Tidal deformation and dynamics of black holes

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Context

The tidal dynamics of compact bodies in general relativity is now the subject of vigourous development.

- The tidal deformation of neutron stars could have measurable effects on gravitational waves produced during inspirals, well before merger occurs. [Flanagan, Hinderer (2008); Postnikov, Prakash, Lattimer (2010); Pannarale *et al* (2011), Lackey *el al* (2012), Damour, Nagar, Villain (2012); Read *et al* (2013); Vines, Flanagan (2013)]
- Tidal interactions are important in extreme mass-ratio inspirals: tidal torquing of the large black hole leads to a significant gain of orbital angular momentum. [Hughes (2001); Martel (2004); Yunes *et al* (2010, 2011)]
- Relativistic theory of Love numbers. [Damour, Nagar (2009); Binnington, Poisson (2009); Damour, Lecian (2009); Landry, Poisson (2014)]
- I-Love-Q relations. [Yagi, Yunes (2013); Doneva, Yazadjiev, Stergioulas, Kokkotas (2013); Maselli *et al* (2013); Haskell *et al* (2014)]
- Tidal invariants for point-particle actions [Bini, Damour, Faye (2012); Dolan, Nolan, Ottewill (2014); Bini, Damour (2014)]

Goal

The main, long-term goal of this work is to develop a relativistic theory of tidal deformations and interactions that is as complete and elegant as the Newtonian theory.

In this talk I shall focus on black holes.

Outline

- Tides on Newtonian bodies, in four easy steps
- Tides on black holes, in the same four easy steps
- Conclusion

Newtonian tides: Setting and assumptions

We consider a self-gravitating body ("the body") in a generic tidal environment created by remote external matter.

The body has a mass M and radius R. It is spherical in isolation.

The body may be rotating, but we ignore the rotational deformation.

The tidal forces are **weak** and the deformation is **small**.

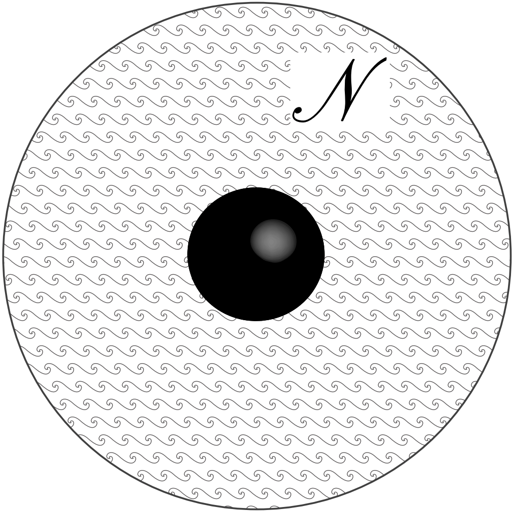
The external time scales are long compared with the time scales associated with internal processes in the body; the tides are **slow**.

We work in the noninertial frame of the moving body, with its origin at the centre-of-mass.

We focus our attention on a domain $\mathscr{N}$ that does not extend far beyond the body.

The domain $\mathscr{N}$

The local domain $\mathscr{N}$ excludes all external matter.



1. Characterize the tidal environment

The Newtonian potential in $\mathscr{N}$ is decomposed as

$$U = U_{\rm body} + U_{\rm ext}$$
$$\nabla^2 U_{\rm body} = -4\pi\rho, \qquad \nabla^2 U_{\rm ext} = 0$$

Because the external matter is remote, the external potential can be Taylor-expanded about the body's centre-of-mass,

$$U_{\rm ext}(t,\boldsymbol{x}) = U_{\rm ext}(t,\boldsymbol{0}) + g_a(t)x^a - \frac{1}{2}\mathcal{E}_{ab}(t)x^a x^b + \cdots$$
$$g_a(t) = \partial_a U_{\rm ext}(t,\boldsymbol{0}) = \text{body's CM acceleration}$$
$$\boldsymbol{\mathcal{E}_{ab}(t)} = -\partial_{ab} U_{\rm ext}(t,\boldsymbol{0}) = \text{tidal tensor}$$

The tidal tensor is **not determined** by the field equations restricted to $\mathscr{N}$; it provides a characterization of a generic tidal environment.

