The action for higher spin black holes

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Fields and their interactions

Fields with spins lower or equal than two ($s \leq 2$) interact happily with each other and with themselves,

	$s=0$ ϕ	$s=\frac{1}{2}$ Ψ	$s=1$ A_μ	$s=\frac{3}{2}$ Ψ_μ	$s=2$ $g_{\mu\nu}$
ϕ	✓	✓	✓	✓	✓
Ψ		✓	✓	✓	✓
A_μ			✓	✓	✓
Ψ_μ				✓	✓
$g_{\mu\nu}$					✓

but the situation changes dramatically for fields with $s > 2$:

Free higher spin field theories (Fronsdal equations)

Free equations of motion for higher spin fields can be built in a very symmetric and algorithmic way,

spin	equation	gauge symmetry
$s = 1$	$\Box A_\sigma - \partial_\sigma \partial^\nu A_\nu = 0$	$\delta A_\sigma = \partial_\sigma \epsilon$
$s = 2$	$\Box h_{\sigma\nu} - \partial_\sigma \partial^\rho h_{\rho\nu} - \partial_\nu \partial^\rho h_{\sigma\rho}$ $+\partial_\sigma \partial_\nu h = 0$	$\delta h_{\sigma\nu} = \partial_\sigma \epsilon_\nu + \partial_\nu \epsilon_\sigma$
$s = 3$	$\Box h_{\sigma\nu\sigma} - \partial_\sigma \partial^\rho h_{\rho\nu\sigma} \ldots\ldots = 0$	$\delta h_{\sigma\nu\rho} = \partial_\sigma \epsilon_{\nu\rho} + \partial_\rho \epsilon_{\sigma\nu} + \partial_\nu \epsilon_{\rho\sigma}$
$\vdots$	$\vdots$	$\vdots$

but adding interactions and/or self-interactions is a difficult problem.

Interactions – preserving gauge invariance.

- Self interactions for vector ($s = 1$) and tensor ($s = 2$) fields are not restricted (by gauge invariance):

$$
\begin{aligned}
I_{s=1} &= \int \Big(F_{\sigma\nu} F^{\sigma\nu} + f(F_{\sigma\nu} F^{\sigma\nu}) \Big) \\
I_{s=2} &= \int \sqrt{g} (R + f(R) + \cdots)
\end{aligned}
$$

  (Of course, at $s = 2$, $\sqrt{g}R$ is already an interacting theory.)

- Self-interactions of $s > 2$ higher spin fields described by symmetric tensors

$$
g_{\mu\nu\rho}, \quad g_{\mu\nu\rho\sigma}, \dots
$$

  are severely restricted. The only known interacting action (Vasiliev) involves the whole tower of fields with all s.

In three dimensions life is easier

The magic is provided by the Chern-Simons action,

$$I[A_\mu] = \frac{k}{4\pi} \int \left(AdA + \frac{2}{3} A^3 \right), \qquad A = A_\mu dx^\mu \in \mathcal{G}.$$

- This action has a cubic A^3 interaction.
- Gauge and diffeomorphism (trivial) invariant

$$\delta_\lambda A^a_\mu = D_\mu \lambda^a, \qquad \delta_\xi A^a_\mu = F^a_{\mu\nu} \xi^\nu \approx 0$$

- Possess non-trivial solutions on topologically non-trivial manifolds. And applications to knot theory
- Provides a gauge field theory formulation of three-dimensional gravity

