On the Einstein-Grossmann Collaboration 100 Years ago

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Program

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- Starting point in August 1912; programmatic aspects
- Coupling of material systems to gravitational fields
- In search for the gravitational field equation
- Final phase in Nov. 1915

Einstein's work on gravitation before summer 1912

1907: Equivalence principle (in special form); redshift; light deflection (by the Earth): in Chap. V ("Principle of Relativity and Gravitation") of CPAE, Vol. 2 Doc. 47 .

With the EP Einstein went beyond SR; became the **guiding thread**. [Later recollections show that E. had tried before a special relativistic scalar theory of gravity.]

Until 1911 no further publications about gravity. But: *"Between 1909-1912 while I had to teach theoretical physics at the Zürich and Prague Universities I pondered ceaselessly on the problem"*.

1911: Einstein realizes that gravitational light deflection should be experimentally observable; takes up vigorously the problem of gravitation.

Begins to *"work like a horse"* in developing a coherent **theory of the static gravitational fields** $\longrightarrow$ variable velocity of light; non-linear field equation ($\rightarrow$ EP holds only in infinitesimally small regions). Modification of equations of electrodynamics and thermodynamics by static gravitational fields.

Begins to investigate the **dynamical** gravitational field.

Starting point in August 1912

$g_{\mu\nu}$ is the relativistic generalization of Newton's potential: field equations ???
Einstein meets Marcel Grossmann:

I was made aware of these [works by Ricci and Levi-Civita] by my friend Grossmann in Zürich , when I put the problem to investigate generally covariant tensors, whose components depend only on the derivatives of the coefficients of the quadratic fundamental invariant. He at once caught fire, although as a mathematician he had a somewhat sceptical stance towards physics. (...) He went through the literature and soon discovered that the indicated mathematical problem had already been solved, in particular by Riemann, Ricci and Levi-Civita. This entire development was connected to the Gaussian theory of curved surfaces, in which for the first time systematic use was made of generalized coordinates.

Requirements to be satisfied by the future theory

- The theory reduces to the Newtonian limit for weak fields and slowly moving matter.

- Conservation laws for energy and momentum must hold.

- The equivalence principle must be embodied.

- The theory respects a generalized principle of relativity to accelerating frames, taking into account that gravitation and inertia are described by one and the same field $g_{\mu\nu}$. Einstein expressed this by the requirement of general covariance of the basic equations (to become a much debated subject).

Coupling of matter to gravity (Part 1)

• Einstein generalizes the eq. of motion for a point particle from the static case to

$$\delta \int ds = 0, \qquad ds^2 = g_{\mu\nu} dx^\mu dx^\nu;$$

writes the geodesic equation in the form

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0;$$

• guesses the energy-momentum conservation for dust

$$\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}g_{\mu\lambda}T^{\lambda\nu}) - \frac{1}{2}\partial_\mu g_{\alpha\beta}T^{\alpha\beta} = 0, \qquad T^{\mu\nu} = \rho_0 u^\mu u^\nu$$

($g := \det(g_{\mu\nu})$). Einstein checks general covariance of this equation.

In search of the gravitational field equations

Soon, Einstein begins to look for candidate field equations. The pages before 27 of the Zürich Notebook show that he was not yet acquainted with the absolute calculus of Ricci and Levi-Civita. On p. 26 he considers for the case $-g = 1$ the equation

$$g^{\alpha\beta}\partial_\alpha\partial_\beta g^{\mu\nu} = \kappa T^{\mu\nu},$$

and substitutes the left hand side into the last eq., but that produces third derivatives and leads to nowhere.

