Dynamics of Bianchi spacetimes

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Bianchi cosmological models : presentation

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Raisons d'être :

- natural **finite dimensional** class of spacetimes ;
- **BKL conjecture :** *generic spacetimes "behave like" spatially homogeneous spacetimes close to their initial singularity*.

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$$M \simeq I \times G \qquad\qquad g = -dt^2 + h_t$$
where $I = (t_-, t_+) \subset \mathbb{R}$,
G is 3-dimensional Lie group,
h_t is a left-invariant riemannian metric on G.

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- A **Bianchi spacetime** amounts to a one-parameter family of left-invariant metrics $(h_t)_{t \in I}$ on a 3-dimensional Lie group G.

Bianchi cosmological models : definitions

We will consider **vacuum type A** Bianchi models.

- **Type A :** G is unimodular.
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The results would certainly also hold in the case where :

- G is not unimodular.
- the energy-momentum tensor corresponds to a non-tilted perfect fluid.

Einstein equation

A Bianchi spacetime can be seen as a one-parameter family of left-invariant metrics $(h_t)_{t\in I}$ on a 3-dim Lie group G

$+$ The space of left-invariant metrics on G is finite-dimensional

$\Longrightarrow$ the Einstein equation $\mathrm{Ric}(g) = 0$ is a system of ODEs.

Einstein equation : coordinate choice

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- e_1, e_2, e_3 are tangent to $\{\cdot\} \times G$ and left-invariant ;
- $\nabla_{e_0} e_i = 0$ for $i = 1, 2, 3$;
- (e_1, e_2, e_3) is orthonormal for h_t ;
- $[e_1, e_2] = n_3(t) e_3$;
  $[e_2, e_3] = n_1(t) e_1$;
  $[e_3, e_1] = n_2(t) e_2$;
- the second fundamental form of h_t is diagonal in (e_1, e_2, e_3).

Why taking an ortho**normal** frame ?

- One studies the behavior of the **structure constants** n_1, n_2, n_3 instead of the behavior of **metric coefficients** $h_t(e_i, e_j)$;

Why taking an ortho**normal** frame ?

- One studies the behavior of the **structure constants** n_1, n_2, n_3 instead of the behavior of **metric coefficients** $h_t(e_i, e_j)$;
- Key advantage : the various 3-dimensional Lie groups are treated altogether.

Variables

- The three structure constants $n_1(t)$, $n_2(t)$, $n_3(t)$;
- The three diagonal components $\sigma_1(t)$, $\sigma_2(t)$, $\sigma_3(t)$ of the traceless second fundamental form ;
- The mean curvature of $\theta(t)$.

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Actually, it is convenient to replace

- n_i and σ_i by $N_i = \dfrac{n_i}{\theta}$ and $\Sigma_i = \dfrac{\sigma_i}{\theta}$
- t by τ such that $\dfrac{d\tau}{dt} = -\dfrac{\theta}{3}$.

(Hubble renormalisation ; the equation for θ decouples).

The phase space

With these variables, the phase space $\mathcal{B}$ is a (non-compact) four dimensional submanifold in $\mathbb{R}^6$.

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$$\mathcal{B} = \left\{(\Sigma_1, \Sigma_2, \Sigma_3, N_1, N_2, N_3) \in \mathbb{R}^6 \mid \Sigma_1 + \Sigma_2 + \Sigma_3 = 0\,,\ \Omega = 0\right\}$$

where

$$\Omega = 6 - (\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2) + \frac{1}{2}(N_1^2 + N_2^2 + N_3^2) - (N_1 N_2 + N_1 N_3 + N_2 N_3).$$

Wainwright-Hsu equations

$$\frac{d}{d\tau}\begin{pmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \\ N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} (2-q)\Sigma_1 - R_1 \\ (2-q)\Sigma_2 - R_2 \\ (2-q)\Sigma_3 - R_3 \\ -(q+2\Sigma_1)N_1 \\ -(q+2\Sigma_2)N_2 \\ -(q+2\Sigma_3)N_3 \end{pmatrix}.$$

where

$$\begin{aligned} q &= \frac{1}{3}\left(\Sigma_1^2+\Sigma_2^2+\Sigma_3^2\right) \\ R_i &= \frac{1}{3}\left(2N_i^2 - N_j^2 - N_k^2 + 2N_jN_k - N_iN_j - N_iN_k\right). \end{aligned}$$

Wainwright-Hsu equations

We denote by $X_{\mathcal{B}}$ the vector field on $\mathcal{B}$ corresponding to this system of ODEs.