$SL(2,\Re)\times SL(2,\Re)$ and three-dimensional gravity

Consider two $SL(2,\Re)$ Chern-Simons fields,

$$A_\mu = \begin{pmatrix} a_\mu & b_\nu \\ c_\mu & -a_\mu \end{pmatrix}, \qquad \bar{A}_\mu = \begin{pmatrix} \bar{a}_\mu & \bar{b}_\nu \\ \bar{c}_\mu & -\bar{a}_\mu \end{pmatrix}$$

then, the following equality follows (Achúcarro-Townsend 1986)

$$I[A] - I[\bar{A}] = \frac{1}{16\pi G}\int d^3x\sqrt{g}(R+\Lambda).$$

The dictionary between $A, \bar{A}$ and metric variables is

$$g_{\mu\nu} = \mathrm{Tr}(e_\mu e_\nu) \qquad \text{where} \qquad e_\mu = A_\mu - \bar{A}_\mu$$

$$\Gamma^\mu{}_{\lambda\rho} = (e^{-1}\, w\, e + e^{-1}\partial e)^\mu{}_{\lambda\rho} \quad \text{where} \quad w_\mu = A_\mu + \bar{A}_\mu$$

$$k = \frac{\ell}{4G}$$

$SL(N,\Re) \times SL(N,\Re)$ and higher spin fields

Let $A_\sigma, \bar{A}_\sigma$ be two $SL(N,\Re)$ Chern-Simons fields and

$$e_\mu = A_\mu - \bar{A}_\mu.$$

Define now $N-1$ metrics (Cayley-Hamilton theorem)

$$\begin{aligned}
g_{\mu\nu} &= \mathrm{Tr}(e_\mu e_\nu) \\
g_{\mu\nu\rho} &= \mathrm{Tr}(e_{(\mu} e_\nu e_{\rho)}) \\
g_{\mu\nu\rho\sigma} &= \mathrm{Tr}(e_{(\mu} e_\nu e_\rho e_{\sigma)}) \\
&\vdots \\
g_{\sigma_1\sigma_2\ldots\sigma_N} &= \mathrm{Tr}(e_{(\sigma_1} e_{\sigma_2} \cdots e_{\sigma_N)})
\end{aligned}$$

1. These fields satisfy Fronsdal equations, when linearized, on the AdS background. Thus Chern-Simons theory provides interactions for higher spin gauge fields, preserving gauge invariance.
2. Asymptotic symmetries are W_N algebras (Henneaux et al, Campoleoni et al (2010))

Black holes

Not a lot is known yet about these theories.... But black holes have been found.

For $N = 3$, $g_{\mu\nu}$ and $g_{\mu\nu\rho}$ have the structure:

$$
\begin{aligned}
g_{\sigma\nu} dx^\sigma dx^\nu &= f_2(r) dt^2 + \frac{dr^2}{f_2(r)} + r^2 d\phi^2, \\
g_{\sigma\nu\rho} dx^\sigma dx^\nu dx^\rho &= d\phi \left(f_3(r) dt^2 + \frac{dr^2}{\chi(r) f_3(r)} + z_3^2(r) d\phi^2 \right)
\end{aligned}
$$

where $f_2(r)$ and $f_3(r)$ vanishes at the same point.

See, for example, Gutperle-Kraus (2011) and Castro el at (2012).

Our plan

1. Topological characterization of solutions.
2. Regularity conditions ($\rightarrow$ Hawking temperature)
3. The Euclidean on-shell action ('free energy') for black holes.

We shall not discuss the emergence of W_N algebras. See the extensive recent –and not too recent– literature for details:

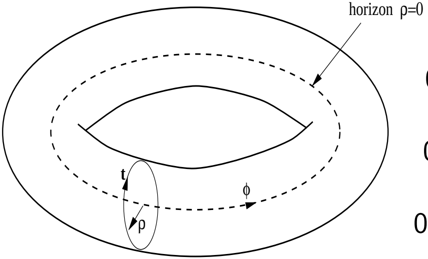
SL($N, \Re$) Chern-Simons $\rightarrow$ 2d affine algebras $|_{\text{reduced}} \rightarrow$

W_N algebras

3d Euclidean black holes live on a solid torus

Example: $$ds^2 = +(r^2 - M)dt^2 + \frac{dr^2}{r^2 - M} + r^2 d\phi^2$$

In the Euclidean geometry, the time coordinate is compact.