1. **Einstein studies the Ricci tensor as a candidate**

On p. 27, referring to Grossmann, Einstein writes down the expression for the fully covariant Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$. Next, he forms by contraction the Ricci tensor $R_{\mu\nu}$. The resulting terms involving second derivatives consist, beside $g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu}$, of three additional terms. Einstein writes below their sum: "should vanish" ["sollte verschwinden"]. The reason is that he was looking for a field equation of the following general form:

$$\Gamma^{\mu\nu}[g] = \kappa T^{\mu\nu},$$

with

$$\Gamma^{\mu\nu}[g] = \partial_\alpha(g^{\alpha\beta}\partial_\beta g^{\mu\nu}) + \text{terms that vanish in linear approximation.}$$

To simplfy the explicit lengthy expressions for $R_{\mu\nu}$ in terms of $g_{\mu\nu}$, Einstein finally used coordinates that satisfy the *harmonic condition*

$$\Box x^\alpha = 0, \quad \Box := \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)$$

or $\Gamma^\alpha = 0$, where

$$\Gamma^\alpha := g^{\mu\nu}\Gamma^\alpha{}_{\mu\nu} = -\partial_\mu g^{\mu\alpha} - \frac{1}{2}g^{\alpha\beta}g^{\mu\nu}\partial_\beta g_{\mu\nu}.$$

Einstein notes that now the only term with second derivatives is $-(1/2)g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu}$, and, therefore the result is of the desired form:

In harmonic coordinates*:

$$^{(h)}R_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + H_{\mu\nu}(g, \partial g),$$

where $H_{\mu\nu}(g, \partial g)$ is a rational expression of $g_{\mu\nu}$ and $\partial_\alpha g_{\mu\nu}$ (with denominator g) that vanishes in the linear approximation. This is, of course, a familiar result for us which plays an important role in GR (for instance, in studying the Cauchy problem).

This seems to look good, and Einstein begins to analyse the linear weak field approximation of the field equation

$$\boxed{R_{\mu\nu} = \kappa T_{\mu\nu}.}$$

*In general coordinates the Ricci tensor is given by

$$R_{\mu\nu} = {}^{(h)}R_{\mu\nu} + \frac{1}{2}(g_{\alpha\mu}\partial_\nu \Gamma^\alpha + g_{\alpha\nu}\partial_\mu \Gamma^\alpha).$$

2. **The weak field approximation**

The linearized harmonic coordinate condition becomes for $h_{\mu\nu} := g_{\mu\nu} - \eta_{\mu\nu}$ ($\eta_{\mu\nu}$: Minkowski metric)

$$\partial_\mu (h^{\mu\alpha} - \frac{1}{2}\eta^{\mu\alpha} h) = 0$$

($h := h^\mu{}_\mu$, indices are now raised and lowered with the Minkowski metric). This is nowadays usually called the *Hilbert condition*, but Einstein imposed it already in 1912. The field equation becomes

$$\Box h_{\mu\nu} = -2\kappa T_{\mu\nu}.$$

Einstein takes for $T_{\mu\nu}$ his earlier expression for dust.

But now he runs into a **serious problem**: the trace $T := T^\mu{}_\mu$ must be a constant!

From $\partial^\nu T_{\mu\nu} = 0$ in the weak field limit, it follows that $\Box(\partial^\nu h_{\mu\nu}) = 0$, hence the harmonic coordinate condition requires $\Box(\partial_\nu h) = 0$, and therefore the trace of the the field equation implies $\Box h = -2\kappa T = const.$, $T := T^\mu{}_\mu$. For dust this requires that $T = -\rho_0 = const.$ This is, of course, unacceptable. One would not even be able to describe a star, with a smooth distribution of matter localized in a finite region of space.

Non-linear version of this difficulty: Field equation plus $\nabla^\nu T_{\mu\nu} = 0$ imply, using the contracted Bianchi identity $\nabla^\nu R_{\mu\nu} = \frac{1}{2}\partial_\mu R$, that $R = const.$, thus the trace of the field equation leads again to $T = const.$ Einstein discovered this, without knowing the Bianchi identity, in fall 1915, when he reconsidered the candidate field equation. (To be discussed.)

