Quantum Gravity (?) from CDT

A. Görlich, J. Jurkiewicz, R. Loll and J. A.\textsuperscript{1}

\textsuperscript{1}Niels Bohr Institute, Copenhagen, Denmark

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Main goal (at least in 80ties) for QG

- Obtain the background geometry \( \langle g_{\mu\nu} \rangle \) we observe
- Study the fluctuations around the background geometry

What lattice gravity (CDT) offers:

- A non-perturbative QFT definition of QG
- A background independent formulation
- An emergent background geometry \( \langle g_{\mu\nu} \rangle \)
- The possibility to study the quantum fluctuations around this emergent background geometry.
Problems to confront for a lattice theory

(1) How to face the non-renormalizability of quantum gravity (this is a problem for any field theory of quantum gravity, not only lattice theories)

(2) Provide evidence of a continuum limit (where the continuum field theory has the desired properties)

(3) If rotation is performed to Euclidean signature, how does one deal with the unboundedness of the Euclidean Einstein-Hilbert action?

(4) If there exists no continuum field theory of gravity, can a lattice theory be of any use?
Effective QFT of gravity

We believe gravity exists as an effective QFT for $E^2 \ll 1/G$.

True for other non-renormalizable theories

Weak interactions

$$\mathcal{L} = \bar{\psi} \partial \psi + G_F \bar{\psi}(\cdot) \psi \bar{\psi}(\cdot) \psi$$

Nonlinear sigma model

$$\mathcal{L} = (\partial \pi)^2 + \frac{1}{F^2_\pi} \frac{(\pi \partial \pi)^2}{1 - \pi^2 / F^2_\pi}$$

Good for $E^2 \ll 1/G_F$ and $E^2 \ll F^2_\pi$. 

Effective QFT of gravity

Lowest order quantum correction to the gravitational potential of a point particle:

\[
\frac{G}{r} \rightarrow \frac{G(r)}{r}, \quad G(r) = G \left(1 - \omega \frac{G}{r^2} + \cdots\right), \quad \omega = \frac{167}{30\pi}.
\]

The gravitational coupling constant becomes scale dependent and transferring from distance to energy we have

\[
G(E) = G(1 - \omega GE^2 + \cdots) \approx \frac{G}{1 + \omega GE^2}.
\]
Effective QFT of the electric charge

Same calculation in QED

\[
\frac{e^2}{r} \rightarrow \frac{e^2(r)}{r}, \quad e^2(r) = e^2 \left( 1 - \frac{e^2}{6\pi^2} \ln(mr) + \cdots \right), \quad mr \ll 1.
\]

The electric charge is also scale dependent and has a Landau pole

\[
e^2(E) = e^2 \left( 1 + \frac{e^2}{6\pi^2} \ln(E/m) + \cdots \right) \approx \frac{e^2}{1 - \frac{e^2}{6\pi^2} \ln(E/m)}.
\]
\[ GE^2 \ll 1 \quad \rightarrow \quad G(E)E^2 \ll 1 \]

BUT

\[ G(E)E^2 < 1 \quad (\ll 1 ?). \]

Suddenly seems as if quantum gravity has become an (almost) reliable quantum theory at all energy scales.

The behavior can be described the \( \beta \) function for QG. For the dimensionless coupling constant \( \tilde{G}(E) = G(E)E^2 \)

\[ E \frac{d\tilde{G}}{dE} = \beta(\tilde{G}), \quad \beta(\tilde{G}) = 2\tilde{G} - 2\omega \tilde{G}^2. \]

Two fixed points \( (\beta(\tilde{G}) = 0) \): \( \tilde{G} = 0 \) and \( \tilde{G} = 1/\omega \).
Generic situation for asymptotic free theories in $d$ dimensions, extended to $d + \varepsilon$ dimensions.

$$\beta(g) \rightarrow \varepsilon g + \beta(g)$$
The four-Fermi action, the nonlinear sigma model and QG are all renormalizable theories in 2d, with a negative $\beta$-function and have a $2 + \varepsilon$ expansion. For QG first explored by Kawai et al.