$0 < \rho < \infty \qquad \rho = r - M^{1/2}$

$0 \leq t < \beta,$ contractible loop

$0 \leq \phi < 2\pi,$ non-contractible loop

The three dimensional spacetime topology can be seen as a

$$\text{torus} \times \Re^+ = \underline{\text{disc} \times S_1.}$$

Interesting (not zero), regular, solutions

Interesting solutions $A_\mu = \{A_t, A_r, A_\phi\} \in SL(N, \Re)$ must satisfy:

1. The Chern-Simons equations of motion $F_{\mu\nu} = 0$
2. Must have a **non-trivial** holonomy along ϕ:

$$Pe^{\oint A_\phi d\phi} \neq 1$$

   If this holonomy was trivial, the solution can be set to zero by a gauge transformation.

3. Must have a **trivial** holonomy along t.

$$Pe^{\oint A_t dt} = 1.$$

   If this holonomy is not trivial, the field will be singular, because the time cycle is contractible.

Solutions are then characterized by conditions on A_t and A_ϕ.

Note, that A_t and A_ϕ are coupled through $F_{t\phi} = 0$.

Building the general solution in radial gauge $A_r = 0$

- $F_{\mu\nu} = 0$ in the gauge $A_r = 0$ imply

$$A_t(t,\phi), \qquad A_\phi(t,\phi), \qquad \partial_t A_\phi - \partial_\phi A_t + [A_t, A_\phi] = 0.$$

- Furthermore, for black holes, we consider static and spherically symmetric fields. That is, we take A_t, A_ϕ to be constant matrices. The equations reduce to:

$$[A_t, A_\phi] = 0$$

In summary, our game will be to find constant $SL(N, \Re)$ matrices A_ϕ, A_t that commute, and satisfy the holonomy conditions.

$$Pe^{\oint A_\phi d\phi} = e^{2\pi A_\phi} \neq 1, \qquad Pe^{\oint A_t dt} = e^{A_t} = 1$$

A_ϕ and charges

Let A_ϕ be a general $SL(N,\Re)$ matrix,

$$A_\phi = \left(\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & .. & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & \dots & \dots & a_{NN} \end{array} \right), \qquad a_{ij} \in \Re$$

with

$$\mathrm{Tr}(A_\phi) = 0, \qquad Pe^{\int_0^{2\pi} A_\phi} \neq 1$$

The coefficients a_{ij} are not really relevant, but only the $N-1$ Casimirs or **charges**,

$$Q_2 = \mathrm{Tr}(A_\phi^2), \quad Q_3 = \mathrm{Tr}(A_\phi^3), \quad \dots \quad Q_N = \mathrm{Tr}(A_\phi^N),$$

All physical quantities will depend on these charges. See below.

A_t and chemical potentials

For a given A_ϕ we seek A_t such that $[A_t, A_\phi] = 0$, for *all* charges. One concludes that A_t must be a function of A_ϕ,

$$A_t = f(A_\phi).$$

Furthermore, the most general function is (Cayley-Hamilton theorem)

$$A_t \;=\; \sigma_2 A_\phi + \sigma_3 A_\phi^2 + \cdots + \sigma_N A_\phi^{N-1} - \text{Trace}$$

Besides the $N-1$ charges, A_t brings in $N-1$ new parameters $\sigma_2, \sigma_3, ...\sigma_N$. Solutions are characterized by pairs $\{\sigma_1, Q_1\}$, $\{\sigma_2, Q_2\}$... which turn out to be canonically conjugated.

Finally, the trivial holonomy condition (regularity)

$$Pe^{\oint A_t} = 1,$$

imply exactly $N-1$ equations that fix the chemical potentials σ_n as functions of the charges Q_n, or the other way around.