Remark. From his studies of static gravity in Prague, Einstein was convinced that in the (weak) static limit the metric must be of the form $(g_{\mu\nu}) = \text{diag}(g_{00}(\mathbf{x}), 1, 1, 1)$, thus *spatially flat*. But then $\Box h = const.$ would imply that $\triangle g_{00} = const.$ If the function g_{00} is bounded on $\mathbb{R}^3$, then $g_{00}(\mathbf{x})$ would have to be a constant.*

*A non-linear version of this remark may be of some interest. If the metric is assumed to be static with flat spatial sections, then we obtain in coordinates adapted to the static Killing field for the curvature scalar

$$R = -\frac{2}{\varphi} \triangle \varphi,$$

with $g_{00} =: -\varphi^2$. Since R is constant, we obtain the equation $\triangle \varphi = \Lambda \varphi$, where the constant Λ is equal to $-\kappa T/2$. For 'normal' matter Λ is non-negative. If $\Lambda > 0\,(T \neq 0)$ we conclude that $\varphi = 0$. Since φ must be everywhere positive, it follows that a bounded φ has to be a constant, hence **only the Minkowski metric remains**.

3. **Einstein's modified linearized field equation**

Now, something very interesting happens. Einstein avoids the first problem by modifying the linearized field equation to

$$\Box(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h) = -2\kappa T_{\mu\nu} \quad \Longleftrightarrow \quad \Box h_{\mu\nu} = -2\kappa(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T).$$

Then the harmonic coordinate condition is compatible with $\partial_\nu T^{\mu\nu} = 0$. Remarkably, this is the linearized equation of the final theory (in harmonic coordinates). One wonders why Einstein did not try at this point the analogous substitution $R_{\mu\nu} \longrightarrow R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ or $T_{\mu\nu} \longrightarrow T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T$ in the full non-linear equation.

Before we discuss the probable reasons for this, we go on with his research notes.

a) **Energy-momentum conservation for weak fields**

In linearized approximation

$$\partial_\nu T_\mu{}^\nu - \frac{1}{2}\partial_\mu h_{\alpha\beta} T^{\alpha\beta} = 0.$$

Einstein replaces in the second term $T^{\alpha\beta}$ by $(-1/2\kappa)$ times the left hand side of the modified field equation. This is rewritten as a total divergence by performing several partial integrations:

$$\Box(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h) h^{\mu\nu}{}_{,\sigma} = -4\kappa t_\sigma{}^\lambda{}_{,\lambda} \, ,$$

$$t_\sigma{}^\lambda = -\frac{1}{4\kappa}\Big[h_{\mu\nu}{}^{,\lambda} h^{\mu\nu}{}_{,\sigma} - \frac{1}{2}\delta^\lambda_\sigma h_{\mu\nu,\rho} h^{\mu\nu,\rho} - \frac{1}{2}(h^{,\lambda} h_{,\sigma} - \frac{1}{2}\delta^\lambda_\sigma h_{,\rho} h^{,\rho})\Big].$$

With this substitution the second term in (*) also becomes a total divergence, and Einstein obtains the *conservation law*

$$\partial_\nu (T_\mu{}^\nu + t_\mu{}^\nu) = 0.$$

b) **The problem with the Newtonian limit**

The problem with the Newtonian limit was, it appears, one of the main reasons why Einstein abandoned the general covariance of the field equations. Apparently, the modified field equations did not reduce to the correct limit. That it leads to the Poisson equation for $g_{00}(\mathbf{x})$ is fine, but because of the harmonic coordinate condition the metric **can not be spatially flat**. Einstein found this unacceptable. He was convinced, I recall, that for (weak) static gravitational fields the metric must be of the form $(g_{\mu\nu}) = \mathrm{diag}(g_{00}(\mathbf{x}), 1, 1, 1)$, as he already noted on p. 1 of his research notes.