2. Describe the body's deformation

The deformation of the body is measured by its quadrupole-moment tensor,

$$U_{\text{body}} = \frac{M}{r} + \frac{3}{2} Q_{ab} \frac{x^a x^b}{r^5} + \cdots$$

To relate Q_{ab} to $\mathcal{E}_{ab}$ requires formulating a model for the body and solving the structure equations (eg, equations of hydrostatic equilibrium) for the perturbed configuration.

Generically,

$$Q_{ab}(t) = -\frac{2}{3} k_2 R^5 \mathcal{E}_{ab}(t - \tau)$$

$\boldsymbol{k_2}$ = gravitational Love number

$\boldsymbol{\tau}$ = viscous delay

The Love number k_2 and viscous delay τ depend on the details of internal structure, composition, dissipation mechanism, etc.

3. Deduce dynamical consequences

The tidal interaction leads to an exchange of angular momentum between the body and the external matter.

For a tidal environment in a state of rigid rotation of angular frequency $\Omega_{\rm tide}$ around the body's rotation axis,

$$\mathcal{E}_{11} = \mathcal{E}_0 + \mathcal{E}_2 \cos(2\Omega_{\rm tide} t), \qquad \mathcal{E}_{12} = \mathcal{E}_2 \sin(2\Omega_{\rm tide} t), \qquad \mathcal{E}_{13} = 0$$

$$\mathcal{E}_{22} = \mathcal{E}_0 - \mathcal{E}_2 \cos(2\Omega_{\rm tide} t), \qquad \mathcal{E}_{23} = 0, \qquad \mathcal{E}_{33} = -2\mathcal{E}_0$$

Tidal torquing

$$\frac{dS}{dt} = \frac{8}{3}(k_2\tau)R^5(\mathcal{E}_2)^2(\Omega_{\rm tide} - \Omega_{\rm body})$$

$\Omega_{\rm body}$ = body's intrinsic angular velocity

The body **spins down** when $\Omega_{\rm tide} < \Omega_{\rm body}$; it **spins up** when $\Omega_{\rm tide} > \Omega_{\rm body}$.

4. Specify the tidal environment

In order to apply the general theory, $\mathcal{E}_{ab}$ must be specified.

This requires leaving the domain $\mathscr{N}$ and identifying the source of the tidal environment.

When the body is a member of a two-body system with a companion of mass M' at position $\boldsymbol{r}(t)$,

$$U_{\text{ext}}(t, \boldsymbol{x}) = \frac{M'}{|\boldsymbol{x} - \boldsymbol{r}(t)|}$$

and $\mathcal{E}_{ab} = -\partial_{ab} U_{\text{ext}}(t, \boldsymbol{0})$ is easily computed.

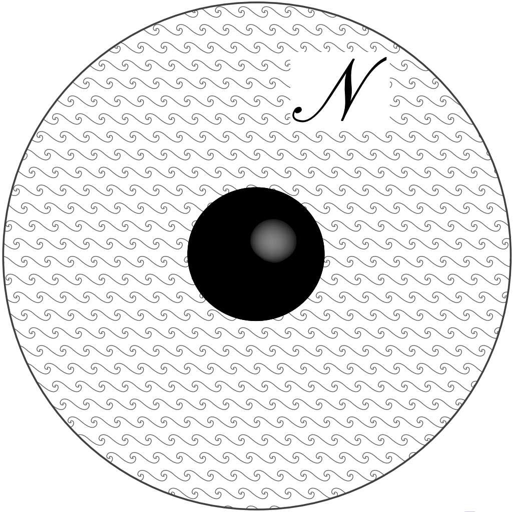
For a system in circular motion with orbital radius r,

$$\mathcal{E}_0 = -\frac{M'}{2r^3}, \qquad \mathcal{E}_2 = -\frac{3M'}{2r^3}, \qquad \Omega_{\text{tide}} = \sqrt{\frac{M+M'}{r^3}} = \Omega_{\text{orbital}}$$

This can then be substituted into the general formulae.