Example $N = 3$

A good parametrization for A_ϕ is:

$$A_\phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ Q_3 & Q_2 & 0 \end{pmatrix}, \qquad A_t = \sigma_2 A_\phi + \sigma_3 \left(A_\phi^2 - \frac{1}{3}\mathrm{Tr}(A_\phi^2) \right)$$

$(Q_2 = \mathrm{Tr}(A_\phi^2),\ Q_3 = \mathrm{Tr}(A_\phi^3).)$

The condition $Pe^{\oint A_t dt} = 1$ becomes:

$$\begin{aligned} 0 &= -16\,\sigma_3^3\,Q_2^3 + 72\,\sigma_3(1+\sigma_2)^2\,Q_2^2 + 54\,\sigma_3^2\,(1+\sigma_2)\,Q_3\,Q_2 \\ &\quad +27\,\sigma_3^3\,Q_3^2 + 27\,(1+\sigma_2)^3\,Q_3\,, \\ 8\,\pi^2 &= 4\,Q_2\,(1+\sigma_2)^2 + 6\,Q_3\,(1+\sigma_2)\sigma_3 + \frac{8}{3}\,Q_2^2\,\sigma_3^2\,. \end{aligned}$$

These 2 equations express Q_2, Q_3 as functions of σ_2, σ_3. Then, the black hole has two independent parameters (spin 2 and spin 3 charge).

Integrability and free energy

From these relations Gutperle and Kraus discover a "coincidence":

$$\frac{\partial Q_2}{\partial \sigma_3} \; = \; \frac{-16\,\sigma_3\,Q_2^2 + 9\,(1+\sigma_2)\,Q_3}{8\,\sigma_3^2\,Q_2 - 3\,(1+\sigma_2)^2} \; = \; \frac{\partial Q_3}{\partial \sigma_2}$$

This equality implies that there must exists a function $W(\sigma_2, \sigma_3)$ such that,

$$Q_2 = \frac{\partial W(\sigma_2, \sigma_3)}{\partial \sigma_2}, \qquad Q_3 = \frac{\partial W(\sigma_2, \sigma_3)}{\partial \sigma_3},$$

What does W mean?

The partition function

Gutperle and Kraus conjectured the identification,

$$e^{-W(\sigma_2,\sigma_3)} = \mathrm{Tr}_{\mathcal{H}}\ e^{-(\sigma_2 Q_2+\sigma_3 Q_3)},$$

$W(\sigma_2, \sigma_3)$ is the free energy and

$$Q_2 = \frac{\partial W(\sigma_2,\sigma_3)}{\partial\sigma_2}, \qquad Q_3 = \frac{\partial W(\sigma_2,\sigma_3)}{\partial\sigma_3},$$

is not a "coincidence" but derived from the partition function.

- We will see that this idea is indeed correct. In the semiclassical limit, we shall prove that the on-shell action

$$\frac{k}{4\pi}\int\left(AdA+\frac{2}{3}A^3\right)\Big|_{\text{solution}} \equiv W(\sigma_2,\sigma_3)$$

is in fact the solution to the integrability condition.

Plugging a solution into its action seems easy...

- Often the solution in globally defined coordinates is not known. Black holes, for example, need two patches.

For free theories, this is easily solved:

$$
\begin{aligned}
I[\Phi] &= \frac{1}{2}\int dx\sqrt{g}\left(-g^{\sigma\nu}\frac{\partial\Phi}{\partial x^{\sigma}}\frac{\partial\Phi}{\partial x^{\nu}} - m^2\Phi^2\right), \\
&= \frac{1}{2}\int dx\sqrt{g}\Phi\left(\Box\Phi - m^2\Phi\right) - \int_{r\to\infty} d\Sigma^{\nu}\,\Phi\partial_{\nu}\Phi, \\
&= 0 - \int_{r\to\infty} d\Sigma^{\nu}\,\Phi\partial_{\nu}\Phi
\end{aligned}
$$

- The bulk part is zero for all solutions. We only need the field in one patch (at infinity) to find $I[\Phi]_{\text{on-shell}}$

But for an interacting theory, life is not as easy.

