Enhanced terms in the time delay and the direction of light propagation. Discussion for some solar system experiments

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Séminaire IHES, 6 Mars 2014

Contents

Introduction

General framework : "metric theories of gravity"
$\longrightarrow$ post-Newtonian parameters β, γ,... in GR:
γ linked to the "curvature" of space and light propagation
β linked to the "non-linearity" of the field

- The best current value for γ is due to the CASSINI Mission:

$$\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5} \qquad \text{(Bertotti } et\ al \text{ 2003)}$$

- Recent improvements on γ (and β) from INPOP13a (Verma *et al* 2013).

- Scalar field in primordial cosmology $\longrightarrow$ expected violations of GR:

$\gamma - 1 < 0$ and, possibly, in the range $10^{-7} - 10^{-9}$ (Damour & Lilley 2008)

- $\longrightarrow$ Many projects designed to measure γ at this level in the solar system: LATOR, ASTROD, ODYSSEY, SAGAS, GAME, ...

Interest of the time transfer functions (TTFs)

The basic ingredients:

- Space-time $(\mathcal{V}_4, g)$, $g =$ physical metric $(+ - - -)$.
- Geometric approximation : light is propagating along null geodesics of g.
- $\mathcal{V}_4$ covered by a single quasi-Cartesian coordinate system : $x^\alpha = (x^0, \mathbf{x})$, $x^0 = ct$, $g_{00} > 0$.
- Light rays emitted at $\mathbf{x}_A$ at instant t_A and received at $\mathbf{x}_B$ at instant t_B.

In many tests of GR, the relevant tool is the expression of the light travel time $t_B - t_A$ between $\mathbf{x}_A$ and $\mathbf{x}_B$ as a function of $(\mathbf{x}_A, t_B, \mathbf{x}_B)$ (resp. $(t_A, \mathbf{x}_A, \mathbf{x}_B)$):

$$t_B - t_A = \mathcal{T}_r(\mathbf{x}_A, t_B, \mathbf{x}_B) = \mathcal{T}_e(t_A, \mathbf{x}_A, \mathbf{x}_B).$$

$\mathcal{T}_r$ (resp. $\mathcal{T}_e$) = “reception (resp. emission) time transfer function”.

Interest of the time transfer functions (TTFs)

- Synchronization of distant clocks.

- Doppler tracking and time/frequency transfers

$$\frac{\nu_B}{\nu_A} = \frac{(g_{00} + 2g_{0i}\beta^i + g_{ij}\beta^i\beta^j)_A^{1/2}}{(g_{00} + 2g_{0i}\beta^i + g_{ij}\beta^i\beta^j)_B^{1/2}} \frac{1 - \frac{\partial \mathcal{T}_r}{\partial t_B} - c\boldsymbol{\beta}_B.\boldsymbol{\nabla}_{\mathbf{x}_B}\mathcal{T}_r}{1 + c\boldsymbol{\beta}_A.\boldsymbol{\nabla}_{\mathbf{x}_A}\mathcal{T}_r},$$

where $\boldsymbol{\beta}_{A/B} = (\beta^i_{A/B}) = \frac{1}{c}\left(\frac{dx^i_{A/B}}{dt}\right)$.

- Direction of light propagation at $\mathbf{x}_A$ and $\mathbf{x}_B$ (astrometry)

$$\widehat{\underline{\mathbf{l}}}_A = \left(\frac{l_i}{l_0}\right)_A = c\boldsymbol{\nabla}_{\mathbf{x}_A}\mathcal{T}_r, \qquad \widehat{\underline{\mathbf{l}}}_B = \left(\frac{l_i}{l_0}\right)_B = -c\frac{\boldsymbol{\nabla}_{\mathbf{x}_B}\mathcal{T}_r}{1 - \frac{\partial \mathcal{T}_r}{\partial t_B}},$$

where $l_\alpha = g_{\alpha\beta}dx^\beta/d\lambda$ along the ray.

General considerations

Determining the TTFs in a given space-time is inextricably complicated!

- Infinity of null geodesics passing through $\mathbf{x}_A$ and $\mathbf{x}_B$ (cf. Schwarzschild, e.g.).
- Even in Schwarschild space-time, no full expression of any TTF.

Current assumptions :

- $(\mathcal{V}_4, g)$ asymptotically flat
- weak field:

$$g_{\mu\nu}(x, G) = \eta_{\mu\nu} + \sum_{n=1}^{\infty} g_{\mu\nu}^{(n)}(x, G), \quad g_{\mu\nu}^{(n)}(x, G) = G^n h_{\mu\nu}^{(n)}(x)$$

- light rays are "quasi-Minkowskian" null geodesics:

$$\Longrightarrow \quad \mathcal{T}_r(\mathbf{x}_A, t_B, \mathbf{x}_B, G) = \frac{|\mathbf{x}_B - \mathbf{x}_A|}{c} + \sum_{n=1}^{\infty} \mathcal{T}_r^{(n)}(\mathbf{x}_A, t_B, \mathbf{x}_B, G).$$

General considerations

General methods at your disposal:

- Integration of the null geodesic equations (Shapiro 1964, Richter & Matzner 1983, Brumberg 1987, Kopeikin & Schäfer 1999, Klioner 2003)
- Iterative determination of Synge's world function (John 1975, Linet & Teyssandier 2002, Le Poncin-Lafitte *et al* 2004)
- Iterative solution of an eikonal equation (Teyssandier & Le Poncin-Lafitte 2008)

A lot of results at order G

$\longrightarrow$ contributions due to the non-sphericity or/and the motions of bodies.

