The Geroch group in Einstein spaces

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Highlights

Motivations

From four to three dimensions and back to four

The sigma-model

Conservation laws, integrability and solution generation

Outlook

Framework

Solution-generating algebraic methods for Einstein's equations

- Always: give a deeper perspective on
  - the structure of the space of solutions
  - integrability properties
- Often:
  - assume extra symmetry
  - based on a mini-superspace analysis of the eoms
- Sometimes: provide new solutions

Here

Explore Geroch's approach for $R_{ab} = \Lambda g_{ab}$

- Originally: $R_{ab} = 0$ [Ehlers '59; Geroch '71]
  - $(\mathcal{M}, \mathrm{g}, \xi) \to (\mathcal{S}, \mathrm{h}) \to (\mathcal{S}, \mathrm{h'}) \to (\mathcal{M}, \mathrm{g'}, \xi')$
  - h → h': algebraic action of $SL(2, \mathbb{R})$
  - no integrability discussion
- Before: Ernst method with 2 Killings [Ernst '68]
- After: general integrability properties with 2 Killings → 2-dim sigma-models (Lax pairs, inverse scattering, ...)
  - powerful and complementary wrt algebraic (Geroch) [Belinskii, Zakharov '78; Maison '79; Bernard, Regnault '01]
  - no mention of Λ: hard problem [Astorino '12]

Results [Leigh, Petkou, Petropoulos, Tripathy '14]

Unified treatment for $\Lambda = 0$ or $\neq 0$ thanks to the conformal mode κ

- Mapping to a 3-dim sigma-model: $(\kappa, \omega, \lambda)$-target space conformal to $\mathbb{R} \times H_2$
- Geroch's $SL(2, \mathbb{R}) \equiv$ isometry – partly broken by the potential
  - reduced algebraic solution-generating action
  - no effect on integrability
- Mini-superspace analysis: h on $\mathcal{S} \propto \mathbb{R} \times \mathcal{S}_2$
  - particle motion on $\mathbb{R} \times H_2$ at zero energy
  - integrability using Hamilton–Jacobi
  - Λ: constant of motion as m and n

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4-dim $\mathcal{M}$

With $g = g_{ab}dx^a dx^b$ $(-+++)$ and a time-like Killing field ξ

- norm: $\lambda = \|\xi\|^2 < 0$
- twist 1-form: $\Omega = -2\mathrm{i}_\xi \star \mathrm{d}\xi$

Assuming $\mathrm{Ric} = \Lambda \mathrm{g}$

$$\mathrm{d} \star \mathrm{d}\xi = 2\Lambda \star \xi$$
$$\Downarrow$$
$$\mathrm{d}\Omega = 0$$

Locally scalar twist $\Omega = \mathrm{d}\omega$

3-dim $\mathcal{S}$

$\mathcal{S}$: coset space obtained by modding out the group generated by ξ

- Natural pos. def. metric/projector: $h_{ab} = g_{ab} - \frac{\xi_a \xi_b}{\lambda}$
- Natural fully antisymmetric tensor: $\eta_{abc} = \frac{-1}{\sqrt{-\lambda}} \eta_{abcd} \xi^d$
- One-to-one correspondence between tensors on $\mathcal{S}$ and tensors T on $\mathcal{M}$ s.t. $\mathrm{i}_\xi T = 0$ and $\mathcal{L}_\xi T = 0$:

$$T^{\mathcal{S}}{}_{a_1 \ldots a_p}{}^{b_1 \ldots b_q} = h^{m_1}_{a_1} \ldots h^{m_p}_{a_p} h^{b_1}_{n_1} \ldots h^{b_q}_{n_q} T^{\mathcal{M}}{}_{m_1 \ldots m_p}{}^{n_1 \ldots n_q}$$