diff-invariant Hamiltonian theories. Time quantization

For static solutions, the Hamiltonian action provides an alternative

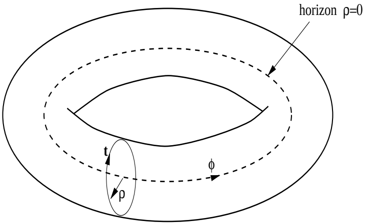
$$\begin{aligned} I &= \int (p_i \dot{q}^i - N\mathcal{H} - N^i \mathcal{H}_i) + B_\infty - B_+ \\ &= \quad 0 \quad + B_\infty - B_+ \end{aligned}$$

The bulk part is zero on-shell ($\mathcal{H} = 0$) for static fields ($\dot{q} = 0$). Good.

- $B_\infty = \beta(M + \Omega J)$ + other charges. Well understood.
- This procedure, however, still needs the field in two patches; infinity and horizon.
- In fact, B_+ arises because Schwarzschild coordinates are singular at horizon. In all known cases (to me) B_+ =Entropy.
- However, a general expression for B_+ valid for any theory is not known. For higher spin black holes, is not known yet.

The trick: Angular quantization

Consider now the angle ϕ as 'time': angular foliation.



The foliation is now regular everywhere

$$
\begin{aligned}
I &= \int (p_i \partial_\phi q^i - \tilde{N}\tilde{\mathcal{H}} - \tilde{N}^i \tilde{\mathcal{H}}_i) + B'_\infty \\
&= 0 + B'_\infty
\end{aligned}
$$

- Spherical symmetry ($\partial_\phi q = 0$) plus the constraints make the bulk part again equal to zero.
- We are left with a term on only one patch, at infinity. Easily calculated.

Chern-Simons action in angular quantization.

A 2+1 angular decomposition: $x^\mu = \phi, x^\alpha$.

$$
\begin{aligned}
I_{CS} &= \frac{k}{4\pi}\int \epsilon^{\sigma\nu\rho}\left(A_\sigma\partial_\rho A_\nu + \frac{2}{3}A_\sigma A_\nu A_\rho\right) \\
&= \frac{k}{4\pi}\int \epsilon^{\alpha\beta}\left(A_\alpha\partial_\phi A_\beta + A_\phi F_{\alpha\beta}\right) - \frac{k}{4\pi}\int_\infty dt d\phi \mathrm{Tr}(A_t A_\phi) \\
&= -\frac{k}{2}\mathrm{Tr}(A_t A_\phi).
\end{aligned}
$$

- We have 4 parameters $Q_2, Q_3, \sigma_2, \sigma_3$, plus two relations. Which is the right choice?

$$
I_{CS}(\sigma_2,\sigma_3)? \quad I_{CS}(\sigma_2,Q_3)? \quad I_{CS}(\sigma_3,Q_2)? \quad I_{CS}(Q_2,Q_3)?
$$

- This information is encoded in the variation of the action.

Action variation

- Varying the Chern-Simons action we have,

$$\delta I_{CS} = \frac{k}{4\pi}\int F\delta A - \frac{k}{4\pi}\int_{\infty} dt d\phi \, \mathrm{Tr}(A_t \delta A_\phi - A_\phi \delta A_t)$$

The boundary term tells us what is fixed. Plugging

$$A_t = \sigma_2 A_\phi + \sigma_3 A_\phi^2 + \cdots + \sigma_N A_\phi^{N-1} - (\mathrm{Trace})$$

to obtain (after some rearrangements),

$$\delta I_{CS} = (eom) + k\sum_n Q_n \delta\sigma_n - \delta\left(\frac{k}{2}\sum_n (2-n)\sigma_n Q_n\right)$$

So, I_{CS} is neither a function of $\{Q_2, Q_3, \ldots\}$ nor $\{\sigma_2, \sigma_3, \ldots\}$.

