The resolution of the bounded L^2 curvature conjecture in general relativity

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Evolution problem for EE

$(\mathcal{M}, \mathbf{g})$ Lorentzian, $\mathbf{R}$ curvature tensor of $\mathbf{g}$

Einstein Vacuum equations: $\mathbf{Ric}_{\alpha\beta} = 0$

Wave coordinates: $\square_{\mathbf{g}} x^\alpha = \frac{1}{\sqrt{|\mathbf{g}|}} \partial_\beta (\mathbf{g}^{\beta\gamma} \sqrt{|\mathbf{g}|} \partial_\gamma) x^\alpha = 0, \alpha = 0, 1, 2, 3$

$\square_{\mathbf{g}} \mathbf{g}_{\alpha\beta} = \mathcal{N}_{\alpha\beta}(\mathbf{g}, \partial \mathbf{g}), \alpha, \beta = 0, 1, 2, 3$

Cauchy data: (Σ_0, g_0, k) where $\Sigma_0 = \{t = 0\}$, $\mathbf{g}(0, .) = g_0$, $\partial_t \mathbf{g}(0, .) = k$

Local existence if (Σ_0, g_0, k) sufficiently smooth (Y. C. Bruhat, 1952)

Question: Under which regularity do we have local existence for EE?

Two examples of solutions to EE with singularities

$$\mathbf{g} = -\left(1 - \frac{2m}{r}\right)(dt)^2 + \left(1 - \frac{2m}{r}\right)^{-1}(dr)^2 + r^2\left((d\theta)^2 + (\sin\theta)^2(d\phi)^2\right)$$

Example 1: The Schwarzschild solution has a singularity at $r = 0$.
This singularity lies inside the black hole region $r \leq 2m$

$$\mathbf{g} = -\frac{q^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{q^2}\left(d\phi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{q^2}{\Delta}(dr)^2 + q^2(d\theta)^2$$

$$\Delta = r^2 + a^2 - 2mr,\ q^2 = r^2 + a^2(\cos\theta)^2,\ \Sigma^2 = (r^2 + a^2)q^2 + 2mra^2(\sin\theta)^2$$

Example 2: The Kerr solution has a singularity at $r = 0$, $\theta = \pi/2$. If $a > m$, there is no horizon and this singularity is said to be naked

The weak cosmic censorship conjecture

Are singularities of solutions to the EE a limit to the physical theory?

No, as long as an observer cannot have access to them, i.e. as long as they are hidden by a black hole!

Conjecture [R. Penrose (1969)]: Singularities are generically hidden by a black hole

More precisely: for generic asymptotically flat initial data, the maximal Cauchy development of the corresponding solution to EE has a complete future null infinity

In which norm should the genericity be measured?

The special case of spherical symmetry

Theorem [Christodoulou (1999)]: Weak cosmic censorship holds true generically for the EE coupled to a scalar field in spherical symmetry

Two main ingredients:

- A very rough local existence result
- A sharp criterion for the formation of trapped surfaces

To obtain a trapped surface using the sharp criterion, one needs perturbations of the initial data with a jump along the backward null cone. This requires to make sense of very rough solutions

Try to extend both ingredients to the EE in the absence of symmetry. In this talk, we focus on the first ingredient

The bounded L^2 curvature theorem

Theorem [KRS (2012)]: Let (Σ_0, g_0, k) *with* $R \in L^2(\Sigma_0)$ *and* $\nabla k \in L^2(\Sigma_0)$. *Then, local existence for the EE holds*

Further motivations:

- First local existence result for EE which exploits the full nonlinear structure of the equation
- The assumptions $R \in L^2(\Sigma_0)$, $\nabla k \in L^2(\Sigma_0)$ are invariant
- $\mathbf{R} \in L^2$ is a fundamental quantity controlling true singularity formation
- There is some criticality in this problem: the control of the Eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ requires $\mathbf{R} \in L^2$

Strategy of the proof

A *Recast the EE as a quasilinear Yang-Mills theory*

B *Prove appropriate bilinear estimates for solutions to* $\Box_{\mathbf{g}}\phi = 0$

C *Construct a plane wave representation for the solutions of* $\Box_{\mathbf{g}}\phi = 0$

D *Prove a sharp* $L^4(\mathcal{M})$ *Strichartz estimate for the solutions of* $\Box_{\mathbf{g}}\phi = 0$

Achieve Steps B, C and D only assuming L^2 bounds on $\mathbf{R}$

This requires to exploit the full structure of the Einstein equations

EE as a quasilinear Yang-Mills theory

Let e_α an orthonormal frame on $\mathcal{M}$, i.e. $\mathbf{g}(e_\alpha, e_\beta) = \mathbf{m}_{\alpha\beta}$

Let $(\mathbf{A}_\mu)_{\alpha\beta} := (\mathbf{A})_{\alpha\beta}(\partial_\mu) = \mathbf{g}(\mathbf{D}_\mu e_\beta, e_\alpha)$

$$\mathbf{R}(e_\alpha, e_\beta, \partial_\mu, \partial_\nu) = \partial_\mu(\mathbf{A}_\nu)_{\alpha\beta} - \partial_\nu(\mathbf{A}_\mu)_{\alpha\beta} + (\mathbf{A}_\nu)_\alpha{}^\lambda(\mathbf{A}_\mu)_{\lambda\beta} - (\mathbf{A}_\mu)_\alpha{}^\lambda(\mathbf{A}_\nu)_{\lambda\beta}$$

$$\mathbf{D}^\mu \mathbf{R}_{\alpha\beta\mu\nu} = 0 \text{ (consequence of Bianchi identities + EE)}$$

$$(\Box_{\mathbf{g}}\mathbf{A})_\nu - \mathbf{D}_\nu(\mathbf{D}^\mu \mathbf{A}_\mu) = \mathbf{D}^\mu([\mathbf{A}_\mu, \mathbf{A}_\nu]) + [\mathbf{A}^\mu, \mathbf{D}_\mu \mathbf{A}_\nu - \mathbf{D}_\nu \mathbf{A}_\mu] + \mathbf{A}^3$$

We choose the Coulomb gauge $\nabla^j A_j = 0$

We need a procedure to scalarize the tensorial wave equation and to project on divergence free vectorfields without destroying the null structure

Building a plane wave representation for $\Box_{\mathbf{g}}(\phi) = 0$

Let a plane wave $e^{i\lambda u(t,x,\omega)}$ with $\lambda \in [0, +\infty)$ and $\omega \in \mathbb{S}^2$ parameters corresponding to Fourier variables in $\mathbb{R}^3$ in spherical coordinates

$$\Box_{\mathbf{g}}(e^{i\lambda u}) = \Big(-\lambda^2 \mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u + i\lambda \Box_{\mathbf{g}} u\Big) e^{i\lambda u}$$

For u a solution of the Eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$, we have:

$$\Box_{\mathbf{g}}(e^{i\lambda u}) = i\lambda \Box_{\mathbf{g}} u e^{i\lambda u}$$

This yields in general an approximate solution to $\Box_{\mathbf{g}}(\phi) = 0$. We then superpose these plane waves to generate any initial data