General considerations

However, higher order terms for the mass monopole contributions will be needed in future experiments (Minazzoli & Chauvineau 2011 and refs. therein).

Several works devoted to the calculation of TTF in static, spherically symmetric (sss) metrics (John 1975, Richter & Matzner 1983, Brumberg 1987, Le Poncin-Lafitte *et al* 2004, Teyssandier & Le Poncin 2008, Klioner & Zschocke 2010). In sss space-time

$$\mathcal{T}_r(\mathbf{x}_A, t_B, \mathbf{x}_B) \equiv \mathcal{T}_e(t_A, \mathbf{x}_A, \mathbf{x}_B) = \mathcal{T}(\mathbf{x}_A, \mathbf{x}_B).$$

- Until recently, $\mathcal{T}(\mathbf{x}_A, \mathbf{x}_B)$ was known only up to order G^2.
- However, appearance of enhanced terms (Bertotti and Ashby, 2010)

  $\longrightarrow$ order G^3 is necessary.
- New methods developped to determine $\mathcal{T}$ at any order (Linet & Teyssandier 2013).

Expression of the static, spherically symmetric metric

For a central body of mass M ($m = GM/c^2$), in isotropic coordinates:

$$ds^2 = \mathcal{A}(r)(dx^0)^2 - \mathcal{B}^{-1}(r)\,\delta_{ij}dx^i dx^j .$$

Assumption : $\exists r_h \sim m \quad \forall r > r_h \quad \mathcal{A}(r) > 0,\ \mathcal{B}^{-1}(r) > 0$ and

$$\mathcal{A}(r) = 1 - \frac{2m}{r} + 2\beta\frac{m^2}{r^2} - \frac{3}{2}\beta_3\frac{m^3}{r^3} + \beta_4\frac{m^4}{r^4} + \sum_{n=5}^{\infty}\frac{(-1)^n n}{2^{n-2}}\beta_n\frac{m^n}{r^n},$$

$$\mathcal{B}^{-1}(r) = 1 + 2\gamma\frac{m}{r} + \frac{3}{2}\epsilon\frac{m^2}{r^2} + \frac{1}{2}\gamma_3\frac{m^3}{r^3} + \frac{1}{16}\gamma_4\frac{m^4}{r^4} + \sum_{n=5}^{\infty}(\gamma_n - 1)\frac{m^n}{r^n}.$$

In GR:

$$\beta = \beta_3 = \beta_4 = \beta_5 = \cdots = 1, \qquad \gamma = \epsilon = \gamma_3 = \gamma_4 = \cdots = 1.$$

Expression of the static, spherically symmetric metric

A null geodesic of g is also a null geodesic of the conformal metric

$$d\tilde{s}^2 = (dx^0)^2 - \mathcal{U}(r)\,\delta_{ij}dx^i dx^j,$$

where

$$\mathcal{U}(r) = \frac{1}{\mathcal{A}(r)\mathcal{B}(r)} = 1 + 2(1+\gamma)\frac{m}{r} + 2\kappa\frac{m^2}{r^2} + 2\sum_{n=3}^{\infty}\kappa_n\frac{m^n}{r^n}.$$

The parameters κ and κ_3 are given by

$$\kappa = 2(1+\gamma) - \beta + \frac{3}{4}\epsilon,$$

$$\kappa_3 = 2\kappa - 2\beta(1+\gamma) + \frac{3}{4}\beta_3 + \frac{1}{4}\gamma_3.$$

In GR: $\kappa = \frac{15}{4}, \quad \kappa_3 = \frac{9}{2}.$

A natural, useful idea...

Assume that the ray has a unique pericentre $P(r = r_P)$, has no apocentre and passes through P. The null geodesic equations lead to

$$t_B - t_A = \frac{1}{c}\left[\sqrt{r_A^2 - r_P^2} + \sqrt{r_B^2 - r_P^2}\right]$$

$$+\frac{(1+\gamma)m}{c}\left[\ln\frac{(r_A + \sqrt{r_A^2 - r_P^2})(r_B + \sqrt{r_B^2 - r_P^2})}{r_P^2} + \sqrt{\frac{r_A - r_P}{r_A + r_P}} + \sqrt{\frac{r_B - r_P}{r_B + r_P}}\right]$$

$$+\frac{m^2}{cr_P}\left\{2\kappa\left[\arccos\frac{r_P}{r_A} + \arccos\frac{r_P}{r_B}\right] - \frac{(1+\gamma)^2}{4}\left[5\sqrt{\frac{r_A - r_P}{r_A + r_P}} + 5\sqrt{\frac{r_B - r_P}{r_B + r_P}}\right.\right.$$

$$\left.\left. - \left(\frac{r_A - r_P}{r_A + r_P}\right)^{3/2} - \left(\frac{r_B - r_P}{r_B + r_P}\right)^{3/2}\right]\right\} + O\left(\frac{m^3}{cr_P^2}\right),$$

where $r_A = |\mathbf{x}_A|$ and $r_B = |\mathbf{x}_B|$.