The vaccum type A Bianchi spacetimes can be seen as the orbits of $X_{\mathcal{B}}$.

Dynamics of $X_{\mathcal{B}}$

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- a non-uniformly hyperbolic chaotic map of the circle ;
- original questions on continued fractions ;
- problems of "linearization" (or existence of "normal forms") ;
- delicate problems concerning the absolute continuity of stable manifols in Pesin theory ;
- "Bowen's eye-like phenomena" yielding to non-convergence of Birkhoff sums.

Dynamics of $X_{\mathcal{B}}$

Fundamental remark. — The classification of Lie algebras gives rise to an $X_{\mathcal{B}}$-invariant stratification of the phase space $\mathcal{B}$.

Bianchi classification

Name	N_1	N_2	N_3	$\mathfrak{g}$
I	0	0	0	$\mathbb{R}^3$
II	$+$	0	0	heis_3
VI_0	$+$	$-$	0	$\mathrm{so}(1,1) \ltimes \mathbb{R}^2$
VII_0	$+$	$+$	0	$\mathrm{so}(2) \ltimes \mathbb{R}^2$
VIII	$+$	$+$	$-$	$\mathrm{sl}(2,\mathbb{R})$
IX	$+$	$+$	$+$	$\mathrm{so}(3,\mathbb{R})$

Type I models ($\mathfrak{g} = \mathbb{R}^3$, $N_1 = N_2 = N_3 = 0$)

- The subset of $\mathcal{B}$ corresponding to type I Bianchi spacetimes is a euclidean circle : the *Kasner circle* $\mathcal{K}$.

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- The subset of $\mathcal{B}$ corresponding to type I Bianchi spacetimes is a euclidean circle : the *Kasner circle* $\mathcal{K}$.
- Every point of $\mathcal{K}$ is a fixed point for the flow.

Type I models ($\mathfrak{g} = \mathbb{R}^3$, $N_1 = N_2 = N_3 = 0$)

- For every $p \in \mathcal{K}$, the derivative $DX_{\mathcal{B}}(p)$ has :
  - two distinct negative eignevalues,
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  - a zero eigenvalue,
  - a positive eigenvalue.

- Except if p is one of the three special points T_1, T_1, T_3, in which case $DX_{\mathcal{B}}(p)$ has :
  - a negative eigenvalue,
  - a triple-zero eigenvalue.

Type II models ($\mathfrak{g} = \mathfrak{hein}_3$, one of the N_i's is non-zero)

- The subset $\mathcal{B}_{\text{II}}$ of $\mathcal{B}$ corresponding to type II models is the union of three ellipsoids which intersect along the Kasner circle.

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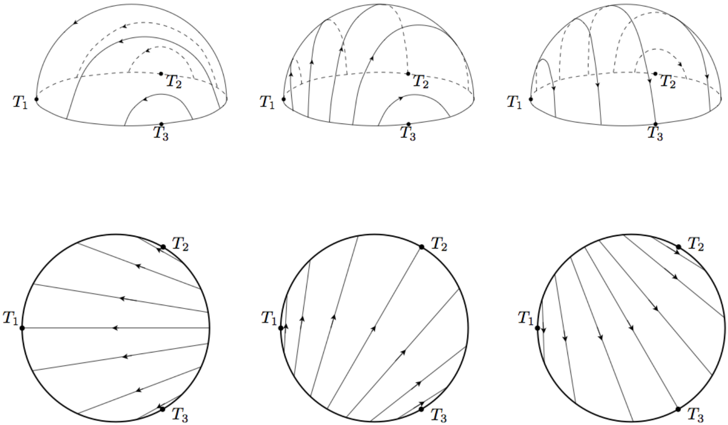
- Every type II orbit converges to a point of $\mathcal{K}$ in the past, and converges to another point of $\mathcal{K}$ in the future.
- The orbits on one ellipsoid "take off" from one third of $\mathcal{K}$, and "land on" the two other thirds.

