
Models from different geometric realizations of $\mathbb{A}$

Can start with other formulations of null superparticles

- Pure spinor version (Berkovits) $S = \int P \cdot \bar{\partial}X + p_\alpha \bar{\partial} \theta^\alpha + \ldots$
- In $d = 4$ have (super) Twistor space $\mathbb{T} := \mathbb{C}^4|\mathbb{N}$

$$\mathbb{A} = T^*\mathbb{P}\mathbb{T} := \{(Z, W) \in \mathbb{T} \times T^* | Z \cdot W = 0\} / \{Z \cdot \partial Z - W \cdot \partial W\}$$

$$S = \int_\Sigma W \cdot \bar{\partial}Z + a Z \cdot W$$

$\leadsto$ Twistor-strings [Witten, Berkovits & Skinner].

- In 4d have full ambitwistor representation [Geyer, Lipstein, M. 1404.6219]

$$S = \int_\Sigma Z \cdot \bar{\partial}W - W \cdot \bar{\partial}Z + aZ \cdot W$$

Not twistor string: $(Z, W) \in K^{1/2}$ gives simpler 4d formulae with no moduli. Nonchiral, working with no supersymmetry.

Can adapt also to geometry of null infinity, $\mathbb{A} = T^*\mathbb{I}$ and connect to BMS symmetries & conformal scattering theory.
The string paradigm gives

\[ \mathcal{M}_n = \text{\includegraphics[width=1.5cm]{loop1}} + \text{\includegraphics[width=1.5cm]{loop2}} + \ldots + \text{\includegraphics[width=1.5cm]{loop3}} + \ldots \]

Can we make sense of this at 1 loop, i.e., on a torus?

Need critical model with all anomalies cancelling, i.e., type II super-gravity.
On torus $\Sigma_q = \mathbb{C}/\{\mathbb{Z} \oplus \mathbb{Z}\tau\}$, $q = e^{2\pi i \tau}$, solve

$$\bar{\partial} P = 2\pi i \sum_i k_i \delta(z - z_i) dz$$

with

$$P = 2\pi i \ell dz + \sum_i k_i \left( \frac{\theta'_1(z - z_i)}{\theta_1(z - z_i)} + \sum_{j \neq i} \frac{\theta'_1(z_{ij})}{n \theta_1(z_{ij})} \right) dz .$$

zero-modes $\ell \in \mathbb{R}^d \leftrightarrow$ loop momenta.

**Scattering eqs:**

$$\text{Res}_{z_i} P^2 := k_i \cdot P(z_i) = 0, \quad i = 2, \ldots, n, \quad P(z_0)^2 = 0 .$$

Gives amplitude formula

$$\mathcal{M}^{(1)}_{SG} = \int I_q d^d \ell \, d\tau \, \bar{\delta}(P^2(z_0)) \prod_{i=2}^n \bar{\delta}(k_i \cdot P(z_i)) dz_i .$$

Localizes on discrete set of solutions to scattering eqs. With $I_q = 1$, conjectured to be permutations sum of $n$-gons.
\sum \{\text{residues at } P^2(z_0) = 0\} = \{\text{residue at } q = 0\} \text{ so}

\[ M_{n}^{(1)} = \int I_q d^d \ell \frac{dq}{q} \frac{1}{P^2(z_0)} \prod_{i=2}^{n} \delta(k_i \cdot P(z_i)) dz_i , \]

\[ = - \int I_0 d^d \ell \frac{1}{\ell^2} \prod_{i=2}^{n} \delta(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i^2} , \]
Off-shell scattering eqs and \(n\)-gon conjecture

At \(q = 0\)

\[
P(z) = P(\sigma) = \ell \frac{d\sigma}{\sigma} + \sum_{i=1}^{n} \frac{k_i \, d\sigma}{\sigma - \sigma_i}.
\]

Set \(S = P^2 - \ell^2 \frac{d\sigma^2}{\sigma^2}\), gives off-shell scattering equations:

\[
0 = \text{Res}_{\sigma_i} S = k_i \cdot P(\sigma_i) = \frac{k_i \cdot \ell}{\sigma_i} + \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j}.
\]

The \(n\)-gon conjecture becomes

\[
\mathcal{M}_{n-gon}^{(1)} = - \int d^{2n+2} \ell \frac{1}{\ell^2} \prod_{i=2}^{n} \delta(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i^2},
\]

which yields

\[
\mathcal{M}_{n}^{(1)} = (-1)^n \sum_{\sigma \in S_n} \prod_{i=1}^{n-1} \frac{1}{\ell \cdot K_{\sigma_i} + \frac{1}{2} K_{\sigma_i}^2}, \quad K_{\sigma_i} = \sum_{j=1}^{i} k_{\sigma_i(j)}
\]