$\Longrightarrow$ Determining γ at the level 10^{-8} $\longleftrightarrow$ $t_B - t_A$ at the level 0.7 ps.

which is a false good idea...

So, is there a problem?   YES!

This expansion doesn't solve our problem : we don't know $r = r_P$ as a function of $\mathbf{x}_A$ and $\mathbf{x}_B$.

Fortunately, several other methods may be used (see above).

Preference for procedures natively adapted to an emitter and a receiver both located at a finite distance of the origin of coordinates.

New methods for determining $\mathcal{T}$ for sss metrics

First method: integration of an eikonal equation (Teyssandier & Le Poncin 2008, Linet & Teyssandier 2013)

- From

$$\widehat{\underline{\mathbf{l}}}_A = \left(\frac{l_i}{l_0}\right)_A = c\boldsymbol{\nabla}_{\mathbf{x}_A}\mathcal{T}, \qquad \widehat{\underline{\mathbf{l}}}_B = \left(\frac{l_i}{l_0}\right)_B = -c\boldsymbol{\nabla}_{\mathbf{x}_B}\mathcal{T}.$$

and $g^{\alpha\beta}l_\alpha l_\beta = 0$, it is inferred that $\mathcal{T}(\mathbf{x}, \mathbf{x}_B)$ obeys the eikonal equation

$$c^2|\boldsymbol{\nabla}_{\mathbf{x}}\mathcal{T}(\mathbf{x}, \mathbf{x}_B)|^2 = \mathcal{U}(r) = 1 + 2(1+\gamma)\frac{m}{r} + 2\sum_{n=2}^{\infty}\kappa_n\left(\frac{m}{r}\right)^n$$

for any $\mathbf{x}$ ($r = |\mathbf{x}|$) and any $|\mathbf{x}_B|$ such that $r > r_h$ and $r_B > r_h$.

- We are looking for light rays such that

$$\mathcal{T}(\mathbf{x}_A, \mathbf{x}_B) = \frac{|\mathbf{x}_B - \mathbf{x}_A|}{c} + \sum_{n=1}^{\infty}\mathcal{T}^{(n)}(\mathbf{x}_A, \mathbf{x}_B).$$

New methods for determining $\mathcal{T}$ for sss metrics

The $\mathcal{T}^{(n)}(\mathbf{x}_A, \mathbf{x}_B)$'s are given by

$$\mathcal{T}^{(n)}(\mathbf{x}_A, \mathbf{x}_B) = \frac{1}{c}|\mathbf{x}_B - \mathbf{x}_A| F^{(n)}(\mathbf{x}_A, \mathbf{x}_B),$$

where the functions $F^{(n)}$ are determined by the recurrence law

$$F^{(1)}(\mathbf{x}_A, \mathbf{x}_B) = (1+\gamma)m \int_0^1 \frac{d\xi}{|\mathbf{z}(\xi)|},$$

$$F^{(n)}(\mathbf{x}_A, \mathbf{x}_B) = \kappa_n m^n \int_0^1 \frac{d\xi}{|\mathbf{z}(\xi)|^n} - \frac{c^2}{2}\sum_{p=1}^{n-1}\int_0^1 \left[\boldsymbol{\nabla}_{\mathbf{x}}\mathcal{T}^{(p)}(\mathbf{x}_A, \mathbf{x}).\boldsymbol{\nabla}_{\mathbf{x}}\mathcal{T}^{(n-p)}(\mathbf{x}_A, \mathbf{x})\right]_{\mathbf{x}=\mathbf{z}(\xi)} d\xi,$$

for $n \geq 2$, with integrations on the straight segment joining $\mathbf{x}_A$ and $\mathbf{x}_B$

$$\mathbf{z}(\xi) = \mathbf{x}_A + \xi(\mathbf{x}_B - \mathbf{x}_A).$$

New methods for determining $\mathcal{T}$ for sss metrics

The recurrence law implies

- the uniqueness of $\mathcal{T}$ for quasi-Minkowskian light rays

- a property of analyticity:

  The $\mathcal{T}^{(n)}(\mathbf{x}_A, \mathbf{x}_B)$'s are analytic functions if $\mathbf{x}_A \neq \mathbf{x}_B$ and $\mathbf{n}_B \neq -\mathbf{n}_A$.

  $\longrightarrow$ of crucial interest for the second method.