- Consider instead,

$$\begin{aligned} W &\equiv I_{CS} + \frac{k}{2}\sum_n (2-n)\sigma_n Q_n \\ \delta W &= k\sum_n Q_n \delta\sigma_n \quad \to \quad W(\sigma_2, \sigma_3) \end{aligned}$$

$N = 3$

$$
\begin{aligned}
W(\sigma_2, \sigma_3) &= \frac{k}{4\pi}\int \left(AdA + \frac{2}{3}A^3\right) - \left(2k\sigma_3 Q_3\right) \\
&= 2k\left(3\sigma_2\sigma_3 Q_3 + \frac{4}{3}\sigma_3^2 Q_2^2 + 2(\sigma_2^2 - 1)Q_2 - \sigma_3 Q_3\right).
\end{aligned}
$$

- And the crucial check (a fantastic partial derivative exercise)

$$
\frac{\partial W}{\partial \sigma_2} = kQ_2, \qquad \frac{\partial W}{\partial \sigma_3} = kQ_3
$$

as discovered by Gutperle and Kraus.

Regularity and Invertibility. Giving up the gauge $A_r = 0$.

- On the torus, the vector field $\frac{\partial}{\partial t}$ has a fixed point at the horizon: A_t should vanish there for $A_t dt$ to be well-defined. But our A_t is constant!
- Also, if $A_r = 0$ the metric is not invertible.

These two problems can be tackled by changing the gauge. Consider $r-$dependent group elements $g_1(r), g_2(r)$,

$$A \to g_1^{-1} A \, g_1 + g_1^{-1} dg_1, \qquad \bar{A} \to g_2^{-1} \bar{A} \, g_2 + g_2^{-1} dg_2,$$

The fields are still static and spherically symmetric.

- Everything we said before (gauge invariant) still holds!
- $A_r, \bar{A}_r$ are now different from zero and the metric

$$e_\mu = A_\mu - \bar{A}_\mu, \qquad g_{\mu\nu} = \mathrm{Tr}(e_\mu e_\nu)$$

is invertible.

Regularity is more subtle:

- We cannot impose both $A_t = 0$ and $\bar{A}_t = 0$ at the horizon, while keeping the solution static and spherically symmetric.
- But we can impose half of the conditions:

$$e_t = A_t - \bar{A}_t = 0 \qquad (\text{at } r = r_0)$$

while leaving the spin$-N$ connection to be singular

$$w_t = A_t + \bar{A}_t \neq 0, \qquad (\text{at } r = r_0).$$

- The curvature will be regular.

- and, since e_t vanishes at the horizon, the metric fields

$$g_{tt} = \mathrm{Tr}(e_t^2), \;\; g_{ttt} = \mathrm{Tr}(e_t^3), \;\; g_{tt\phi} = \mathrm{Tr}(e_t^2 e_\phi), \ldots$$

all vanish at the horizon, and thus are regular as well.

Holonomies and Hawking temperature

Following this construction one builds invertible metrics:

$$\begin{aligned} g_{\sigma\nu} dx^\sigma dx^\nu &= f_2(r) dt^2 + \frac{dr^2}{f_2(r)} + r^2 d\phi^2, \\ g_{\sigma\nu\rho} dx^\sigma dx^\nu dx^\rho &= d\phi \left(f_3(r) dt^2 + \frac{dr^2}{\chi(r) f_3(r)} + z^2(r) d\phi^2 \right) \end{aligned}$$

with a regular horizon and the nice property:

- The holonomy conditions are exactly equivalent to periodicity conditions on t (leading to Hawking's temperature).

For the future

1. In general, the entropy is not equal to Area/4. A nice geometrical formula for S is still missing.
2. The Cardy formula does not work in full generality for all black holes, with arbitrary charges.
3. Rotating solutions. The most general black hole not yet been built.
4. The geometry of higher spin fields is not well understood. It would be nice to have an action involving only the metric-like fields.
5. One would like to define curvatures for $g_{\mu\nu\rho}$, analogous to the Riemann curvature for $g_{\mu\nu}$.
6. AdS/CFT and correlators on black hole backgrounds.

Thank you