"If wise men did not err, fools should despair" (Wolfgang Goethe)

The Einstein-Grossmann field equations

Einstein's difficulties, discussed previously, were among the reasons that he **abandoned general covariance for the field equations**. Another argument had to do with energy-momentum conservation. Generalizing the argument to the full theory, i.e., replacing $T^{\alpha\beta}$ in the second term of the **conservation law** for matter should lead to a conservation law for matter plus gravity of the form

$$\partial_\nu[\sqrt{-g}(T_\mu{}^\nu + t_\mu{}^\nu)] = 0.$$

Now, Einstein thought that the gravitational part $t_\mu{}^\nu$ in a covariant theory should also be a tensor under general coordinate tensor. This is, however, impossible.

Later, by November 1913, Einstein came up with yet another general argument, related to **determinism** ('hole' argument). (To be discussed later.)

Ansatz for the lhs of the field equations $\Gamma^{\mu\nu}[g] = T_{\mu\nu}$:

$$\Gamma^{\mu\nu}[g] = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta g^{\mu\nu}) + H^{\mu\nu}(g, \partial g).$$

Einstein inserts this in the second term of the "conservation law"

$$\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}g_{\mu\lambda}T^{\lambda\nu}) - \frac{1}{2}\partial_\mu g_{\alpha\beta}T^{\alpha\beta} = 0.$$

Tries to determine $H^{\mu\nu}(g, \partial g)$ such that $\partial_\mu g_{\alpha\beta}\Gamma^{\alpha\beta}$ becomes a total divergence. Finds such an object by applying several partial integrations for the contribution of the first term of $\Gamma^{\mu\nu}[g]$:

$$\Gamma^{\mu\nu}[g] = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta g^{\mu\nu}) - g^{\alpha\beta}g_{\sigma\rho}\partial_\alpha g^{\mu\sigma}\partial_\beta g^{\nu\rho} - \kappa t^{\mu\nu},$$

$$-2\kappa t^{\mu\nu} = g^{\alpha\mu}g^{\beta\nu}\partial_\alpha g_{\sigma\rho}\partial_\beta g^{\sigma\rho} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\partial_\alpha g_{\sigma\rho}\partial_\beta g^{\sigma\rho}.$$

With this expression for $t^{\mu\nu}$ the **conservation law for matter plus gravity holds**. [Note. In GR: $-\frac{1}{2}\partial_\mu g_{\alpha\beta}G^{\alpha\beta} = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}t^\nu{}_\mu)$; $t^\nu{}_\mu$ = Einstein pseudo-tensor; $\partial_\nu[\sqrt{-g}(G_\mu{}^\nu + t_\mu{}^\nu)] = 0$ is equivalent to Bianchi identity (κ=1).]

Ableitung der Gravitationsgleichungen [128]

[eq. 200]

$$\underbrace{\frac{\partial g_{\mu\nu}}{\partial x_m}\Big| \frac{\partial}{\partial x_\alpha}\left(\gamma_{\alpha\beta}\sqrt{G}\frac{\partial\gamma_{\mu\nu}}{\partial x_\beta}\right)}_{\text{o}} = \underbrace{\frac{\partial}{\partial x_\alpha}\left(\gamma_{\alpha\beta}\sqrt{G}\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\frac{\partial g_{\mu\nu}}{\partial x_m}\right)}_{+} - \sqrt{G}\gamma_{\alpha\beta}\frac{\partial\gamma_{\mu\nu}}{\partial x_\beta}\frac{\partial^2 g_{\mu\nu}}{\partial x_m\partial x_\alpha}$$

$$\underbrace{-\frac{\partial}{\partial x_m}\left(\gamma_{\alpha\beta}\sqrt{G}\frac{\partial\gamma_{\mu\nu}}{\partial x_\beta}\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right)}_{\times} + \frac{\partial g_{\mu\nu}}{\partial x_\alpha}\frac{\partial}{\partial x_m}\left(\sqrt{G}\gamma_{\alpha\beta}\frac{\partial\gamma_{\mu\nu}}{\partial x_\beta}\right)$$