The validity of the method is ensured only if $\mathbf{x}_A$ and $\mathbf{x}_B$ are such that

$$|\mathbf{x}_A + \xi(\mathbf{x}_B - \mathbf{x}_A)| > r_h \quad \forall \xi \in [0, 1].$$

New methods for determining $\mathcal{T}$ for sss metrics

Second method: integro-differential eq. inferred from the geodesic eqs. (Linet & Teyssandier 2013)

Since $\mathcal{T}(\mathbf{x}_A, \mathbf{x}_B)$ is analytic, we may assume that $\mathbf{x}_A$ and $\mathbf{x}_B$ are such that:

- The light ray joining $\mathbf{x}_A$ and $\mathbf{x}_B$ does not pass through a pericentre.
- At any point of the light ray

$$r_c < r_A \leq r \leq r_B,$$

where r_c is the Euclidean distance between the origin and the line passing through $\mathbf{x}_A$ and $\mathbf{x}_B$, i.e.

$$r_c = \frac{r_A r_B \sqrt{1-\mu^2}}{|\mathbf{x}_B - \mathbf{x}_A|},$$

with

$$\mu = \mathbf{n}_A . \mathbf{n}_B, \qquad \mathbf{n}_A = \frac{\mathbf{x}_A}{r_A}, \qquad \mathbf{n}_B = \frac{\mathbf{x}_B}{r_B}.$$

New methods for determining $\mathcal{T}$ for sss metrics

Second method: integro-differential eq. inferred from the geodesic eqs. (Linet & Teyssandier 2013)

Then

$$c(t_B - t_A) = b \arccos \mu + \int_{r_A}^{r_B} \frac{1}{r}\sqrt{r^2\mathcal{U}(r) - b^2}\,dr$$

where b is the impact parameter (intrinsic quantity). Since b is linked to $\mathcal{T}$ by

$$b = -c\sqrt{1-\mu^2}\,\frac{\partial \mathcal{T}(r_A, r_B, \mu)}{\partial \mu},$$

we have an integro-differential equation for $\mathcal{T}(r_A, r_B, \mu)$.

New methods for determining $\mathcal{T}$ for sss metrics

Assuming that

$$b = r_c \left[1 + \sum_{n=1}^{\infty} q_n \left(\frac{m}{r_c} \right)^n \right],$$

we find

$$\mathcal{T}^{(n)}(r_A, r_B, \mu) = \frac{1}{c} \left(\frac{m}{r_c} \right)^n \int_{r_A}^{r_B} \sum_{k=0}^{3n-4} U_{kn}(q_1, \dots, q_{n-1}) r_c^{3n-k-2} \frac{r^{k-n+1}}{(r^2 - r_c^2)^{\frac{2n-1}{2}}} dr,$$

where the U_{kn}'s are polynomials in $q_1, \dots, q_{n-1}$. Using

$$b = -c\sqrt{1-\mu^2} \frac{\partial \mathcal{T}(r_A, r_B, \mu)}{\partial \mu} \implies q_n = -c \left(\frac{r_c}{m} \right)^n \frac{\sqrt{1-\mu^2}}{r_c} \frac{\partial \mathcal{T}^{(n)}(r_A, r_B, \mu)}{\partial \mu},$$

it may be seen that

$$\mathcal{T}^{(1)}, \dots, \mathcal{T}^{(n-1)} \longrightarrow q_1, \dots, q_{n-1} \longrightarrow \mathcal{T}^{(n)}.$$

The analytic extension theorem $\implies$ expressions of the $\mathcal{T}^{(n)}$'s valid even if the initial assumptions are not met.

Expression of $\mathcal{T}$ up to order G^3

All the integrals can be calculated with any symbolic computer program.

The two methods lead to the same results for $n = 1, 2, 3$:

$$\mathcal{T}^{(1)}(\mathbf{x}_A, \mathbf{x}_B) = \frac{(1+\gamma)m}{c} \ln\left(\frac{r_A + r_B + |\mathbf{x}_B - \mathbf{x}_A|}{r_A + r_B - |\mathbf{x}_B - \mathbf{x}_A|}\right), \qquad \text{(Shapiro 1964)}$$

$$\mathcal{T}^{(2)}(\mathbf{x}_A, \mathbf{x}_B) = \frac{m^2}{r_A r_B} \frac{|\mathbf{x}_B - \mathbf{x}_A|}{c}\left[\kappa \frac{\arccos\mu}{\sqrt{1-\mu^2}} - \frac{(1+\gamma)^2}{1+\mu}\right], \quad \text{(Le Poncin } \textit{et al} \text{ 2004)}$$