Type II models ($\mathfrak{g} = \mathrm{hein}_3$, one of the N_i's is non-zero)



The Kasner map

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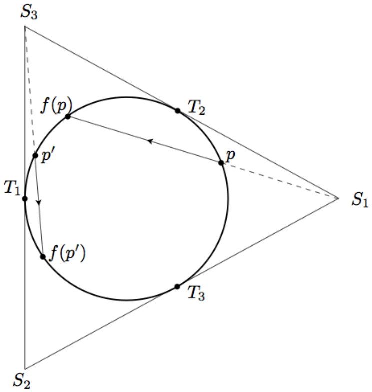
- We restrict to the subset $\mathcal{B}^+$ of $\mathcal{B}$ where the N_i's are non-negative.
- For every $p \in \mathcal{K}$, there is one (and only one) type II orbit "taking off" from p. In the future, this orbit "land on" at some point $f(p) \in \mathcal{K}$.
- This defines a map $f : \mathcal{K} \longrightarrow \mathcal{K}$: the *Kasner map*.

The Kasner map



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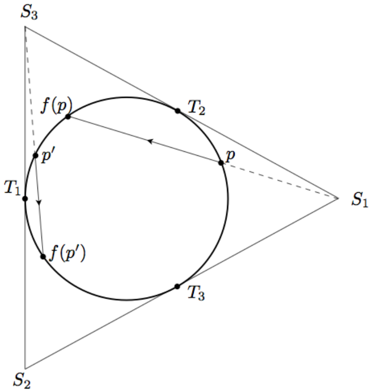

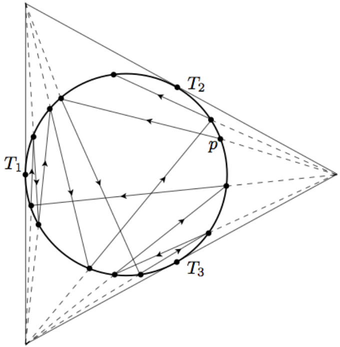

The Kasner map

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- The Kasner map f is topologically conjugated to $\theta \mapsto -2\theta$.
- The Kasner map f is *not* uniformly hyperbolic (its derivative is equal to -1 at the Taub points T_1, T_2, T_3).
- There exists an arc $\mathcal{K}_0$ of $\mathcal{K}$ such that the map induced by f on $\mathcal{K}$ is the Gauss map.

Type VIII and IX models ($\mathfrak{g} = \mathrm{so}(3, \mathbb{R})$ or $\mathrm{sl}(2, \mathbb{R})$, all the N_i's are non-zero)

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Type VIII and IX models ($\mathfrak{g} = \mathrm{so}(3, \mathbb{R})$ or $\mathrm{sl}(2, \mathbb{R})$, all the N_i's are non-zero)

- **Vague conjecture.** The dynamics of of type VIII and IX orbits "reflects" the dynamics of the Kasner map.
- **Example of more precise conjecture.** Almost every type IX orbit accumulates on the whole Kasner circle.

Ringström's theorem

Let $\mathcal{A} := \mathcal{K} \cup \mathcal{B}_{\text{II}}$ be the union of all type I and type II orbits.

Theorem (Ringström 2000). $\mathcal{A}$ is attracting all type IX orbits (except for the Taub-NUT type orbits).

Taub-NUT $\iff \exists i, j \in \{1, 2, 3\}$ such that $\Sigma_i = \Sigma_j$ and $N_i = N_j$.
(codimension 2 submanifold of the phase space)

Ringström's theorem

- Ringström's result does not imply that the dynamics of type IX orbits "reflects" the dynamics of the Kasner map.
- For example, it could be possible that every type IX orbit is attracted by the period 3 orbit of f.

Dynamics of type VIII or IX orbits

Let $q \in \mathcal{K}$, and $r \in \mathcal{B}$. I say that the $X_{\mathcal{B}}$ orbit of r *shadows* the f-orbit of q if there exist $t_0 < t_1 < t_2 < \dots$ such that

- $\mathrm{dist}(X_{\mathcal{B}}^{t_n}(r), f^n(q)) \underset{n\to\infty}{\longrightarrow} 0$;
- the distance between the piece of orbit $\{X_{\mathcal{B}}^t(r) \;;\; t_n \leq t \leq t_{n+1}\}$ and the type II orbit connecting $f^n(q)$ to $f^{n+1}(q)$ goes to 0.

The point r is necessarily of type VIII or IX.

Given $q \in \mathcal{K}$, I denote by $W^s(q)$ the set of points r such that the $X_{\mathcal{B}}$ orbit of r shadows the f-orbit of q.

Dynamics of type VIII or IX orbits

Theorem (Béguin 2010) There exists $k_0 \in \mathbb{N}$ with the following property. Consider $q \in \mathcal{K}$ such that the closure of the f-orbit of q does not contain any periodic orbit of period $\leq k_0$.
Then $W^s(q)$ is non-empty.