$$\frac{1}{2}\frac{\partial g_{\sigma\tau}}{\partial x_m}\gamma_{\sigma\tau}\sqrt{G}\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\frac{\partial\gamma_{\mu\nu}}{\partial x_\beta}\gamma_{\alpha\beta} + \frac{\partial\gamma_{\alpha\beta}}{\partial x_m}\sqrt{G}\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\frac{\partial\gamma_{\mu\nu}}{\partial x_\beta} + \frac{\partial g_{\mu\nu}}{\partial x_\alpha}\sqrt{G}\gamma_{\alpha\beta}\frac{\partial^2\gamma_{\mu\nu}}{\partial x_m\partial x_\beta}$$

$$\underline{\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x_m}\gamma_{\mu\nu}\sqrt{G}\frac{\partial g_{\sigma\tau}}{\partial x_\alpha}\frac{\partial\gamma_{\sigma\tau}}{\partial x_\beta}\gamma_{\alpha\beta}} \qquad \frac{\partial\gamma_{\alpha\beta}}{\partial x_m}\underbrace{g_{\alpha\sigma}g_{\beta\tau}\,\Big|\,\gamma_{\alpha'\sigma}\gamma_{\beta'\tau}}\sqrt{G}\frac{\partial g_{\mu\nu}}{\partial x_{\alpha'}}\frac{\partial\gamma_{\mu\nu}}{\partial x_{\beta'}}$$

$$\underbrace{-\frac{\partial g_{\sigma\tau}}{\partial x_m}\sqrt{G}\gamma_{\alpha\sigma}\gamma_{\beta\tau}\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\frac{\partial\gamma_{\mu\nu}}{\partial x_\beta}}_{\text{o}}$$

$$\underbrace{\frac{\partial}{\partial x_\beta}\left(\frac{\partial\gamma_{\mu\nu}}{\partial x_m}\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\sqrt{G}\gamma_{\alpha\beta}\right)}_{+} \qquad - \frac{\partial\gamma_{\mu\nu}}{\partial x_m}\frac{\partial}{\partial x_\beta}\left(\sqrt{G}\gamma_{\alpha\beta}\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right)$$

$$\underbrace{-\frac{\partial\gamma_{\mu\nu}}{\partial x_m}g_{\mu\sigma}g_{\nu\tau}}\,\Big|\,\gamma_{\mu'\sigma}\gamma_{\nu'\tau}\frac{\partial}{\partial x_\beta}\left(\sqrt{G}\gamma_{\alpha\beta}\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right)$$

$$\frac{\partial g_{\sigma\tau}}{\partial x_m}\Big|\frac{\partial}{\partial x_\beta}\left(\gamma_{\mu\sigma}\gamma_{\nu\tau}\sqrt{G}\gamma_{\alpha\beta}\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right) - \sqrt{G}\gamma_{\alpha\beta}\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\frac{\partial \gamma_{\mu\sigma}\gamma_{\nu\tau}}{\partial x_\beta}$$

$$\underbrace{-\frac{\partial}{\partial x_\beta}\left(\frac{\partial\gamma_{\sigma\tau}}{\partial x_\alpha}\gamma_{\alpha\beta}\sqrt{G}\right)}_{\text{o}} + \underbrace{2\sqrt{G}\gamma_{\alpha\beta}g_{\mu\nu}\frac{\partial\gamma_{\mu\sigma}}{\partial x_\alpha}\frac{\partial\gamma_{\nu\tau}}{\partial x_\beta}}_{\text{o}}$$

This result is not unique, contrary to what E & G claim.

Einstein showed explicitly only later in 1913 in his famous Vienna lecture that the Newtonian limit in his sense (with a flat spatial metric) is indeed recovered.