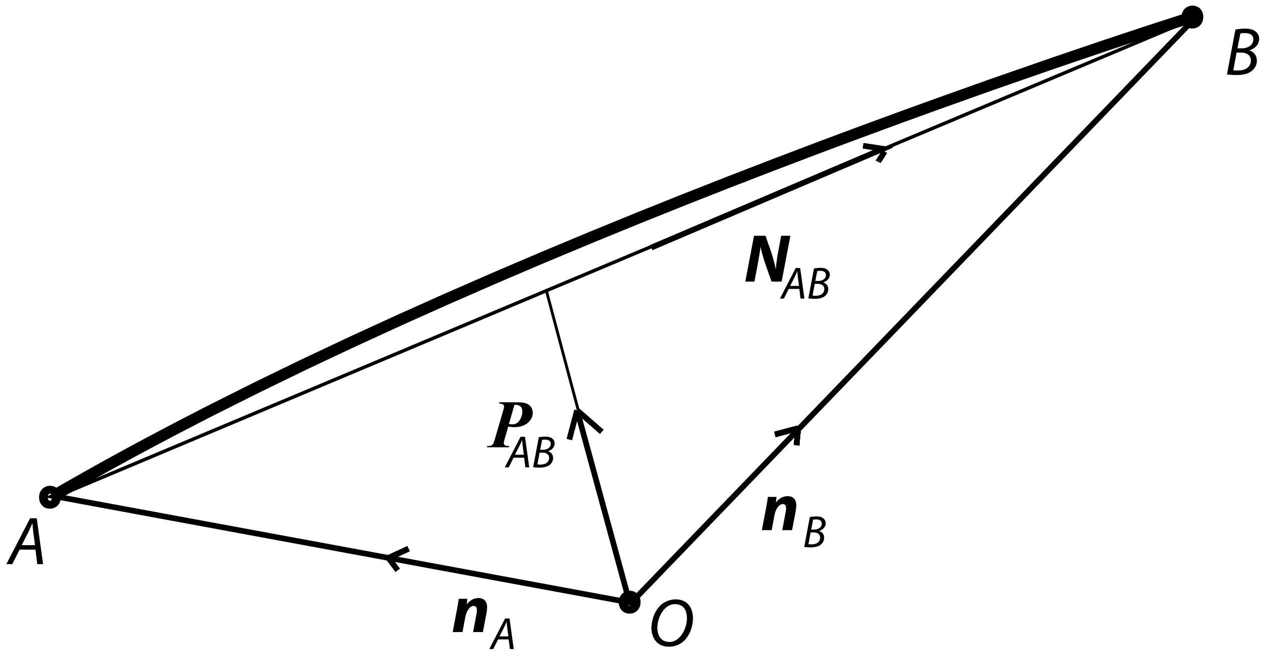
$$\mathcal{T}^{(3)}(\mathbf{x}_A, \mathbf{x}_B) = \frac{m^3}{r_A r_B}\left(\frac{1}{r_A} + \frac{1}{r_B}\right)\frac{|\mathbf{x}_B - \mathbf{x}_A|}{c(1+\mu)}\left[\kappa_3 - (1+\gamma)\kappa_2 \frac{\arccos\mu}{\sqrt{1-\mu^2}} + \frac{(1+\gamma)^3}{1+\mu}\right],$$

where

$$\mu = \mathbf{n}_A.\mathbf{n}_B, \qquad \mathbf{n}_A = \frac{\mathbf{x}_A}{r_A}, \qquad \mathbf{n}_B = \frac{\mathbf{x}_B}{r_B}.$$

Enhanced terms up to order G^3

Case where $\mathbf{x}_A$ and $\mathbf{x}_B$ are in almost opposite directions (*i.e.* $\mu \sim -1$)



$$\mu \sim -1 \implies \frac{1}{1+\mu} \sim \frac{2r_A r_B}{(r_A+r_B)^2}\frac{r_A r_B}{r_c^2},$$

where

$$r_c = \frac{r_A r_B\sqrt{1-\mu^2}}{|\mathbf{x}_B - \mathbf{x}_A|}.$$

Enhanced terms up to order G^3

Asymptotic expressions of the $\mathcal{T}^{(n)}$ in a conjunction $\longrightarrow$ enhanced terms:

$$\mathcal{T}^{(1)}_{enh}(\mathbf{x}_A, \mathbf{x}_B) \sim \frac{(1+\gamma)m}{c} \ln\left(\frac{4r_A r_B}{r_c^2}\right),$$

$$\mathcal{T}^{(2)}_{enh}(\mathbf{x}_A, \mathbf{x}_B) \sim -2\frac{(1+\gamma)^2 m^2}{c(r_A + r_B)} \frac{r_A r_B}{r_c^2},$$

$$\mathcal{T}^{(3)}_{enh}(\mathbf{x}_A, \mathbf{x}_B) \sim 4\frac{(1+\gamma)^3 m^3}{c(r_A + r_B)^2} \left(\frac{r_A r_B}{r_c^2}\right)^2.$$

(cf. Ashby & Bertotti 2010)

Enhanced terms up to order G^3

These expressions are reliable for $n = 1, 2, 3$ for configurations such that

$$\left|\mathcal{T}^{(n)}(\mathbf{x}_A, \mathbf{x}_B)\right| \ll \left|\mathcal{T}^{(n-1)}(\mathbf{x}_A, \mathbf{x}_B)\right|, \quad \text{with } \mathcal{T}^{(0)}(\mathbf{x}_A, \mathbf{x}_B) = \frac{1}{c}|\mathbf{x}_B - \mathbf{x}_A|$$

These conditions are satisfied if

$$\frac{2m}{r_A + r_B}\frac{r_A r_B}{r_c^2} \ll 1 \qquad \text{(C)}$$

Condition (C) is met in the solar system. For $r_B = 1$ au and $r_A \geq r_B$, one has

$$\frac{2m_\odot}{r_A + r_B}\frac{r_A r_B}{r_c^2} \leq 9.12 \times 10^{-4} \times \frac{R_\odot^2}{r_c^2}, \qquad R_\odot = \text{solar radius.}$$

$\Longrightarrow$ Our results may be applied to the solar system experiments.