Alternatively one can use the exact renormalization group approach. (Reuter et al., Litim, ......). Philosophy: asymptotic safety (Weinberg).
Continuum limit?
Defining the continuum limit in lattice field theory

Let the lattice coordinate be $x_n = a n$, $a$ being the lattice spacing and $O(x_n)$ an observable.

$$- \log \langle O(x_n) O(x_m) \rangle \sim |n - m|/\xi(g_0) + o(|n - m|).$$

$$\xi(g_0) \propto \frac{1}{|g_0 - g_0^c|^\nu}, \quad a(g_0) \propto |g_0 - g_0^c|^\nu.$$ 

$$m_{ph}a(g_0) = 1/\xi(g_0), \quad e^{-|n-m|/\xi(g_0)} = e^{-m_{ph}|x_n - x_m|}$$

$\langle O(x_n) O(y_m) \rangle$ falls off exponentially like $e^{-m_{ph}|x_n - y_m|}$ for $g_0 \to g_0^c$ when $|x_n - y_m|$, but not $|n - m|$, is kept fixed in the limit $g_0 \to g_0^c$. 
How to define the equivalent of \( \langle \mathcal{O}(x_n)\mathcal{O}(y_m) \rangle \) in a diffeomorphism invariant theory (F. David (1991))

\[
\langle \phi \phi(R) \rangle \equiv \int \mathcal{D}[g_{\mu\nu}] \ e^{-S[g_{\mu\nu}]} \times
\int \int \sqrt{g(x)} \sqrt{g(y)} \ \langle \phi(x)\phi(y) \rangle_{\text{matter}}[g_{\mu\nu}] \delta(R - d_{\mu\nu}(x, y)).
\]

\( \langle \phi(x)\phi(y) \rangle_{\text{matter}}[g_{\mu\nu}] \) denotes the correlator of the matter fields calculated for a fixed geometry, defined by the metric \( g_{\mu\nu}(x) \).

It works in 2d Euclidean QG (Liouville gravity)
Already the discussion about continuum limit of the lattice theories hinted a rotation to Euclidean signature. The Einstein-Hilbert action is unbounded from below, caused by the conformal factor:

\[ \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \]

\[ S[g, \Lambda, G] = -\frac{1}{16\pi G} \int d^4\xi \sqrt{g} \left( R - 2\Lambda \right). \]

\[ S[\tilde{g}, \Lambda, G] = -\frac{1}{16\pi G} \int d^4\xi \sqrt{g} \left( \Omega^2 R + 6\partial^\mu \Omega \partial_\mu \Omega - 2\Lambda \Omega^4 \right). \]

How is this dealt with ?.
Using the lattice regularization called dynamical triangulations (DT) the Euclidean action is bounded for a fixed lattice spacing $a$ and a fixed four-volume $V_4 = N_4 a^4$. However, for $a \to 0$ the unboundedness re-emerges.

$$S[T] = -\kappa_2 N_2(T) + \kappa_4 N_4(T), \quad c_1 < \frac{N_2}{N_4} (= x) < c_2.$$ 

The unbounded configurations corresponds to $x \approx c_2$. But are they important in the non-perturbative path integral?

$$Z = \sum_T e^{-S[T]} = \sum_{N_4} e^{-\kappa_4 N_4} \sum_{N_2} \mathcal{N}(N_2, N_4) e^{\kappa_2 N_2}$$

$$\mathcal{N}(N_2, N_4) e^{\kappa_2 N_2} = P_{N_4}(x), \quad \sum_x P_{N_4}(x) = f(N_4) e^{\kappa_4 c_2 N_4}$$
\[ P_{N_4}(x) \approx A e^{N_4 \left( \kappa_4^c - \alpha(x-x_0)^2 \right)} + \tilde{A} e^{N_4 \left( \tilde{\kappa}_4^c - \tilde{\alpha}(x-\tilde{x}_0)^2 \right)}. \]

Phase transition when \( \kappa_4^c = \tilde{\kappa}_4^c \).
Do we know examples of such entropy driven phase transitions? Yes, the \textit{Kosterlitz-Thouless transition} in the XY model. This Abelian 2d spin model has vortices with energy

\[ E = \kappa \ln(R/a) \]

Saturating the partition function by single vortex configurations:

\[ Z \equiv e^{-F/k_B T} = \sum_{\text{spin configurations}} e^{-E[\text{spin}]/k_B T} \approx \left( \frac{R}{a} \right)^2 e^{-[\kappa \ln(R/a)]/k_B T}. \]

\[ S = k_B \ln(\text{number of configurations}) \] has the same functional form as the vortex energy. Thus

\[ F = E - ST = (\kappa - 2k_B T) \ln(R/a) \]
Examples

- Lattice compact $U(1)$ gauge theory in 3 dimensions has confinement for all values of the coupling constant, due to lattice monopoles. It describes perfectly the non-perturbative physics of the Georgi-Glashow model, i.e. the physics below the scale of Higgs and the W-particle. The formula for the string tension is the same expressed in terms of lattice monopoles masses and continuum monopole masses.

