Dynamics of Bianchi spacetimes

François Béguin
Université Paris-Sud 11 & ÉNS

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Bianchi cosmological models: presentation

**Bianchi spacetimes** are spatially homogeneous (not isotropic) cosmological models.
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**Bianchi spacetimes** are spatially homogeneous (not isotropic) cosmological models.

**Raisons d’être:**

- natural **finite dimensional** class of spacetimes;

- **BKL conjecture**: generic spacetimes “behave like” spatially homogeneous spacetimes close to their initial singularity.
A Bianchi spacetime is a globally hyperbolic spatially homogeneous (but not isotropic) spacetime.
Bianchi cosmological models : definitions

- A **Bianchi spacetime** is a globally hyperbolic spatially homogeneous (but not isotropic) spacetime.

- A **Bianchi spacetime** is a spacetime $(M, g)$ with
  \[ M \simeq I \times G \quad \text{where} \quad I = (t_-, t_+) \subset \mathbb{R}, \]
  \[ g = -dt^2 + h_t \]
  where
  - $G$ is 3-dimensional Lie group,
  - $h_t$ is a left-invariant riemannian metric on $G$. 

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  where \(I = (t_-, t_+) \subset \mathbb{R}\),
  \(G\) is 3-dimensional Lie group,
  \(h_t\) is a left-invariant riemannian metric on \(G\).

- A **Bianchi spacetime** amounts to a one-parameter family of left-invariant metrics \((h_t)_{t \in I}\) on a 3-dimensional Lie group \(G\).
We will consider **vacuum type A** Bianchi models.

- **Type A**: $G$ is unimodular.
- **Vacuum**: $\text{Ric}(g) = 0$. 
Bianchi cosmological models: definitions

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The results would certainly also hold in the case where:

- $G$ is not unimodular.
- the energy-momentum tensor corresponds to a non-tilted perfect fluid.
A Bianchi spacetime can be seen as a one-parameter family of left-invariant metrics \((h_t)_{t \in I}\) on a 3-dim Lie group \(G\).

The space of left-invariant metrics on \(G\) is finite-dimensional.

\[\text{the Einstein equation } \text{Ric}(g) = 0 \text{ is a system of ODEs.}\]
Proposition. — Consider a Bianchi spacetime \((I \times G, -dt^2 + h_t)\). There exists a frame field \((e_0, e_1, e_2, e_3)\) such that:

- \(e_0 = \partial / \partial t\);
- \(e_1, e_2, e_3\) are tangent to \({\cdot} \times G\) and left-invariant;
- \(\nabla e_0 e_i = 0\) for \(i = 1, 2, 3\);
- \((e_1, e_2, e_3)\) is orthonormal for \(h_t\);
- \([e_1, e_2] = n_3(t) e_3\);
- \([e_2, e_3] = n_1(t) e_1\);
- \([e_3, e_1] = n_2(t) e_2\);
- the second fundamental form of \(h_t\) is diagonal in \((e_1, e_2, e_3)\).
Einstein equation : coordinate choice

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Why taking an ortho**normal** frame?

- One studies the behavior of the **structure constants** $n_1, n_2, n_3$ instead of the behavior of **metric coefficients** $h_t(e_i, e_j)$;
Why taking an ortho**normal** frame?

- One studies the behavior of the **structure constants** $n_1, n_2, n_3$ instead of the behavior of **metric coefficients** $h_t(e_i, e_j)$;

- Key advantage: the various 3-dimensional Lie groups are treated altogether.
Variables

- The three structure constants $n_1(t), n_2(t), n_3(t)$;
- The three diagonal components $\sigma_1(t), \sigma_2(t), \sigma_3(t)$ of the traceless second fundamental form;
- The mean curvature of $\theta(t)$.

Actually, it is convenient to replace $n_i$ and $\sigma_i$ by $N_i = n_i \theta$ and $\Sigma_i = \sigma_i \theta$ by $\tau$ such that $d\tau/dt = -\theta^3$. (Hubble renormalisation; the equation for $\theta$ decouples.)
Variables

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Actually, it is convenient to replace

- \( n_i \) and \( \sigma_i \) by \( N_i = \frac{n_i}{\theta} \) and \( \Sigma_i = \frac{\sigma_i}{\theta} \)
- \( t \) by \( \tau \) such that \( \frac{d\tau}{dt} = -\frac{\theta}{3} \).