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Actually, $W^s(q)$ is a three-dimensional injectively immersed manifold which depends continuously on q (when q ranges in a closed f-invariant subset of $\mathcal{K}$ without any orbit of period $\leq k_0$).

Proposition. The set of the points q satisfying the hypothesis of the theorem above is dense in $\mathcal{K}$, but has zero Lebesgue measure.

Dynamics of type VIII or IX orbits

Theorem (Georgi, Häterich, Liebscher, Webster, 2010).
Consider a point $q \in \mathcal{K}$ which is periodic point for f.
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Theorem (Reiterer, Trubowitz, 2010). There is a full Lebesgue measure subsets of points q in $\mathcal{K}$ such that $W^s(q)$ is non-empty.

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Caution. This does not imply that almost every Bianchi spacetime is in $W^s(q)$ for some q.

Dynamics of type VIII or IX orbits

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Question. — Does the union of the $W^s(q)$ has full Lebesgue measure ?

Remark. — For most points, Birkhoff sums should not converge.

Dynamics of type VIII or IX orbits

Informal interpretation of the results. Close to the initial singularity :

- For all Bianchi spacetimes, the spacelike slice $G \times \{t\}$ is curved in only one direction (Ringström).

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Dynamics of type VIII or IX orbits

Informal interpretation of the results. Close to the initial singularity :

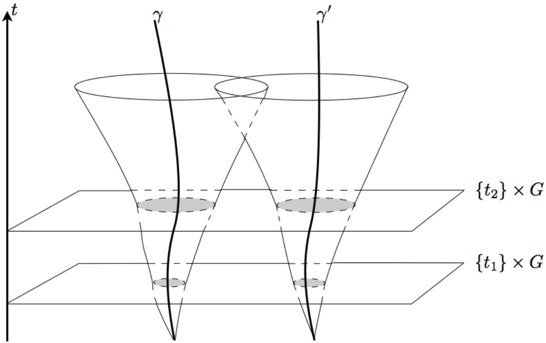
- For all Bianchi spacetimes, the spacelike slice $G \times \{t\}$ is curved in only one direction (Ringström).
- For "many" Bianchi spacetimes, this direction oscillates in a complicated periodic or aperiodic way.
- The way this direction oscillates is sensitive to initial conditions.

Asymptotic silence

A Bianchi spacetime is said to be **asymptotically silent** if "different particles cannot have exchanged information arbitrarily close to the initial singularity".

Asymptotic silence

Formally : for every past inextendible timelike curve γ, the diameter of the set $J^+(\gamma) \cap (\{t\} \times G)$ goes to 0 as $t \to t_-$.



Asymptotic silence

Theorem. For q as in one of the three preceding theorems, the orbits in $W^s(q)$ correspond to asymptotically silent spacetimes.

About the proof of the theorem.

The key is to understand what happens to type IX orbits when they pass *close to the Kasner circle*. Indeed :

- close to the Kasner circle, there should be some "supra-linear contraction-dilatation phenomena" ;
- far from the Kasner circle, everything is "at most linear".

Hartman Grobman theorem.

Consider a vector field X and a point p such that $X(p) = 0$.

Theorem. Assume that $DX(p)$ does not have any purely imaginary eigenvalue.

Then, there is a C^0 local coordinate system on a neighborhood of p, such that X is linear in these coordinates.

Sternberg's theorem

Theorem. Assume that $DX(p)$ does not have any purely imaginary eigenvalue. Assume moreover that the eigenvalues of $DX(p)$ are independent other $\mathbb{Q}$.

Then, there is a C^{∞} local coordinate system on a neighbourhood of p, such that X is linear in these coordinates.

Takens' theorem

Generalization of Sternberg's theorem to the case where $DX(p)$ has some purely imaginary eigenvalues.

There is a C^r local coordinate system on a neighbourhood of p, such that "X depends linearly on the coordinates corresponding to non purely imaginary eigenvalues".

Linearization of the Wainwright-Hsu vector field near of point of $\mathcal{K}$

Let $X_{\mathcal{B}}$ be the Wainwright-Hsu vector field and p be a point of the Kasner circle which is not one of the three Taub points.

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Let $X_{\mathcal{B}}$ be the Wainwright-Hsu vector field and p be a point of the Kasner circle which is not one of the three Taub points.