- Lattice compact $U(1)$ gauge theory in 4 dimensions at the phase transition point describes the low energy physics of certain broken $\mathcal{N} = 1, 2$ supersymmetric field theories. In fact, one can use the supersymmetric symmetry breaking technology of Seiberg et al. scale matching to “post-dict” (unfortunately) the lattice critical exponents.
Basic tool: The path integral

Text-book example: non-relativistic particle in one dimension.

\[ x(t) = \langle x(t) \rangle + y(t) \]

\[ \langle |y| \rangle \propto \sqrt{\hbar/m\omega} \]

In QG we want \( \langle x(t) \rangle \)

\[ \langle |y| \rangle \propto \sqrt{\hbar G} \]

Transition amplitude as a weighted sum over all possible trajectories. On the plot: time is discretized in steps \( a \), trajectories are piecewise linear.
In a continuum limit $a \to 0$

\[
G(x_i, x_f, t) := \int_{\text{trajectories}: x_i \to x_f} e^{iS[x(t)]}
\]

where $S[x(t)]$ is a classical action.

The QG amplitude between the two geometric states

\[
G(g_i, g_f, t) := \int_{\text{geometries}: g_i \to g_f} e^{iS[g_{\mu\nu}(t')]}
\]

To define this path integral we need a geometric cut-off $a$ and a definition of the class of geometries entering.
showcasing piecewise linear geometries via building blocks:
CDT slicing in proper time. Topology of space preserved.

\[ a_t^2 = -\alpha a_s^2, \quad iS_L[\alpha] = -S_E[-\alpha] \]

\[ S_E[-\alpha] = -\left( \kappa_0 + 6\Delta \right) N_0 + \kappa_4 \left( N_{4}^{(2,3)} + N_{4}^{(1,4)} \right) + \Delta \left( N_{4}^{(2,3)} + 2N_{4}^{(1,4)} \right) \]
\[ G(g_i, g_f, t) := \int_{\text{geometries: } g_i \rightarrow g_f} e^{iS[g_{\mu\nu}(t')]} \]

\[ = \lim_{a \rightarrow 0} \sum_{T: T_i^{(3)} \rightarrow T_f^{(3)}} \frac{1}{C_T} e^{iS_T} \]

\[ G_E(g_i, g_f, t, \kappa_0, \kappa_4, \Delta) = \lim_{a \rightarrow 0} \sum_{T: T_i^{(3)} \rightarrow T_f^{(3)}} \frac{1}{C_T} e^{-S_E[T]} \]

\[ \langle x_f | e^{i\hat{H}t} | x_i \rangle \rightarrow \langle x_f | e^{-\hat{H}\tau} | x_i \rangle \]
Scaling in the IR limit?

\[ Z(\kappa_0, \kappa_4) = \sum_{N_4} e^{-\kappa_4 N_4} Z_{N_4}(\kappa_0), \]

where \( Z_{N_4}(\kappa_0) \) is the partition function for a fixed number \( N_4 \) of four-simplices (we ignore \( \Delta \) for simplicity), namely,

\[ Z_{N_4}(\kappa_0) = e^{k_4 N_4} f(N_4, \kappa_0) \]

We want to consider the limit \( N_4 \to \infty \), and fine-tune \( \kappa_4 \to \kappa_4^c \) for fixed \( \kappa_0 \). We expect the physical cosmological constant \( \Lambda \) to be defined by the approach to the critical point according to

\[ \kappa_4 = \kappa_4^c + \frac{\Lambda}{16\pi G} a^4, \quad (\kappa_4 - \kappa_4^c) N_4 = \frac{\Lambda}{16\pi G} V_4, \quad V_4 = N_4 a^4, \]
How can one imagine obtaining an interesting continuum behavior as a function of $\kappa_0$? Assume $f(N_4, \kappa_0)$ has the form (numerical evidence)

$$f(N_4, \kappa_0) = e^{k_1(\kappa_0) \sqrt{N_4}}, \quad \langle e^{-\frac{1}{G} \int_{V_4} \sqrt{g} R} \rangle = e^{c\frac{\sqrt{V_4}}{G}}.$$ 

$$Z(\kappa_4, \kappa_0) = \sum_{N_4} e^{-(\kappa_4 - \kappa_4^c)N_4 + k_1(\kappa_0)\sqrt{N_4}}.$$ 