(Hubble renormalisation; the equation for \( \theta \) decouples).
The phase space

With these variables, the phase space $\mathcal{B}$ is a (non-compact) four dimensional submanifold in $\mathbb{R}^6$. 
The phase space

With these variables, the phase space \( \mathcal{B} \) is a (non-compact) four dimensional submanifold in \( \mathbb{R}^6 \).

\[
\mathcal{B} = \{ (\Sigma_1, \Sigma_2, \Sigma_3, N_1, N_2, N_3) \in \mathbb{R}^6 \mid \Sigma_1 + \Sigma_2 + \Sigma_3 = 0, \ \Omega = 0 \}
\]

where

\[
\Omega = 6 - (\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2) + \frac{1}{2}(N_1^2 + N_2^2 + N_3^2) - (N_1 N_2 + N_1 N_3 + N_2 N_3).
\]
Wainwright-Hsu equations

\[
\frac{d}{d\tau} \begin{pmatrix}
\Sigma_1 \\
\Sigma_2 \\
\Sigma_3 \\
N_1 \\
N_2 \\
N_3
\end{pmatrix}
= \begin{pmatrix}
(2 - q)\Sigma_1 - R_1 \\
(2 - q)\Sigma_2 - R_2 \\
(2 - q)\Sigma_3 - R_3 \\
-(q + 2\Sigma_1)N_1 \\
-(q + 2\Sigma_2)N_2 \\
-(q + 2\Sigma_3)N_3
\end{pmatrix}.
\]

where

\[
q = \frac{1}{3} (\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2)
\]

\[
R_i = \frac{1}{3} \left( 2N_i^2 - N_j^2 - N_k^2 + 2N_jN_k - N_jN_i - N_iN_k \right).
\]
Wainwright-Hsu equations

We denote by $X_B$ the vector field on $\mathcal{B}$ corresponding to this system of ODEs.

The vacuum type A Bianchi spacetimes can be seen as the orbits of $X_B$. 
Dynamics of $X_B$

The dynamics of $X_B$ appears to be rich and interesting. The study of this dynamics yields to:

- a non-uniformly hyperbolic chaotic map of the circle;
- original questions on continued fractions;
- problems of “linearization” (or existence of “normal forms”);
- delicate problems concerning the absolute continuity of stable manifolds in Pesin theory;
- “Bowen’s eye-like phenomena” yielding to non-convergence of Birkhoff sums.
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Fundamental remark. — The classification of Lie algebras gives rise to an $X_B$-invariant stratification of the phase space $B$. 
Bianchi classification

<table>
<thead>
<tr>
<th>Name</th>
<th>$N_1$</th>
<th>$N_2$</th>
<th>$N_3$</th>
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<tr>
<td>VII$_0$</td>
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<td>+</td>
<td>0</td>
<td>$\text{so}(2) \ltimes \mathbb{R}^2$</td>
</tr>
<tr>
<td>VIII</td>
<td>+</td>
<td>+</td>
<td>−</td>
<td>$\text{sl}(2, \mathbb{R})$</td>
</tr>
<tr>
<td>IX</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>$\text{so}(3, \mathbb{R})$</td>
</tr>
</tbody>
</table>
Type I models \((\mathfrak{g} = \mathbb{R}^3, N_1 = N_2 = N_3 = 0)\)

- The subset of \(B\) corresponding to type I Bianchi spacetimes is a euclidean circle: the *Kasner circle* \(K\).
Type I models \((g = \mathbb{R}^3, N_1 = N_2 = N_3 = 0)\)

- The subset of \(B\) corresponding to type I Bianchi spacetimes is a euclidean circle: the *Kasner circle* \(\mathcal{K}\).

- Every point of \(\mathcal{K}\) is a fixed point for the flow.
Type I models ($g = \mathbb{R}^3, N_1 = N_2 = N_3 = 0$)

- For every $p \in \mathcal{K}$, the derivative $DX_B(p)$ has:
  - two distinct negative eigenvalues,
  - a zero eigenvalue,
  - a positive eigenvalue.

- Except if $p$ is one of the three special points $T_1, T_1, T_3$, in which case $DX_B(p)$ has:
  - a negative eigenvalue,
  - a triple-zero eigenvalue.
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Type II models ($g = \text{hein}_3$, one of the $N_i$'s is non-zero)

- The subset $\mathcal{B}_{\text{II}}$ of $\mathcal{B}$ corresponding to type II models is the union of three ellipsoids which intersect along the Kasner circle.
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- Every type II orbit converges to a point of $\mathcal{K}$ in the past, and converges to another point of $\mathcal{K}$ in the future.
The subset $B_{II}$ of $B$ corresponding to type II models is the union of three ellipsoids which intersect along the Kasner circle.

Every type II orbit converges to a point of $K$ in the past, and converges to another point of $K$ in the future.