In collaboration with his lifelong friend Michele Besso, Einstein studied the perihelion motion of Mercury on the basis of the "Entwurf"" theory. The result was 5/12 of what Einstein later (1915) found for GR.

Further remarks on the two Einstein-Grossmann papers

• Einstein generalizes Maxwell's equations to the generally covariant equation we all know. This part has survived in GR.

• Is a Poincaré-invariant scalar theory of gravity possible?

• In a second paper by Einstein and Grossmann, the authors investigate the covariance properties of their field equation, and show that the covariance group is larger than the linear group. As a tool they establish the following variational principle for their field equation:

$$\delta \int \mathcal{L}[g]\sqrt{-g}d^4x = \kappa \int T_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}d^4x \,,$$

with

$$\mathcal{L}[g] = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha g_{\mu\nu}\partial_\beta g^{\mu\nu}.$$

In later developments on the way to GR, variational principles were often used by Einstein, but – before Hilbert – he did not consider the curvature scalar.

The Einstein-Fokker theory

Consistent scalar theory of gravity; non-linear generalization of Nordström's theory that embodies the equivalence principle (actually the strong version). In a non-geometrical, flat-spacetime formulation:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi + \mathcal{L}_{mat}\left[\psi; (1+k\varphi)^2\eta_{\mu\nu}\right](1+k\varphi)^4;$$

in particular, the flat metric $\eta_{\mu\nu}$ in $\mathcal{L}_{mat}$ is replaced by $(1+k\varphi)^2\eta_{\mu\nu}$, $k^2 = \kappa/2$. One can get rid of the Minkowski metric, replacing it by a "physical metric":

$$g_{\mu\nu} = (1+k\varphi)^2\eta_{\mu\nu}\,.$$

For example, only relative to this metric the Compton wave length is constant, i.e., not spacetime dependent.

Einstein and Fokker gave a geometrical formulation of the theory. This can be summarized as follows:

(i) spacetime is conformally flat: Weyl tensor = 0;

(ii) field equation: $R = 24\pi G T$;

(iii) test particles follow geodesics.

In adapted coordinates, with $g_{\mu\nu} = \phi^2 \eta_{\mu\nu}$, one finds

$$R = -6\phi^{-3}\eta^{\mu\nu}\partial_\mu\partial_\nu\phi,$$

and the field equation becomes

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = -4\pi G\phi^3 T.$$

Remarks

• The Einstein-Fokker theory is generally covariant (as emphasized in the original paper), however, *not* generally *invariant*. The object $\tilde{g}_{\mu\nu} = g_{\mu\nu}/(-g)^{1/4}$ is an **absolute tensor density**, in that it is diffeomorphic (as a tensor density) to $\eta_{\mu\nu}$. Therefore, the **invariance group** is the *conformal group*, which is a finite dimensional Lie group.

• Since the scalar theory of Nordström and the generalization by Einstein and Fokker predict **no global light deflection**, Einstein urged in 1913 astronomers to measure the light deflection during the solar eclipse in the coming year in the Crimea. Moreover, both predict -1/6 the Einsteinian value for the **perihelion advance**, in contrast to observation.

The ‘hole’ argument against general covariance

In a lecture given to the Annual Meeting of the Swiss Naturforschende Gesellschaft in September 1913, Einstein stated: “It is possible to demonstrate by a general argument that equations that completely determine the gravitational field cannot be generally covariant with respect to arbitrary substitutions.” He repeated this statement shortly afterwards in his Vienna lecture of September 23, 1913.

Imagine a finite region $\mathcal{D}$ of spacetime – the ‘hole’ – in which the stress energy tensor vanishes. Assume that a metric field g is a solution of generally covariant field equations. Apply now a diffeomorphism φ on g, producing $\varphi_* g$ (push-forward), and choose the diffeomorphism such that it leaves the spacetime region outside $\mathcal{D}$ pointwise fixed.