Discussion for solar system experiments

Determination of γ in a SAGAS-like scenario: $r_A \approx 50$ au, $r_B \approx 1$ au

$$\text{Shapiro's formula} \quad \Longrightarrow \quad \Delta\mathcal{T} \approx \frac{1}{2}\Delta\gamma\,\mathcal{T}^{(1)}$$

$$R_\odot < r_c < 5R_\odot \quad \Longrightarrow \quad 158\ \mu\text{s} > \mathcal{T}^{(1)}_{enh} > 126\ \mu\text{s}$$

$$\Downarrow$$

$$\gamma \text{ at the level } 10^{-8} \quad \Longleftrightarrow \quad \mathcal{T} \text{ at the level } 0.7\ \text{ps}$$

Discussion for solar system experiments

In this configuration (times given in ps):

$r_c/R_\odot$	$\lvert\mathcal{T}_S^{(1)}\rvert$	$\mathcal{T}_{J_2}^{(1)}$	$\mathcal{T}_{enh}^{(2)}$	$\mathcal{T}_\kappa^{(2)}$	$\mathcal{T}_{enh}^{(3)}$
1	10	2	-17616	123	31.5
2	5	0.5	-4404	61.5	2
5	2	0.08	-704.6	24.6	0.05

Conclusion: $\mathcal{T}_{enh}^{(3)}$ must be taken into account for rays almost grazing the Sun.

Moreover, $\mathcal{T}_{enh}^{(3)}$ can be greater than

- the 1rst-order gravitomagnetic effect: $\left|\mathcal{T}_S^{(1)}\right| \sim \frac{2(1+\gamma)GS_\odot}{c^4 r_c}$, with $S_\odot \approx 2\times 10^{41}$ kg m^2 s^{-1};
- the 1rst-order mass quadupole effect: $\mathcal{T}_{J_2}^{(1)} \sim \frac{(1+\gamma)m_\odot}{c} J_{2\odot}\frac{R_\odot^2}{r_c^2}$, with $J_{2\odot} \approx 2\times 10^{-7}$.

Discussion for solar system experiments

This irruption of a G^3-enhanced term is paradoxical. Remember that

$$t_B - t_A = \frac{1}{c}\left[\sqrt{r_A^2 - r_P^2} + \sqrt{r_B^2 - r_P^2}\right]$$

$$+\frac{(1+\gamma)m}{c}\left[\ln\frac{(r_A + \sqrt{r_A^2 - r_P^2})(r_B + \sqrt{r_B^2 - r_P^2})}{r_P^2} + \sqrt{\frac{r_A - r_P}{r_A + r_P}} + \sqrt{\frac{r_B - r_P}{r_B + r_P}}\right]$$

$$+\frac{m^2}{cr_P}\left\{2\kappa\left[\arccos\frac{r_P}{r_A} + \arccos\frac{r_P}{r_B}\right] - \frac{(1+\gamma)^2}{4}\left[5\sqrt{\frac{r_A - r_P}{r_A + r_P}} + 5\sqrt{\frac{r_B - r_P}{r_B + r_P}}\right.\right.$$

$$\left.\left. - \left(\frac{r_A - r_P}{r_A + r_P}\right)^{3/2} - \left(\frac{r_B - r_P}{r_B + r_P}\right)^{3/2}\right]\right\} + O\left(\frac{m^3}{cr_P^2}\right).$$

Numerically : $\frac{m_\odot}{c} = 4.9\ \mu\text{s}, \quad \frac{m_\odot^2}{cR_\odot} = 10.4\ \text{ps}, \quad \frac{m_\odot^3}{cR_\odot^2} = 0.02\ \text{fs}.$

Discussion for solar system experiments

Assume $r_A = r_B = 1$ au and $r_P = R_\odot$

0th-order term : 499 s

1st-order term : 139 μas

2nd-order term : 162 ps $\Longrightarrow$ no enhanced effect predicted

3rd-order term < 1 fs $\Longrightarrow$ no enhanced effect predicted; negligible

How can we surmount this apparent contradiction?

Discussion for solar system experiments

r_P and b are linked by $b = r_P\sqrt{\mathcal{U}(r_P)}$. Eliminating b between this relation and

$$b = r_c\left[1 + q_1\frac{m}{r_c} + q_2\frac{m^2}{r_c^2} + q_3\frac{m^3}{r_c^3} + \cdots\right],$$

and then using

$$q_n = -c\left(\frac{r_c}{m}\right)^n\frac{\sqrt{1-\mu^2}}{r_c}\frac{\partial\mathcal{T}^{(n)}(r_A, r_B, \mu)}{\partial\mu},$$

r_P may be expressed as a function of $\mathbf{x}_A$ and $\mathbf{x}_B$:

$$r_P = r_c\left\{1 - (1+\gamma-q_1)\frac{m}{r_c} - \frac{1}{2}\left[2\kappa - (1+\gamma)^2 - 2q_2\right]\frac{m^2}{r_c^2} \right.$$
$$\left. -\frac{1}{2}\left[2\kappa_3 - 2\kappa q_1 + (1+\gamma)^2 q_1 - 2q_3\right]\frac{m^3}{r_c^3} + \cdots\right\}.$$