Relativistic tides: Setting and assumptions

The same as in the Newtonian theory.



1. Characterize the tidal environment

The metric of a slowly rotating, nearly spherical body is perturbed by a remote distribution of matter, external to the domain $\mathscr{N}$,

$$g_{\alpha\beta} = g^{\text{unpert}}_{\alpha\beta} + p^{\text{body}}_{\alpha\beta} + p^{\text{ext}}_{\alpha\beta}$$

$$\delta G_{\alpha\beta}\left[p^{\text{body}}\right] = 8\pi\delta T_{\alpha\beta}$$

$$\delta G_{\alpha\beta}\left[p^{\text{ext}}\right] = 0$$

The asymptotic behaviour of $p^{\text{ext}}_{\alpha\beta}$ is specified by **two gauge invariant tidal tensors**, $\mathcal{E}_{ab}(t)$ and $\mathcal{B}_{ab}(t)$.

These can be related to the (electric and magnetic) components of the Weyl tensor evaluated at the edge of $\mathscr{N}$.

The tidal tensors are **not determined** by the field equations restricted to $\mathscr{N}$.

2. Determine the body's deformation

The perturbation $p^{\rm body}_{\alpha\beta}$ must be continuous across the surface of a material body; this condition determines the **relativistic Love numbers** $k_2^{\rm el}$ and $k_2^{\rm mag}$, which are gauge invariant.

In the case of a black hole, regularity at the event horizon requires $p^{\rm body}_{\alpha\beta} = 0$, so that $\boldsymbol{k_2^{\rm el} = k_2^{\rm mag} = 0}$; the gravitational Love numbers of a black hole are zero.

Metric of a tidally deformed, slowly rotating black hole

$$
\begin{aligned}
g_{00} &= -1 + \frac{2M}{r} \\
&\quad - \Big(1 - \tfrac{2M}{r}\Big)^2 \mathcal{E}_{ab} x^a x^b - \Big(\tfrac{M^2}{r^2} - 4\tfrac{M^3}{r^3} + 2\tfrac{M^4}{r^4}\Big) \chi \partial_\phi\, \mathcal{E}_{ab} x^a x^b \\
&\quad + \Big(2M - \tfrac{34}{5}\tfrac{M^2}{r} + \tfrac{32}{5}\tfrac{M^3}{r^2}\Big) \chi^p \mathcal{B}_{pa} x^a - \Big(2\tfrac{M^2}{r^3} - \tfrac{8}{3}\tfrac{M^3}{r^4}\Big) \chi_{\langle a} \mathcal{B}_{bc\rangle} x^a x^b x^c
\end{aligned}
$$

where $\chi_a = S_a/M^2 \ll 1$ is the black hole's dimensionless spin.

3. Deduce dynamical consequences

The tidal torquing of a black hole can be calculated on the basis of well-known horizon flux formulae. [Teukolsky, Press (1974); Poisson (2004)]

For a tidal environment in a state of rigid rotation of angular frequency $\Omega_{\rm tide}$ around the black hole's rotation axis,

$$\mathcal{E}_{11} = \mathcal{E}_0 + \mathcal{E}_2 \cos(2\Omega_{\rm tide} v), \qquad \mathcal{E}_{12} = \mathcal{E}_2 \sin(2\Omega_{\rm tide} v), \qquad \mathcal{E}_{13} = 0$$

$$\mathcal{E}_{22} = \mathcal{E}_0 - \mathcal{E}_2 \cos(2\Omega_{\rm tide} v), \qquad \mathcal{E}_{23} = 0, \qquad \mathcal{E}_{33} = -2\mathcal{E}_0$$

$$\mathcal{B}_{11} = 0, \qquad \mathcal{B}_{12} = 0, \qquad \mathcal{B}_{13} = \mathcal{B}_1 \cos(\Omega_{\rm tide} v),$$

$$\mathcal{B}_{22} = 0, \qquad \mathcal{B}_{23} = \mathcal{B}_1 \sin(\Omega_{\rm tide} v), \qquad \mathcal{B}_{33} = 0.$$