Partial fractions + shifts in \(\ell\) gives permutation sum of \(n\)-gons.
For supergravity $I_q = I_q^L I_q^R$ with $I^{L/R} \equiv I^{L/R}(k_i, \epsilon^{L/R}_i, z_i|q)$. At $q = 0$

\[ I_0^{L/R} = 16 \left( \text{Pf}(M_2^{L/R}) - \text{Pf}(M_3^{L/R}) \right) - 2 \partial_{q^{1/2}} \text{Pf}(M_3^{L/R}), \]

So the 1-loop supergravity integrand is

\[ M_n^{(1)} = -\int I_0^L I_0^R \frac{1}{\ell^2} \prod_{i=2}^n \delta(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i^2}. \]

Checked at 4 points algebraically and 5 points numerically.
Super Yang-Mills 1-loop integrand

This leads to conjecture for super Yang-Mills at 1 loop;

\[ M_n^{(1)} = \int \mathcal{I}_0^L \ PT_n \prod_{i=2}^{n} \frac{\delta(k_i \cdot P(\sigma_i))}{\sigma_i} \ d\sigma_i. \]

Here, \( \mathcal{I}_0 \) factor is replaced by cyclic sum of Parke-Taylors running through loop,

\[ PT_n = \sum_{i=1, i \ mod \ n}^{n} \frac{\sigma_0 \infty}{\sigma_0 \ i \sigma_i \ i+1 \sigma_i+1 \ i+2 \ \cdots \ \sigma_i+n \infty}. \]

Checked at 4 and 5 points.

\( PT_n^2 \) integrand also work for bi-adjoint scalar [Bjerrum-Bohr, Bourjailly, Damsgaard] & [He & Yuan]. \( \mapsto \) KLT at 1-loop.
Outlook: All-loop Scattering equations on $\mathbb{CP}^1$

Use residue thms to localize genus $g$ moduli integrals to bdy cpt with $g$ a-cycles contracted $\sim \mathbb{CP}^1$ with $g$ nodes.

Fixes $g$ moduli, remaining $2g - 3 \leftrightarrow 2g$ new marked points.

1-form $P$ becomes

$$P = \sum_{r=1}^{g} \ell_r \omega_r + \sum_{i} k_i \frac{d\sigma}{\sigma - \sigma_i},$$

here $\omega_r$ is basis of $g$ global holomorphic 1-forms on nodal $\mathbb{CP}^1$. Set $S(\sigma) := P^2 - \sum_{r=1}^{g} \ell_r^2 \omega_r^2$, off-shell scattering equations are

$$\text{Res}_{\sigma_i} S = 0, \quad i = 1, \ldots, n + 2g.$$
Outlook: All-loop integrands on $\mathbb{CP}^1$

Leads to proposal for all-loop integrand;

$$\mathcal{M}_n^{(g)} = \int_{(\mathbb{CP}^1)^{n+2g}} d^d g_\ell \frac{\mathcal{I}_0^L \mathcal{I}_0^R}{\text{Vol } G} \prod_{r=1}^{g} \frac{1}{\ell_r^2} \prod_{i=1}^{n+2g} \delta(\text{Res}_{\sigma_i} S(\sigma_i)),$$

where $\mathcal{I}_0 = \begin{cases} \mathcal{I}_0^L \mathcal{I}_0^R, & \text{gravity} \\ \mathcal{I}_0^L PT_n, & \text{Yang-Mills} \\ PT_n PT'_n & \text{biadjoint scalar} \end{cases}.$

Suggests $n$-point $g$-loop integrands have similar complexity to $n + 2g$-point tree amplitudes.
Chiral $\alpha' = 0$ ambitwistor strings use LeBrun’s correspondence to give theories generalizing twistor-strings to CHY formulae.

- Incorporates colour/kinematics Yang-Mills/gravity ideas.
- Extends to many theories from DBI to Nonlinear Sigma models.
- Critical models extend to loops on a Riemann surface.
- Higher genus Riemann surface formulae reduce to simpler formulae on $\mathbb{CP}^1$.
- *Off-shell scattering equations* on $\mathbb{CP}^1$ can be used to find loop integrands for non-critical models.
- Gives canonical choice of loop momenta.

Can we do optimal powercounting for $\mathcal{N} = 8$ SUGRA?
Thank You