- Induced connection on $\mathcal{S}$ – coinciding with Levi–Civita

$$\mathcal{D}_c T_{a_1 \ldots a_p}{}^{b_1 \ldots b_q} = h^{\ell}_{c} h^{m_1}_{a_1} \ldots h^{m_p}_{a_p} h^{b_1}_{n_1} \ldots h^{b_q}_{n_q} \nabla_\ell T_{m_1 \ldots m_p}{}^{n_1 \ldots n_q}$$

  with curvature

$$\mathcal{R}_{abcd} = h^{\,p}_{[a} h^{q}_{b]} h^{\,r}_{[c} h^{s}_{d]} \left(R_{pqrs} + \frac{2}{\lambda} \left(\nabla_p \xi_q \nabla_r \xi_s + \nabla_p \xi_r \nabla_q \xi_s \right) \right)$$

Dynamics for g on $\mathcal{M}$ translates into dynamics for (h, ω, λ) on $\mathcal{S}$

- Dynamics for g on $\mathcal{M}$: $R_{ab} = \Lambda g_{ab}$
- Dynamics for (h, ω, λ) on $\mathcal{S}$:

$$\begin{aligned}
\mathcal{R}_{ab} &= \frac{1}{2\lambda^2}\left(\mathcal{D}_a\omega\mathcal{D}_b\omega - h_{ab}\mathcal{D}^c\omega\mathcal{D}_c\omega\right) + \frac{1}{2\lambda}\mathcal{D}_a\mathcal{D}_b\lambda \\
&\quad - \frac{1}{4\lambda^2}\mathcal{D}_a\lambda\mathcal{D}_b\lambda + \Lambda h_{ab} \\
\mathcal{D}^2\lambda &= \frac{1}{2\lambda}\left(\mathcal{D}^c\lambda\mathcal{D}_c\lambda - 2\mathcal{D}^c\omega\mathcal{D}_c\omega\right) - 2\Lambda\lambda \\
\mathcal{D}^2\omega &= \frac{3}{2\lambda}\mathcal{D}^c\lambda\mathcal{D}_c\omega
\end{aligned}$$

Any new solution (h', ω', λ') on $\mathcal{S}$ translates into a new solution g' on $\mathcal{M}$ with Killing ξ' – a new Einstein space with symmetry

- Define a 2-form on $\mathcal{S}$: $\mathsf{F}' = \frac{1}{(-\lambda')^{3/2}} \star^3_{\mathsf{h}'} \mathsf{d}\omega'$
- Check it is *closed*
- Locally: $\mathsf{F}' = \mathsf{d}\eta'$
- Promote η' on $\mathcal{M}$ by adding a longit. comp. s.t. $\mathsf{i}_\xi \eta' = 1$
- New Killing on $\mathcal{M}$: $\xi' = \eta' \lambda'$
- New Einstein metric on $\mathcal{M}$: $g'_{ab} = h'_{ab} + \frac{\xi'_a \xi'_b}{\lambda'}$

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Introduce a reference metric $\hat{h}$:

$$h_{ab} = \frac{\kappa}{\lambda}\hat{h}_{ab}$$

(in Geroch $\tilde{h}_{ab} = \lambda h_{ab} = \kappa\hat{h}_{ab}$)

Eqs. for $\hat{h}, \kappa, \tau = \omega + i\lambda$ follow from

$$S = \int_{\mathcal{S}} \mathrm{d}^3x \sqrt{\hat{h}}\mathcal{L}$$

$$\mathcal{L} = -\sqrt{-\kappa}\left(\frac{\hat{\mathfrak{D}}^a\kappa\hat{\mathfrak{D}}_a\kappa}{2\kappa^2} + 2\frac{\hat{\mathfrak{D}}^a\tau\hat{\mathfrak{D}}_a\bar{\tau}}{(\tau-\bar{\tau})^2} + \hat{\mathfrak{R}} - 4i\Lambda\frac{\kappa}{\tau-\bar{\tau}}\right)$$

- $\hat{h}_{ab}$: gravity in 3 dim with dilaton-Einstein–Hilbert action
- κ, τ: matter with sigma-model kinetic term plus potential

Symmetries

Kinetic term for the matter fields κ, ω, λ: target space

$$ds^2_{target} = \sqrt{-\kappa}\left(-\frac{d\kappa^2}{\kappa^2} + \frac{d\omega^2 + d\lambda^2}{\lambda^2}\right)$$