3. Deduce dynamical consequences: continued

Tidal torquing of a black hole

$$\frac{dS}{dv} = \frac{128}{45} M^6 \left[\mathcal{E}_2^2 + \frac{1}{4}\mathcal{B}_1^2 + \frac{9}{2}\left(16\mathcal{E}_2^2 + \mathcal{B}_1^2\right)\chi M \Omega_{\rm tide} \right] \left(\Omega_{\rm tide} - \Omega_{\rm H}\right)$$

Comparison with the Newtonian expression

$$\frac{dS}{dt} = \frac{8}{3}(k_2\tau)R^5(\mathcal{E}_2)^2(\Omega_{\rm tide} - \Omega_{\rm body})$$

reveals that

$$(k_2\tau)R^5 = \frac{16}{15}M^6$$

for a black hole.

With $R \sim M$, this implies that $(k_2\tau) \sim M$.

4. Specify the tidal environment

The determination of $\mathcal{E}_{ab}(t)$ and $\mathcal{B}_{ab}(t)$ requires leaving the domain $\mathcal{N}$ and incorporating the external matter that sources the tidal field.

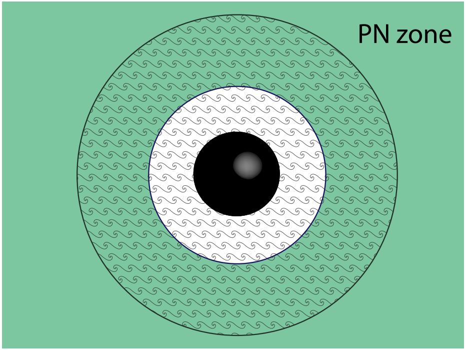
This must be done in general relativity, taking into account the nonlinearity of the field equations.

To make progress it is useful to assume that the **mutual gravity** between the black hole and the external matter is **weak**, so that the metric can be expressed as a post-Newtonian expansion.

Gravity is still strong near the black hole, but at a safe distance the metric becomes post-Newtonian.

4. Specify the tidal environment: BH metric

There is an overlap between $\mathscr{N}$ and the the post-Newtonian zone.



In this overlap, in local harmonic coordinates, the black-hole metric is

$$g_{00} = -1 + \frac{2M}{r} - \frac{2M^2}{r^2} - \left(1 - \frac{2M}{r}\right)\mathcal{E}_{ab}x^a x^b + 2M\chi^p \mathcal{B}_{pa}x^a + (2\text{PN})$$

4. Specify the tidal environment: Matching

This metric can be matched to the post-Newtonian metric that describes the entire system, black hole and external matter.

The matching requires a transformation from the global inertial frame of the post-Newtonian metric to the local frame of the moving black hole.

The matching determines the black hole's motion in the global frame, some *a priori* unknown functions that characterize the black hole in the post-Newtonian metric, and the tidal tensors.

The black hole's equations of motion can be expressed as

$$a_a = \text{geodesic forces} - M\chi^p \mathcal{B}_{pa} + (2\text{PN})$$

The second term, which arises from the dipole term in the black-hole metric, is the **Mathisson-Papapetrou spin force**. It gives rise to a piece of the spin-orbit acceleration, and all of the spin-spin acceleration.