Clearly, g and $\varphi_* g$ are different solutions of the field equations that agree outside $\mathcal{D}$. In other words, generally covariant field equations allow huge families of solutions for one and the same matter distribution (outside the hole). At the time, Einstein found this unacceptable, because this was in his opinion a dramatic failure of what he called the law of causality (now usually called determinism).

It took a long time until Einstein understood that this non-uniqueness is an expression of what we now call *gauge invariance*, analogous to the local invariance of our gauge theories in elementary particle physics. On January 3, 1916 he wrote to Besso: "Everything in the hole argument was correct up to the final conclusion".

Einstein to Besso on November 17, 1915:

> I have worked with great success during these months. *General covariant* gravitational equations. *Motions of the perihelion quantitatively explained*. Role of gravitation in the structure of matter [im Bau der Materie]. You will be amazed. I worked horribly strenuously [schauderhaft angestrengt], [it is] strange that one can endure that. (...) .

Besso passed this card on to **Zangger**: “I enclose the historical card of Einstein, reporting the setting of the capstone of an epoch that began with Newton’s ‘apple’.”

In a particularly instructive detailed technical letter of November 28, 1915 to **Arnold Sommerfeld**, Einstein summarizes his final struggle. Here just two crucial sentences from this important document:

> I realized ... that my previous gravitational field equations were completely untenable. (...) After all confidence thus had been lost in the results and methods of the earlier theory, I saw clearly that only through a connection with the general theory of covariants, i.e., with Riemann's covariant [tensor], could a satisfactory solution be found. (...)

Final phase in Nov. 1915

In what follows, I will rewrite Einstein's arguments without changing the content (calculational streamlining).

May shed some light on the Einstein-Hilbert relation.

Einstein to Lorentz (17.1.1916):

> The basic equations are now finally good, but the derivations abominable; this drawback still has to be removed.

Useful identity (derived by E. in first review paper, Doc. 30, Eq. (55)):

$$\underbrace{G_\mu{}^\alpha{}_{,\alpha}}_{R_\mu{}^\alpha{}_{,\alpha}-(1/2)R_{,\mu}} + \kappa\, t_\mu{}^\alpha{}_{,\alpha} = 0\,. \tag{1}$$

• Doc. 21 (**4. Nov. 1915**), and Addendum Doc. 22 (**11. Nov. 1915**) : $R_{\mu\nu} = \kappa T_{\mu\nu}$.

1. From (1): $\kappa(T_\mu{}^\alpha + t_\mu{}^\alpha)_{,\alpha} = \frac{1}{2}R_{,\mu}$;

2. On the other hand: $\underbrace{\frac{1}{2}g^{\mu\nu}{}_{,\lambda}R_{\mu\nu}}_{E.:=\kappa t_\lambda{}^\nu{}_{,\nu}} = \kappa\frac{1}{2}g^{\mu\nu}{}_{,\lambda}T_{\mu\nu} = -\kappa T_\lambda{}^\nu{}_{,\nu} \Rightarrow$

$$(T_\mu{}^\nu + t_\mu{}^\nu)_{,\nu} = 0\,;$$

1. & 2. imply $R_{,\mu} = 0 \Rightarrow T = const$.

• Doc 25 (**25. Nov. 1915**): $\boxed{G_{\mu\nu} = \kappa T_{\mu\nu}}$.

Problem disappears:
1. From (1): $(T_\mu{}^\alpha + t_\mu{}^\alpha)_{,\alpha} = 0$;

2. In unimodular coord. $\frac{1}{2} g^{\mu\nu}{}_{,\lambda} G_{\mu\nu} = \underline{\kappa t_\lambda{}^\nu{}_{,\nu}} = \kappa g^{\mu\nu}{}_{,\lambda} T_{\mu\nu} = \underline{-\kappa T_\lambda{}^\nu{}_{,\nu}}$.

different interpretation: field equation implies $T_\lambda{}^\nu{}_{;\nu} = 0$.