Search for $\kappa_0^c$ with $k_1(\kappa_0^c) = 0$, with the approach to this point governed by

$$k_1(\kappa_0) \propto \frac{a^2}{G}, \quad \text{i.e.} \quad k_1(\kappa_0) \sqrt{N_4} \propto \frac{\sqrt{V_4}}{G}.$$ 

$$Z(\kappa_4, \kappa_0) \approx \exp \left( \frac{k_1^2(\kappa_0)}{4(\kappa_4 - \kappa_4^c)} \right) = \exp \left( \frac{c}{G\Lambda} \right),$$

as one would naively expect from Einstein’s equations, with the partition function being dominated by a typical instanton contribution, for a suitable constant $c$. 

J. Ambjørn QG from CDT
If we are close to the UV fixed point, we know that $G$ will not be constant when we change scale, but $\hat{G}(a)$ will. Writing $G(a) = a^2 \hat{G}(a) \approx a^2 \hat{G}^*$,

$$\kappa_4 - \kappa_4^c = \frac{\Lambda}{G(a)} a^4 \approx \frac{\Lambda}{\hat{G}^*} a^2,$$

$$k_1(\kappa_0^c) = \frac{a^2}{G(a)} \approx \frac{1}{\hat{G}^*}.$$

The first of these relations now looks two-dimensional because of the anomalous scaling of $G(a)$! Nevertheless, the expectation value of the four-volume is still finite:

$$\langle V_4 \rangle = \langle N_4 \rangle a^4 \propto \frac{\kappa_1^2(\kappa_0^c)}{(\kappa_4 - \kappa_4^c)^2} a^4.$$
Assume now that we have a fixed point for gravity. The gravitational coupling constant is dimensionful, and we can write for the bare coupling constant

\[ G(a) = a^2 \hat{G}(a), \quad a \frac{d\hat{G}}{da} = -\beta(\hat{G}), \quad \beta(\hat{G}) = 2\hat{G} - c\hat{G}^3 + \cdots. \]

The putative non-Gaussian fixed point corresponds to \( \hat{G} \to \hat{G}^\ast \), i.e. \( G(a) \to \hat{G}^\ast a^2 \). In our case it is tempting to identify our dimensionless constant \( k_1 \) with \( 1/\hat{G} \), up to the constant of proportionality. Close to the UV fixed point we have

\[ \hat{G}(a) = \hat{G}^\ast - \tilde{c}a, \quad k_1 = k_1^\ast + \tilde{c}a, \quad \tilde{c} = -\beta'(\hat{G}^\ast). \]

Usually one relates the lattice spacing near the fixed point to the bare coupling constants with the help of some correlation length \( \xi \).
Consider $V_4 = N_4 a^4$ as fixed. It requires the fine-tuning of coupling constants.

$$k_1(N_4) = k_1^c - \tilde{K} N_4^{-\tilde{c}/4}.$$ 

How to determine $k_1(N_4)$?
Phase diagram of CDT

Lifshitz-like diagram....

Phase C: constant magnetization (constant 4d geometry)
Phase B: zero magnetization (no 4d geometry)
Phase A: oscillating magnetization (conformal mode ?)
Volume distribution in (imaginary) time

- **Phase A.** The universe “oscillating” in time direction. The oscillation maybe reflecting the dominance of the conformal mode.

- **Phase B.** Compactification into a 3d Euclidean DT. Only minimal extension in the time direction.

- **Phase C.** Extended de Sitter phase. $d_H = 4$. 

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J. Ambjørn  
QG from CDT
A typical configuration. Distribution of a spatial volume $N_3(t)$ as a function of (imaginary) time $t$.

Quantum fluctuation around a **semiclassical background**?