Discussion for solar system experiments

Substituting this expansion in the above expression of $t_B - t_A$ leads to

$$t_B - t_A = \frac{|\mathbf{x}_B - \mathbf{x}_A|}{c} + \mathcal{T}^{(1)}(\mathbf{x}_A, \mathbf{x}_B) + \mathcal{T}^{(2)}(\mathbf{x}_A, \mathbf{x}_B)$$
$$+\text{enhanced term of order } G^3 + \text{ some regular terms of order } G^3).$$

When $\mu \sim -1$

$$q_1 = (1+\gamma)\frac{r_A + r_B}{|\mathbf{x}_B - \mathbf{x}_A|}\frac{\sqrt{1-\mu}}{\sqrt{1+\mu}} \sim 2(1+\gamma)\frac{\sqrt{r_A r_B}}{r_A + r_B}\frac{\sqrt{r_A r_B}}{r_c},$$
$$q_2 \sim -q_1^2,$$
$$q_3 \sim 2q_1^3.$$

$\longrightarrow$ $r_A = r_B = 1$ au, $r_c \approx R_\odot$ $\Longrightarrow$ $q_1 \approx 430$ $\Longrightarrow$ $r_P - r_c \approx 630$ km

$\longrightarrow$ $r_A = 50$ au, $r_B = 1$ au, $r_c \approx R_\odot$ $\Longrightarrow$ $q_1 \approx 843$ $\Longrightarrow$ $r_P - r_c \approx 1240$ km

So r_P and r_c are significantly different in a configuration of quasi-conjunction.

Compare with $b - r_P \approx (1+\gamma)m_\odot \approx 3$ km.

Enhanced terms in the direction of light propagation

Substituting the $\mathcal{T}^{(n)}(\mathbf{x}_A, \mathbf{x}_B)$ into

$$\widehat{\underline{\mathbf{l}}}_A = \left(\frac{l_i}{l_0}\right)_A = c\boldsymbol{\nabla}_{\mathbf{x}_A}\mathcal{T}_r, \qquad \widehat{\underline{\mathbf{l}}}_B = \left(\frac{l_i}{l_0}\right)_B = -c\boldsymbol{\nabla}_{\mathbf{x}_B}\mathcal{T}_r$$

$\longrightarrow$ expansions of $\widehat{\underline{\mathbf{l}}}_A$ and $\widehat{\underline{\mathbf{l}}}_B$ in series in powers of $\frac{m}{r_c}$

Enhanced terms in the direction of light propagation

$$\widehat{\underline{\mathbf{l}}}_B^{(0)}(\mathbf{x}_A,\mathbf{x}_B) = -\mathbf{N}_{AB},$$

$$\widehat{\underline{\mathbf{l}}}_B^{(1)}(\mathbf{x}_A,\mathbf{x}_B) = -\frac{(1+\gamma)m}{r_c}\frac{r_A}{|\mathbf{x}_B-\mathbf{x}_A|}\left[\sqrt{1-\mu^2}\,\mathbf{N}_{AB}-(1-\mu)\mathbf{P}_{AB}\right],$$

$$\widehat{\underline{\mathbf{l}}}_B^{(2)}(\mathbf{x}_A,\mathbf{x}_B) = -\frac{m^2}{r_c^2}\frac{r_A^2}{|\mathbf{x}_B-\mathbf{x}_A|^2}\left\{\left[\cdots\right]\mathbf{N}_{AB} + \left[\cdots+(1+\gamma)^2\left(\frac{r_B}{r_A}+1\right)\frac{(1-\mu)^{3/2}}{\sqrt{1+\mu}}\right]\mathbf{P}_{AB}\right\},$$

$$\widehat{\underline{\mathbf{l}}}_B^{(3)}(\mathbf{x}_A,\mathbf{x}_B) = -\frac{m^3}{r_c^3}\frac{r_A^3(1-\mu)}{|\mathbf{x}_B-\mathbf{x}_A|^3}\left\{\left\{(1+\gamma)^3\left[2+\frac{r_B}{r_A}(1-\mu)\right]\frac{\sqrt{1-\mu}}{\sqrt{1+\mu}}+\cdots\right\}\mathbf{N}_{AB} - \left[2(1+\gamma)^3\left(1+\frac{r_B}{r_A}\right)^2\frac{1-\mu}{1+\mu}+\cdots\right]\mathbf{P}_{AB}\right\},$$

where

$$\mathbf{N}_{AB} = \frac{\mathbf{x}_B-\mathbf{x}_A}{|\mathbf{x}_B-\mathbf{x}_A|}, \qquad \mathbf{P}_{AB} = \mathbf{N}_{AB}\times\left(\frac{\mathbf{n}_A\times\mathbf{n}_B}{|\mathbf{n}_A\times\mathbf{n}_B|}\right).$$

Enhanced terms in the direction of light propagation

When $\mu \sim -1$

- no enhanced term in $\underline{\widehat{\mathbf{l}}}_B^{(1)}$

- enhanced terms at higher orders:

$$\left(\underline{\widehat{\mathbf{l}}}_B^{(2)}\right)_{enh} \sim -\frac{4(1+\gamma)^2 m^2}{r_c^2}\frac{r_B}{r_c}\frac{r_A^2}{(r_A+r_B)^2}\mathbf{P}_{AB},$$

$$\left(\underline{\widehat{\mathbf{l}}}_B^{(3)}\right)_{enh} \sim \frac{16(1+\gamma)^3 m^3}{r_c^3}\frac{r_B^2}{r_c^2}\frac{r_A^3}{(r_A+r_B)^3}\mathbf{P}_{AB}.$$