Proposition. If the three non-zero eigenvalues of $DX_{\mathcal{B}}(p)$ are independant over $\mathbb{Q}$, then there is a C^∞ local coordinate system (x, x', y, z) on a neighbourhood of p, such that

$$X(x, x', y, z) = \lambda^s(y)x\frac{\partial}{\partial x} + \lambda^{s'}(y)x'\frac{\partial}{\partial x'} + \lambda^u(y)z\frac{\partial}{\partial z}$$

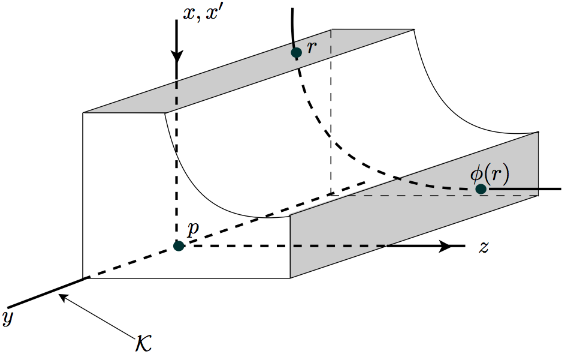
with $\lambda^s(y) < \lambda^{s'}(y) < 0 < \lambda^u(y)$.

Characterization of linearizable points

Proposition. For $p \in \mathcal{K}$, the following conditions are equivalent :

1. the non-zero eigenvalues of $DX(p)$ are independent over $\mathbb{Q}$;
2. the orbit of p under the Kasner map is not pre-periodic.

Dulac map close to a "good" point $p \in \mathcal{K}$



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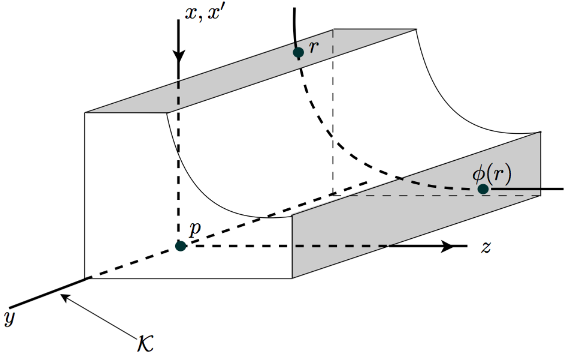

$$X(x, x', y, z) = \lambda^s(y)x\frac{\partial}{\partial x} + \lambda^{s\prime}(y)x'\frac{\partial}{\partial x'} + \lambda^u(y)z\frac{\partial}{\partial z}$$

$$\Phi(1, x', y, z) = \left(z^{-\lambda_s(y)/\lambda_u(y)}\ ,\ x'.z^{-\lambda_s{}'(y)/\lambda_u(y)}\ ,\ y\ ,\ 1\right).$$

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Important observation. The negative eigenvalues are stronger than the positive ones :

$$-\lambda_s(y)/\lambda_u(y) > 1 \qquad -\lambda_s{}'(y)/\lambda_u(y) > 1.$$

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$$-\lambda_s(y)/\lambda_u(y) > 1 \qquad -\lambda_s{'}(y)/\lambda_u(y) > 1.$$

Consequence. Φ can be extended on $M \cap \{z = 0\}$ as a C^1 map.

If $z(q) = 0$, then $d\Phi(q).\frac{\partial}{\partial x'} = d\Phi(c).\frac{\partial}{\partial z} = 0$

$$d\Phi(q).\frac{\partial}{\partial y} = \frac{\partial}{\partial y}$$

Dulac map near a point $p \in \mathcal{K}$

- The distance from an orbit to the attractor $\mathcal{A} = \mathcal{B}_{\mathsf{I}} \cup \mathcal{B}_{\mathsf{II}}$ is contracted when the orbit passes close to the Kasner circle $\mathcal{K}$. This contraction is "super-linear".

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- The drift in the direction tangent to $\mathcal{A}$ is neglectible as compared to this contraction.

$\Longrightarrow$ No matter what happens far from the Kasner circle ! (this will never compensate the "super-linear contraction".)

End of the proof

- One constructs a section.
- One shows that the return map is hyperbolic (or rather can be extended to a hyperbolic map).
- One applies a stable manifold theorem.

Control of a set of orbits with positive Lebesgue measure ?

- No linéarization results apply. One needs to prove "by brute force" some estimates of the contraction, the drift...

- One needs to control the size of the neighbourhood of p where the estimates hold. This size goes to zero exponentially fast as p approaches a Taub point.

- One needs to show that "many" orbits fall each time in the neighbourhoods where the estimates holds. Uses some results on the continued fraction development of almost every point.

- One needs to adapt Pesin theory.