The orbits on one ellipsoid “take off” from one third of $K$, and “land on” the two other thirds.
Type II models ($g = \text{hein}_3$, one of the $N_i$'s is non-zero)
The Kasner map

- We restrict to the subset $\mathcal{B}^+$ of $\mathcal{B}$ where the $N_i$'s are non-negative.
The Kasner map

- We restrict to the subset $B^+$ of $B$ where the $N_i$'s are non-negative.

- For every $p \in \mathcal{K}$, there is one (and only one) type II orbit “taking off” from $p$. In the future, this orbit “land on” at some point $f(p) \in \mathcal{K}$.

- This defines a map $f : \mathcal{K} \longrightarrow \mathcal{K}$: the Kasner map.
The Kasner map
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The Kasner map defines a chaotic dynamical system on the circle.
The Kasner map defines a chaotic dynamical system on the circle.

- The Kasner map $f$ is topologically conjugated to $\theta \mapsto -2\theta$.
- The Kasner map $f$ is *not* uniformly hyperbolic (its derivative is equal to -1 at the Taub points $T_1, T_2, T_3$).
- There exists an arc $\mathcal{K}_0$ of $\mathcal{K}$ such that the map induced by $f$ on $\mathcal{K}$ is the Gauss map.
Type \text{VIII} and \text{IX} models \((g = \text{so}(3, \mathbb{R}) \text{ or } \text{sl}(2, \mathbb{R})\), all the \(N_i\)'s are non-zero)
Type VIII and IX models \((g = \mathfrak{so}(3, \mathbb{R}) \text{ or } \mathfrak{sl}(2, \mathbb{R}))\), all the \(N_i\)'s are non-zero)

- **Vague conjecture.** The dynamics of type VIII and IX orbits “reflects” the dynamics of the Kasner map.
Type \( \text{VIII} \) and \( \text{IX} \) models \((g = \text{so}(3, \mathbb{R}) \text{ or } \text{sl}(2, \mathbb{R})\), all the \( N_i \)'s are non-zero\)

- **Vague conjecture.** The dynamics of type \( \text{VIII} \) and \( \text{IX} \) orbits “reflects” the dynamics of the Kasner map.

- **Example of more precise conjecture.** Almost every type \( \text{IX} \) orbit accumulates on the whole Kasner circle.
Ringström’s theorem

Let $\mathcal{A} := \mathcal{K} \cup \mathcal{B}_I$ be the union of all type I and type II orbits.

**Theorem (Ringström 2000).** $\mathcal{A}$ is attracting all type IX orbits (except for the Taub-NUT type orbits).

$\text{Taub-NUT} \iff \exists i, j \in \{1, 2, 3\} \text{ such that } \Sigma_i = \Sigma_j \text{ and } N_i = N_j.$

(codimension 2 submanifold of the phase space)
Ringström’s theorem

- Ringström’s result does not imply that the dynamics of type IX orbits “reflects” the dynamics of the Kasner map.

- For example, it could be possible that every type IX orbit is attracted by the period 3 orbit of \( f \).
Dynamics of type VIII or IX orbits

Let $q \in \mathcal{K}$, and $r \in \mathcal{B}$. I say that the $X_B$ orbit of $r$ shadows the $f$-orbit of $q$ if there exist $t_0 < t_1 < t_2 < \ldots$ such that

- $\text{dist}(X_{B}^{t_n}(r), f^{n}(q)) \xrightarrow{n \to \infty} 0$;

- the distance between the piece of orbit $\{X_{B}^{t}(r) ; t_n \leq t \leq t_{n+1}\}$ and the type II orbit connecting $f^{n}(q)$ to $f^{n+1}(q)$ goes to 0.

The point $r$ is necessarily of type VIII or IX.

Given $q \in \mathcal{K}$, I denote by $W^{s}(q)$ the set of points $r$ such that the $X_B$ orbit of $r$ shadows the $f$-orbit of $q$. 
Dynamics of type VIII or IX orbits

**Theorem (Béguin 2010)** There exists $k_0 \in \mathbb{N}$ with the following property. Consider $q \in \mathcal{K}$ such that the closure of the $f$-orbit of $q$ does not contain any periodic orbit of period $\leq k_0$. Then $W^s(q)$ is non-empty.
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Actually, $W^s(q)$ is a three-dimensional injectively immersed manifold which depends continuously on $q$ (when $q$ ranges in a closed $f$-invariant subset of $\mathcal{K}$ without any orbit of period $\leq k_0$).
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**Proposition.** The set of the points $q$ satisfying the hypothesis of the theorem above is dense in $\mathcal{K}$, but has zero Lebesgue measure.
Dynamics of type VIII or IX orbits

Theorem (Georgi, Häterich, Liebscher, Webster, 2010). Consider a point \( q \in \mathcal{K} \) which is periodic point for \( f \). Then \( W^s(q) \) is non-empty.