4. Specify the tidal environment: Tidal moments

For a two-body system, for spins aligned or antialigned with the orbital angular momentum, and for circular motion, the tidal moments are determined to be

$$\mathcal{E}_0 = -\frac{M'}{2r^3}\left[1 + \frac{1}{2}qv^2 - 6q'\chi'v^3 + O(v^4)\right]$$

$$\mathcal{E}_2 = -\frac{3M'}{2r^3}\left[1 + \frac{1}{2}(3q + 4q')v^2 - 2q'\chi'v^3 + O(v^4)\right]$$

$$\mathcal{B}_1 = -\frac{3M'}{r^3}v\left[1 - q'\chi'v + O(v^2)\right]$$

$$\Omega_{\rm tide} = \sqrt{\frac{M+M'}{r^3}}\left[1 - \frac{1}{2}(3 + qq')v^2 - \frac{1}{2}\bar{\chi}v^3 + O(v^4)\right] \neq \Omega_{\rm orbital}$$

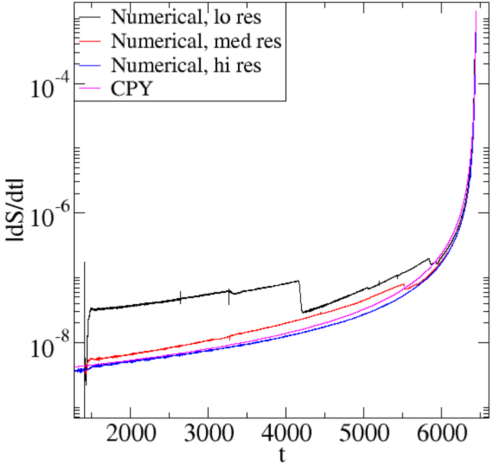
with $r =$ orbital radius, $(M, \chi) =$ mass and spin of the black hole, $(M', \chi') =$ mass and spin of the companion, $v^2 = (M + M')/r$, $q = M/(M + M')$, $q' = M'/(M + M')$, and $\bar{\chi} = q(2q + q')\chi + 3qq'\chi'$.

Comparison with numerical relativity

The Caltech-Cornell-CITA collaboration has recently made a **preliminary measurement** of the tidal torquing of a black hole during a simulated inspiral. This can be compared with the 1.5PN tidal torquing formula.

[Chatziioannou, Poisson, Yunes, Scheel (2014)]

$$16 > r/(M + M') > 2$$



Conclusion

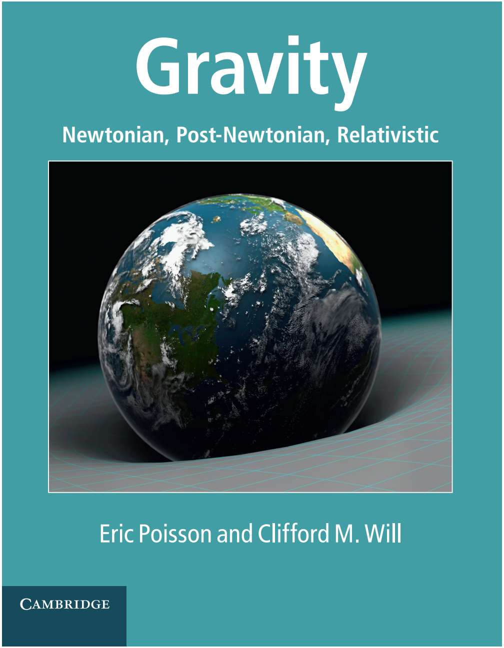
The Newtonian theory of tidal deformations and dynamics is undergoing a generalization to relativistic gravity.

A meaningful description of the tidal deformation of a relativistic body has been achieved; Love numbers have been ported to general relativity.

The tidal dynamics of black holes is well developed; it displays a remarkable similarity with the Newtonian theory of viscous bodies.

The tidal tensors of a slowly rotating black hole immersed in a post-Newtonian tidal environment have been determined to 1.5PN order.

The 1.5PN tidal torquing formula was compared with preliminary numerical results from the Caltech-Cornell-CITA collaboration.


Gravity
Newtonian, Post-Newtonian, Relativistic
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