- Conformal to $\mathbb{R} \times H_2$
- Conformal isometry group: $\mathbb{R}$ generated by $\zeta = \frac{1}{2}\kappa\partial_\kappa$
- Isometry group: $SL(2, \mathbb{R})$ generated by

$$\xi_+ = \partial_\omega \quad \xi_- = \left(\lambda^2 - \omega^2\right)\partial_\omega - 2\omega\lambda\partial_\lambda \quad \xi_2 = \omega\partial_\omega + \lambda\partial_\lambda$$

$$[\xi_+, \xi_-] = -2\xi_2 \quad [\xi_+, \xi_2] = \xi_+ \quad [\xi_2, \xi_-] = \xi_-$$

Potential for the matter fields κ, ω, λ:

$$\mathcal{V} = \sqrt{-\kappa}\left(\hat{\mathfrak{R}} - 2\Lambda\frac{\kappa}{\lambda}\right)$$

Λ breaks ξ_- and ξ_2

Next

- *Integrability properties and solution generation*
  - *Assume a further Killing for g: 2-dim Ernst-like sigma model (Lax pairs, inverse scattering, ...)*
  - *Freeze $\hat{h}$: 1-dim sigma model – particle motion (Hamilton–Jacobi)*
- *Role of the dilaton-like field κ*

Mini-superspace analysis

Freeze $\hat{h}$ to $\mathbb{R} \times \mathcal{S}_2$ – motivation: Taub–NUT, Schwarzschild

$$d\hat{s}^2 = d\sigma^2 + d\Omega^2$$

- $\mathrm{d}\Omega^2$: 2-dim, σ-independent $\to \hat{\mathcal{R}}_{ab}\mathrm{d}x^a\mathrm{d}x^b = \frac{\hat{\mathcal{R}}}{2}\mathrm{d}\Omega^2$
- Matter: $\kappa(\sigma), \omega(\sigma)$ and $\lambda(\sigma)$

Impose in equations and check consistency

- In $\hat{h}_{ab}$ equations
  - Trace part: κ-equation (as in the generic case)
  - Transverse part: consistency condition

$$\hat{\mathfrak{R}} = \frac{2}{(\tau - \bar{\tau})^2}\dot{\tau}\dot{\bar{\tau}} + 4i\Lambda\frac{\kappa}{\tau - \bar{\tau}} + \frac{1}{2\kappa^2}\dot{\kappa}^2$$

  - extended symmetry: $\hat{\mathfrak{R}} = 2\ell$, $\ell = 1, 0, -1$
  - constraint (first-order equation)
- Dynamics: particle motion on ds^2_{target} with V subject to $H = 0$

$$L = \frac{\sqrt{-\kappa}}{2}\left[-\left(\frac{\dot{\kappa}}{\kappa}\right)^2 - 4\frac{\dot{\tau}\dot{\bar{\tau}}}{(\tau - \bar{\tau})^2} - 4\left(\ell - 2i\Lambda\frac{\kappa}{\tau - \bar{\tau}}\right)\right]$$

In summary

$$\begin{array}{c} \textit{4-dim Einstein space with symmetry } (\mathcal{M}, g, \xi) \\ \downarrow \xi \\ \textit{3-dim sigma-model } (\mathcal{S}, \hat{h}, \kappa, \tau) \\ \Big\downarrow \textit{extra } \hat{h} \textit{ isometries} \begin{cases} \mathbb{R} \times S^2 \\ \mathbb{R}^3 \\ \mathbb{R} \times H_2 \end{cases} \\ \textit{1-dim "time"-}\sigma \textit{ particle dynamics} \end{array}$$

Case under investigation: 1 extra Killing field for $h \Rightarrow$ 3-dim sigma-model $\rightarrow$ 2-dim sigma-model (Ernst-like with dilaton)

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At $\Lambda = 0$*: Geroch*

The full Lagrangian is $SL(2,\mathbb{R})$*-invariant*

- Algebraic scan of the space of solutions

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \quad \kappa \text{ frozen}$$