1. Deflection of light in a LATOR-like experiment: $r_A = r_B = 1$ au

For a ray passing near the Sun

$$\Delta\chi^{(2)} \sim \left| \left(\widehat{\underline{\mathbf{l}}}_B^{(2)} \right)_{enh} - \left(\widehat{\underline{\mathbf{l}}}_A^{(2)} \right)_{enh} \right| \sim \frac{4(1+\gamma)^2 m^2}{r_c^2} \frac{r_A r_B}{r_c(r_A + r_B)}$$

and

$$\Delta\chi^{(3)} \sim \left| \left(\widehat{\underline{\mathbf{l}}}_B^{(3)} \right)_{enh} - \left(\widehat{\underline{\mathbf{l}}}_A^{(3)} \right)_{enh} \right| \sim \frac{16(1+\gamma)^3 m^3}{r_c^3} \frac{r_A r_B}{(r_A + r_B)^2} \frac{r_A r_B}{r_c^2}.$$

For $r_c = R_\odot$,

$$\Delta\chi^{(2)} \approx 1.6\,\text{mas},$$

$$\Delta\chi^{(3)} \approx 3\,\mu\text{as} \qquad \text{(compare with Hees } et\ al\ \text{2013).}$$

Enhanced terms in the direction of light propagation

2. Deflection of light coming from infinity, with $r_B = 1$ au (GAME, *e.g.*)

$$\Delta\chi^{(2)} \sim \frac{4(1+\gamma)^2 m^2}{r_c^2}\frac{r_B}{r_c} \qquad \text{(Klioner \& Zschocke 2010)}$$

and

$$\Delta\chi^{(3)} \sim \frac{16(1+\gamma)^3 m^3}{r_c^3}\left(\frac{r_B}{r_c}\right)^2 .$$

For $r_B = 1$ au and $r_c = R_\odot$,

$$\Delta\chi^{(2)} \approx 3.2\,\text{mas},$$

$$\Delta\chi^{(3)} \approx 12\,\mu\text{as}.$$

Enhanced terms in the direction of light propagation

The choice of r_c is not truly relevant.

Using the impact parameter b and inverting the expansion

$$b = r_c \left[1 + \sum_{n=1}^{\infty} q_n \left(\frac{m}{r_c} \right)^n \right],$$

we get for the dominant terms in $\widehat{\underline{\mathbf{l}}}_B . \mathbf{P}_{AB}$

$$\widehat{\underline{\mathbf{l}}}_B . \mathbf{P}_{AB} \approx \frac{2(1+\gamma)m}{b} \frac{r_A}{r_A + r_B} + \frac{\kappa \pi m^2}{b^2} \frac{r_A}{r_A + r_B} + \frac{m^3}{b^3} \frac{r_A}{r_A + r_B} \left\{ 4\kappa_3 + 2(1+\gamma)\kappa \left[1 + \frac{r_A^2}{(r_A + r_B)^2} \right] \right\} + O\left(\frac{m^4}{b^4} \right).$$

$\Longrightarrow$ The enhanced contributions are "absorbed" by using b.

Enhanced terms in the direction of light propagation

This conclusion

- could be expected from the total bending of light

$$\Delta\widehat{\chi} = \underbrace{\frac{2(1+\gamma)m}{b}}_{\substack{\uparrow \\ 1.75\text{ as}}} + \underbrace{\frac{\kappa\pi m^2}{b^2}}_{\substack{\uparrow \\ 11\ \mu\text{as}}} + \underbrace{\frac{m^3}{b^3}\left[4\kappa_3 + 4(1+\gamma)\kappa - \frac{2}{3}(1+\gamma)^3\right]}_{\substack{\uparrow \\ 0.1\text{ nas}}} + O\left(\frac{m^4}{b^4}\right),$$

with

for a light ray grazing the Sun

- is maintained using r_P instead of b. One has

$$b = r_P\sqrt{\mathcal{U}(r_P)} = r_P\left\{1 + \frac{(1+\gamma)m}{r_P} + \frac{1}{2}\left[2\kappa - (1+\gamma)^2\right]\frac{m^2}{r_P^2} + O\left(\frac{m^3}{r_P^3}\right)\right\}.$$

Conclusion

For quasi-Minkowskian light rays:

- Two new methods available for directly calculating the TTF and the direction of light propagation in s.s.s. symmetric space-times, at any given order in G.
- These calculations can be performed with any symbolic computer program.
- They can be extended to more general cases (stationary metrics, e.g.).
- It is explicitly checked that the two procedures lead to the same expressions of TTF up to order G^3.
- We confirm the occurrence of enhanced terms of order G^3 which must be taken into account for modelling
  - time measurements reaching 1 ps
  - light deflection at the level 1 μas
- They can be absorbed by using expansions in series in powers of $\frac{m}{r_P}$ or $\frac{m}{b}$ instead of $\frac{m}{r_c}$.

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