Caution. This does not imply that almost every Bianchi spacetime is in \( W^s(q) \) for some \( q \).
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**Theorem (Reiterer, Trubowitz, 2010).** There is a full Lebesgue measure subsets of points $q$ in $\mathcal{K}$ such that $W^s(q)$ is non-empty.

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Dynamics of type VIII or IX orbits

**Conjecture.** — La réunion des $W^s(q)$ pour $q \in \mathcal{K}$ est de mesure positive dans $\mathcal{B}$.
Dynamics of type \text{VIII} or \text{IX} orbits

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Question. — Does the union of the $W^s(q)$ has full Lebesgue measure?
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Question. — Does the union of the $W^s(q)$ has full Lebesgue measure?

Remark. — For most points, Birkhoff sums should not converge.
Dynamics of type VIII or IX orbits

Informal interpretation of the results. Close to the initial singularity:

- For all Bianchi spacetimes, the spacelike slice $G \times \{t\}$ is curved in only one direction (Ringström).
Dynamics of type VIII or IX orbits

Informal interpretation of the results. Close to the initial singularity:

- For all Bianchi spacetimes, the spacelike slice $G \times \{t\}$ is curved in only one direction (Ringström).

- For “many” Bianchi spacetimes, this direction oscillates in a complicated periodic or aperiodic way.
Dynamics of type VIII or IX orbits

Informal interpretation of the results. Close to the initial singularity:

- For all Bianchi spacetimes, the spacelike slice $G \times \{t\}$ is curved in only one direction (Ringström).

- For “many” Bianchi spacetimes, this direction oscillates in a complicated periodic or aperiodic way.

- The way this direction oscillates is sensitive to initial conditions.
A Bianchi spacetime is said to be **asymptotically silent** if “different particles cannot have exchanged information arbitrarily close to the initial singularity”.
Asymptotic silence

Formally: for every past inextendible timelike curve $\gamma$, the diameter of the set $J^+(\gamma) \cap (\{t\} \times G)$ goes to 0 as $t \to t_-$. 
Asymptotic silence

Theorem. For $q$ as in one of the three preceding theorems, the orbits in $W^s(q)$ correspond to asymptotically silent spacetimes.
About the proof of the theorem.

The key is to understand what happens to type IX orbits when they pass close to the Kasner circle. Indeed:

- close to the Kasner circle, there should be some ”supra-linear contraction-dilatation phenomena”;
- far from the Kasner circle, everything is ”at most linear”.
Hartman Grobman theorem.

Consider a vector field $X$ and a point $p$ such that $X(p) = 0$.

**Theorem.** Assume that $DX(p)$ does not have any purely imaginary eigenvalue.

Then, there is a $C^0$ local coordinate system on a neighborhood of $p$, such that $X$ is linear in these coordinates.
Theorem. Assume that $DX(p)$ does not have any purely imaginary eigenvalue. Assume moreover that the eigenvalues of $DX(p)$ are independent other $\mathbb{Q}$.

Then, there is a $C^\infty$ local coordinate system on a neighbourhood of $p$, such that $X$ is linear in these coordinates.
Generalization of Sternberg's theorem to the case where $DX(p)$ has some purely imaginary eigenvalues.

There is a $C^r$ local coordinate system on a neighbourhood of $p$, such that “$X$ depends linearly on the coordinates corresponding to non purely imaginary eigenvalues”.
Linearization of the Wainwright-Hsu vector field near of point of $\mathcal{K}$

Let $X_B$ be the Wainwright-Hsu vector field and $p$ be a point of the Kasner circle which is not one of the three Taub points.
Linearization of the Wainwright-Hsu vector field near of point of $\mathcal{K}$

Let $X_B$ be the Wainwright-Hsu vector field and $p$ be a point of the Kasner circle which is not one of the three Taub points.