- Integrable with space of solutions: m, n

$$\begin{cases} SO(2) \subset SL(2,\mathbb{R}) : \text{rotation in } (m,n) \\ N \subset SL(2,\mathbb{R}) : \text{homothetic transformation in } (m,n) \end{cases}$$

At $\Lambda \neq 0$: generalization

Summary

- Only ξ_+ leaves L invariant
- Integrability unaltered ($SL(2,\mathbb{R})$ not crucial)
- ξ_+ and ξ_2 generate constants of motion
- Constants of motion: Λ, m, n
- Under $N \subset SL(2,\mathbb{R})$: $(\Lambda, m, n) \to (a^2\Lambda, m/a, n/a)$
- $\kappa(\sigma)$ depends on Λ, m, n: freezing $\kappa \to$ missing solutions

In some detail

Change time $d\hat{r} = \frac{(-\kappa)^{3/2}}{-\lambda} d\sigma$ *and go to the Hamiltonian*

$$\hat{H} = \frac{\lambda}{2} p_\kappa^2 - \frac{\lambda^3}{2\kappa^2}(p_\omega^2 + p_\lambda^2) + 2\ell \frac{\lambda}{\kappa} - 2\Lambda$$

contraint to $\hat{H} = 0$

- Λ no longer any role in the symmetry: reduced to $\xi_+ \forall \Lambda$
- $SL(2, \mathbb{R})$ algebra on the phase space:

$$\hat{F}_+ = p_\omega \qquad \hat{F}_2 = \omega p_\omega + \lambda p_\lambda + 2\Lambda\ \hat{r}$$
$$\hat{F}_- = -2\omega\lambda p_\lambda - (\omega^2 - \lambda^2) p_\omega - 4\Lambda\omega\hat{r}$$

- Action on $\hat{H}$:

$$\left\{\hat{H}, \hat{F}_+\right\} = 0 \quad \left\{\hat{H}, \hat{F}_2\right\} = -\hat{H} - 2\Lambda$$
$$\left\{\hat{H}, \hat{F}_-\right\} = 2\omega\hat{H} + 4\Lambda\left(\omega + \frac{\hat{r}\lambda^3 p_\omega}{\kappa^2}\right)$$

- Conserved quantities:

$$\frac{d\hat{F}_+}{d\hat{r}} = 0 \qquad \frac{d\hat{F}_2}{d\hat{r}} = -\hat{H}$$
$$\frac{d\hat{F}_-}{d\hat{r}} = 2\omega\hat{H} + 4\Lambda\frac{\hat{r}\lambda^3 p_\omega}{\kappa^2}$$

Under the constraint $\hat{H} = 0$: $\hat{F}_+$ and $\hat{F}_2$ conserved

Hamilton–Jacobi integration

Hamilton–Jacobi:

$$\hat{H}\left(\frac{\partial S}{\partial q^i}, q^i\right) + \frac{\partial S}{\partial \hat{r}} = 0$$

not fully separable but integrable – irrespective of Λ

- With $q^i = (\kappa, \omega, \lambda)$
  - find principal solution $S\left(q^i, \hat{r}; \alpha_i\right)$
  - use $\beta^i = \frac{\partial S}{\partial \alpha_i}$ to get $q^i = q^i(\hat{r}; \alpha_j, \beta^k)$
  - use $p_i = \frac{\partial S}{\partial q^i}$ to get $p_i = p_i(\hat{r}; \alpha_j, \beta^k)$

- Partial separation: 2 commuting first integrals $\hat{F}_+$ and $\hat{H}$ with values 2ν and $\hat{E}$

$$S = W + 2\nu\omega - \hat{E}\hat{r}$$

with $W(\kappa, \lambda; \alpha_i)$ solving a pde wrt κ, λ and

$$\alpha_1 = \hat{E} + 2\Lambda \quad \alpha_2 = \nu \quad \alpha_3 = \alpha$$

$\hat{E}$ set to zero at the end

Relevant constants

$$(\alpha_1, \alpha_2, \beta^3) \Leftrightarrow (\Lambda, n, m)$$

the others can be reabsorbed in various redefinitions – Λ: effective constant of motion relaxing the Hamiltonian constraint