Configuration consists of a “stalk” of the cut-off size and a “blob”. Center of the blob can shift. **We fix the “center of mass” to be at zero time.**
\[ \langle N_3(i) \rangle \propto N_4^{3/4} \cos^3 \left( \frac{i}{s_0 N_4^{1/4}} \right) \]
The semiclassical distribution can be obtained from the minisuperspace effective action of Hartle and Hawking

$$S_{\text{eff}} = \frac{1}{24\pi G} \int dt \sqrt{g_{tt}} \left( \frac{g^{tt} \dot{V}_3^2(t)}{V_3(t)} + k_2 V_3^{1/3}(t) - \lambda V_3(t) \right) ,$$

The discretization of this action is (and we have reconstructed it from the date (the 3-volume–3-volume correlations))

$$S_{\text{discr}} = k_1 \sum_i \left( \frac{(N_3(i + 1) - N_3(i))^2}{N_3(i)} + \tilde{k}_2 N_3^{1/3}(i) - \tilde{\lambda} N_3(i) \right) ,$$

$$G = \frac{a^2 \sqrt{C_4} \ s_0^2}{k_1 \ 3\sqrt{6}} .$$
Quantum fluctuations

The classical solution to the minisuperspace action is

\[ \sqrt{g_{tt}} \, V_3^{cl}(t) = V_4 \, \frac{3}{4B} \cos^3 \left( \frac{t}{B} \right) \]

where \( \tau = \sqrt{g_{tt}} \, t \), \( V_4 = 8\pi^2 R^4 / 3 \) and \( \sqrt{g_{tt}} = R / B \).

Writing \( V_3(t) = V_3^{cl}(t) + x(t) \) we can expand the action around this solution

\[ S(V_3) = S(V_3^{cl}) + \frac{1}{18\pi G V_4} \int dt \, x(t) \hat{H} x(t) \]

where the Hermitian operator \( \hat{H} \) is:

\[ \hat{H} = -\frac{d}{dt} \frac{1}{\cos^3(t/B)} \frac{d}{dt} - \frac{4}{B^2 \cos^5(t/B)} \]
In the **quadratic approximation** the volume fluctuations are:

$$C(t, t') := \langle x(t)x(t') \rangle \sim \hat{H}^{-1}(t, t').$$

\(\hat{C}\) and \(\hat{H}\) have the same eigenfunctions.

\(C(t, t')\) can be **measured** as

$$C(i, i') = \left\langle \left( N_3(i) - \langle N_3(i) \rangle \right) \left( N_3(i') - \langle N_3(i') \rangle \right) \right\rangle,$$

and its eigenfunctions can be found and compared to the ones calculated from \(\hat{H}\).
No parameters are put in! (expect $t_i/B = i/s_0 N_4^{1/4}$)

We conclude that the quadratic approximation to the minisuperspace action describes the measured quantum fluctuations well.
For a specific value of the bare coupling constants \( \kappa_0 = 2.2, \Delta = 0.6 \) we have high-statistics measurements for \( N_4 \) ranging from 45,500 to 362,000 four-simplices.

Largest universe corresponds to approx. \( 10^4 \) hyper-cubes.

We have \( G = \text{const.} \, a^2 / k_1 \) and we have measured \( k_1 \).

\[
G \approx 0.23a^2, \quad \ell_P \approx 0.48a, \quad \ell_P \equiv \sqrt{G}.
\]

From \( V_4 = 8\pi^2 R^4 / 3 = C_4 N_4 a^4 \), we obtain that

\[
R = 3.1a
\]

The linear size \( \pi R \) of the quantum de Sitter universes studied here lies in the range of 12-21 \( \ell_P \) for the \( N_4 \) used.
$\ell_P = \sqrt{G} \propto \frac{a}{\sqrt{k_1(\kappa_0, \Delta)}}$ i.e. $k_1(\kappa_0, \Delta) \rightarrow 0$.

BUT IS IT POSSIBLE?
Summary and perspectives

- We have obtained the (Euclidean) minisuperspace action from first principles. *(The self-organized de Sitter space)*
- We have an effective field theory of (something we call) QG down to a few Planck scales.

- Investigate a possible UV fixed point (points, the B-C line). Possibly Hořava-Lifshitz gravity.
- Couple matter to the system and investigate cosmological implications.
- Measure the wave function of the universe

$$
\langle x | e^{-t\hat{H}} | y \rangle \rightarrow \Psi_0(y)\Psi_0(x) e^{-tE_0}
$$