**Proposition.** If the three non-zero eigenvalues of $DX_B(p)$ are independent over $\mathbb{Q}$, then there is a $C^\infty$ local coordinate system $(x, x', y, z)$ on a neighbourhood of $p$, such that

$$X(x, x', y, z) = \lambda^s(y)x \frac{\partial}{\partial x} + \lambda^{s'}(y)x' \frac{\partial}{\partial x'} + \lambda^u(y)z \frac{\partial}{\partial z}$$

with $\lambda^s(y) < \lambda^{s'}(y) < 0 < \lambda^u(y)$. 
Characterization of linearizable points

**Proposition.** For $p \in K$, the following conditions are equivalent:
1. the non-zero eigenvalues of $DX(p)$ are independent over $\mathbb{Q}$;
2. the orbit of $p$ under the Kasner map is not pre-periodic.
Dulac map close to a “good” point \( p \in \mathcal{K} \)

\[
\Phi(1, x', y, z) = \left( z - \frac{\lambda s(y)}{\lambda u(y)}, x', z - \frac{\lambda s'(y)}{\lambda u(y)}, y, 1 \right).
\]

\[
x, x' \quad p \quad r \quad \phi(r)
\]

\[
y \quad \mathcal{K} \quad z
\]
Dulac map close to a “good” point $p \in \mathcal{K}$

\[ X(x, x', y, z) = \lambda^s(y)x \frac{\partial}{\partial x} + \lambda^{s'}(y)x' \frac{\partial}{\partial x'} + \lambda^u(y)z \frac{\partial}{\partial z} \]

\[ \Phi(1, x', y, z) = \left( z^{-\lambda^s(y)/\lambda^u(y)} , x'z^{-\lambda^{s'}(y)/\lambda^u(y)} , y , 1 \right). \]
Dulac map close to a “good” point \( p \in \mathcal{K} \)

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\Phi(1, x', y, z) = \left( z^{-\lambda_s(y)/\lambda_u(y)}, x'.z^{-\lambda_s'(y)/\lambda_u(y)}, y, 1 \right).
\]

**Important observation.** The negative eigenvalues are stronger than the positive ones:

\[-\lambda_s(y)/\lambda_u(y) > 1 \quad -\lambda_s'(y)/\lambda_u(y) > 1.\]
Dulac map close to a “good” point $p \in \mathcal{K}$

$$\Phi(1, x', y, z) = \left( z^{-\lambda_s(y)/\lambda_u(y)}, x'z^{-\lambda'_s(y)/\lambda_u(y)}, y, 1 \right).$$

**Important observation.** The negative eigenvalues are stronger than the positive ones:

$$-\frac{\lambda_s(y)}{\lambda_u(y)} > 1 \quad -\frac{\lambda'_s(y)}{\lambda_u(y)} > 1.$$

**Consequence.** $\Phi$ can be extended on $M \cap \{ z = 0 \}$ as a $C^1$ map.

If $z(q) = 0$, then $d\Phi(q) \cdot \frac{\partial}{\partial x'} = d\Phi(c) \cdot \frac{\partial}{\partial z} = 0$

$$d\Phi(q) \cdot \frac{\partial}{\partial y} = \frac{\partial}{\partial y}$$
Dulac map near a point $p \in \mathcal{K}$

- The distance from an orbit to the attractor $\mathcal{A} = \mathcal{B}_I \cup \mathcal{B}_\parallel$ is contracted when the orbit passes close to the Kasner circle $\mathcal{K}$. This contraction is “super-linear”.

- The drift in the direction tangent to $\mathcal{A}$ is negligible compared to this contraction.

- No matter what happens far from the Kasner circle! (this will never compensate the “super-linear contraction”.)
Dulac map near a point $p \in \mathcal{K}$

- The distance from an orbit to the attractor $\mathcal{A} = \mathcal{B}_1 \cup \mathcal{B}_2$ is contracted when the orbit passes close to the Kasner circle $\mathcal{K}$. This contraction is “super-linear”.

- The drift in the direction tangent to $\mathcal{A}$ is neglectible as compared to this contraction.
Dulac map near a point $p \in \mathcal{K}$

- The distance from an orbit to the attractor $\mathcal{A} = \mathcal{B}_I \cup \mathcal{B}_{II}$ is contracted when the orbit passes close to the Kasner circle $\mathcal{K}$. This contraction is “super-linear”.

- The drift in the direction tangent to $\mathcal{A}$ is neglectible as compared to this contraction.

\[
\Rightarrow \text{ No matter what happens far from the Kasner circle! (this will never compensate the “super-linear contraction”.)}
\]
End of the proof

- One constructs a section.

- One shows that the return map is hyperbolic (or rather can be extended to a hyperbolic map).

- One applies a stable manifold theorem.
Control of a set of orbits with positive Lebesgue measure?

- No linéarization results apply. One needs to prove ”by brute force” some estimates of the contraction, the drift...

- One needs to control the size of the neighbourhood of $p$ where the estimates hold. This size goes to zero exponentially fast as $p$ approaches a Taub point.

- One needs to show that “many” orbits fall each time in the neighbourhoods where the estimates holds. Uses some results on the continued fraction development of almost every point.

- One needs to adapt Pesin theory.