Note that the identity (1), together with the identity

$$\kappa\, t_\mu{}^\nu{}_{,\nu} = \frac{1}{2} g^{\alpha\beta}{}_{,\mu}\, G_{\alpha\beta}$$

($= \frac{1}{2} g^{\alpha\beta}{}_{,\mu}\, R_{\alpha\beta}$ in unimodular coordinates), is equivalent to the contracted Bianchi identity.

On Einstein's approach to the field equations in Doc. 30, 20 March, 1916 (first review)

I rewrite Einstein's arguments, presented in CPAP, Vol. 6, Doc. 30, without changing the content.

Use well-known identity between the Einstein tensor ${G_\mu}^\nu$ and Einstein's pseudo-tensor ${t_\mu}^\nu$ in unimodular coordinates (always used in this Appendix):

$$ {G_\mu}^\alpha + \kappa \, {t_\mu}^\alpha = \frac{1}{2} {U_\mu}^{\alpha\beta}{}_{,\beta} \, , $$

with the super-potential (Freud)

$$ {U_\mu}^{\alpha\beta} = g_{\mu\sigma} H^{\sigma\rho\alpha\beta}{}_{,\rho} \, , \quad H^{\sigma\rho\alpha\beta} = g^{\sigma\alpha} g^{\rho\beta} - g^{\sigma\beta} g^{\rho\alpha} \, . $$

(Not complicated to derive this identity with the tools developed in Sect. 15 of the cited document.)

Vacuum equation $R_{\mu\nu} = 0$ can be written as

$$\frac{1}{2} U_\mu{}^{\alpha\beta}{}_{,\beta} = \kappa \, t_\mu{}^\alpha \, .$$

Equivalent to what Einstein does in a first step. $U_\mu{}^{\alpha\beta}{}_{,\beta\alpha} \equiv 0 \;\; \Rightarrow$ $t_\mu{}^\alpha{}_{,\alpha} = 0$; $t_\mu{}^\alpha$ is interpreted by Einstein as the energy-momentum complex (pseudo- tensor) of the gravitational field.

In the presence of matter, Einstein replaces $t_\mu{}^\alpha$ by the sum $t_\mu{}^\alpha + T_\mu{}^\alpha$:

$$\boxed{\frac{1}{2} U_\mu{}^{\alpha\beta}{}_{,\beta} = \kappa \, (t_\mu{}^\alpha + T_\mu{}^\alpha) \, .}$$

Field eqs. guarantee the conservation law $(t_\mu{}^\alpha + T_\mu{}^\alpha)_{,\alpha} = 0$. By the identity above this form is equivalent to $G_\mu{}^\alpha = \kappa\, T_\mu{}^\alpha$, with the correct trace term.

Note that the identity $U_\mu{}^{\alpha\beta}{}_{,\beta\alpha} \equiv 0$ is equivalent to the contracted Bianchi identity; use also the identity

$$\kappa\, t_\mu{}^\nu{}_{,\nu} = \frac{1}{2} g^{\alpha\beta}{}_{,\mu}\, G_{\alpha\beta}$$

($= \frac{1}{2} g^{\alpha\beta}{}_{,\mu}\, R_{\alpha\beta}$ in unimodular coordinates). Contracted Bianchi identity was not yet known to Einstein (but is implicit in Doc. 30 since it also contains $\underbrace{G_\mu{}^\alpha{}_{,\alpha}} + \kappa\, t_\mu{}^\alpha{}_{,\alpha} = 0$).

Einstein to Hilbert on December 20, 1915:

> "On this occasion I feel compelled to say something else to you that is of much more importance to me. There has been a certain ill-feeling between us, the cause of which I do not wish to analyze. I have struggled against the feeling of bitterness attached to it, and this with success. I think of you again with unmixed congeniality and I ask that you try to do the same with me. Objectively it is a shame when two real fellows who have managed to extricate themselves somewhat from this shabby world do not give one another pleasure."

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