General solution κ, ω, λ with the reference $\hat{h}$

4-dim metric g: general (A)dS Schwarzschild Taub–NUT

$$-\frac{\Delta\lambda}{(m^2+\ell^2 n^2)\kappa}\left(dT + 4n\sqrt{m^2+\ell^2n^2}\, f_\ell\left(\frac{\chi}{2}\right)d\psi\right)^2 + \overbrace{\frac{\kappa}{\lambda}\underbrace{\left(\underbrace{\frac{dr^2}{\Delta}}_{d\sigma^2}+d\Omega^2\right)}_{\hat{h}}}^{h}$$

$\hat{r}$ traded for r and $f_\ell(\chi) = \sin^2\chi, \chi^2, \sinh^2\chi$ for $\ell = 1, 0, -1$

- $\Delta = \ell(r^2 - n^2) - 2mr - {}^{\Lambda}/_{3}\left(r^4 + 6r^2n^2 - 3n^4\right)$
- $\kappa = -{}^{\Delta}/_{m^2+\ell^2n^2}$
- $\omega = -{}^{2n}/_{3(m^2+\ell^2n^2)}\left(\Lambda r + \frac{3\ell r - 3m - 4\Lambda n^2 r}{r^2+n^2}\right)$
- $\lambda = -{}^{\Delta}/_{(m^2+\ell^2n^2)(r^2+n^2)}$

Back to Geroch: role of κ

Reference metrics:

$$h = \frac{\kappa \hat{h}}{\lambda} = \frac{\tilde{h}}{\lambda}$$

- In Geroch ($\Lambda = 0$): define $\cosh \sigma = {}^{r-m}/\sqrt{m^2+n^2}$
  - $-\kappa = \sinh^2 \sigma$
  - $-\tilde{\mathrm{h}} = -\kappa \hat{\mathrm{h}} = \sinh^2 \sigma \left(\mathrm{d}\sigma^2 + \mathrm{d}\Omega^2 \right)$

  independent of (m, n): the space of solutions is scanned while keeping $\tilde{\mathrm{h}}, \hat{\mathrm{h}}, \kappa$ frozen
- Here ($\Lambda \neq 0$):
  - $\kappa(\sigma)$ and $\kappa \hat{\mathrm{h}}(\sigma)$ depend *explicitly* on (m, n)
  - freezing $\tilde{\mathrm{h}} = \kappa \hat{\mathrm{h}}$ *à la* Geroch forbids scanning the space of solutions

Crucial role of the dilaton-like field κ for Einstein spaces

Algebraic solution generation

$\hat{F}_+$ and $\hat{F}_2$ generate $N \subset SL(2, \mathbb{R})$: $\tau \to \tau' = a(a\tau + b)$

- Affects ω by a shift: irrelevant
- Affects λ via

$$(\Lambda, m, n) \to (a^2\Lambda, {}^m/_a, {}^n/_a)$$

(homothetic transformation)

$\hat{F}_-$ is no longer an invariance generator – no algebraic relationship amongst solutions $(\kappa, \omega, \lambda)$ and $(\kappa', \omega', \lambda')$ obtained by rotating (m, n) to (m', n')

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Geroch non-compact $SL(2, \mathbb{R})$ group: tool for handling the dynamics of Einstein spaces with symmetry in a 3-dim sigma-model approach

- In general only a subgroup provides an algebraic mapping in the space of solutions: no role for $SO(2) \subset SL(2, \mathbb{R})$
- Mini-superspace integrability analysis: symmetry reduction does not affect integrability
    - role of the conformal mode κ for scanning the mass–nut space
    - Λ: constant of motion (relaxing the Hamiltonian constraint)
    - (Λ, m, n) transform homothetically under $N \subset SL(2, \mathbb{R})$
- Beyond mini-superspace: standard Lax-pair and inverse-scattering methods under investigation