Nekhoroshev theorem and stability of the planetary problem

(1) N-body problem

Evolution of N particles, with positions $q_i \in \mathbb{R}^3$ and masses $m_i > 0$ $(1 \leq i \leq N)$ in gravitational interaction. Newton's equations:

$$(*) \quad m_i \ddot{q}_i = \sum_{j \neq i} m_j m_i \frac{q_j - q_i}{\|q_j - q_i\|^3}, \quad 1 \leq i \leq N$$

defined on $U = \{ q = (q_1, \dots, q_N) \in \mathbb{R}^{3N} \mid q_i \neq q_j, i \neq j \}$

Hamiltonian formulation: $\quad p_i = m_i \dot{q}_i$

$$H(q,p) = \frac{1}{2} \sum_i \frac{\|p_i\|^2}{m_i} - \sum_{j<i} \frac{m_j m_i}{\|q_j - q_i\|}$$

$$(*) \quad \dot{q}_i = \partial_{p_i} H(q,p) \; ; \; \dot{p}_i = -\partial_{q_i} H(q,p), \quad 1 \leq i \leq N$$

$N = 2$: integrable system (Kepler, Newton, ...)

$N \geq 3$: not integrable (Bruns, Poincaré, ...)

(2) Planetary problem

$1+N$ body problem, with masses $m_0, \varepsilon m_1, \dots, \varepsilon m_N$ where ε is "small", $\varepsilon > 0$.

For the solar system, we are interested in $\varepsilon \sim 10^{-3}$.

$$(*)\quad \begin{cases} \ddot{q}_0 = \varepsilon \sum_{j \geq 1} m_j \dfrac{q_j - q_0}{\|q_j - q_0\|^3} \\ \ddot{q}_i = m_0 \dfrac{q_0 - q_i}{\|q_0 - q_i\|^3} + \varepsilon \sum_{\substack{j \geq 1 \\ j \neq i}} m_j \dfrac{q_j - q_i}{\|q_j - q_i\|^3} \qquad 1 \leq i \leq N \end{cases}$$

For $\varepsilon = 0$, we have N uncoupled 2-body problem (Kepler problem, we can assume $q_0 = 0$): for negative energies each particle describibles an ellipse (with a focus at the origin). The system is integrable, and represented by a Hamiltonian H_K.
For $\varepsilon > 0$, the Hamiltonian is $H = H_K + \varepsilon H_P$.

(3) Angle-action coordinates (Delaunay, ...)

$$H : T^*U \simeq U \times \mathbb{R}^{3N} \longrightarrow \mathbb{R}$$

There exists a symplectic change of coordinates

$$(q,p) \in U \times \mathbb{R}^{3N} \longmapsto (\Lambda, \Gamma, \Theta, \ell, g, \theta) \in B \times \mathbb{T}^{3N}$$

where $B \subseteq \mathbb{R}^{3N}$ is an open ~~set~~ set, such that

$$H(\Lambda, \Gamma, \Theta, \ell, g, \theta) = H_K(\Lambda) + \varepsilon H_P(\Lambda, \Gamma, \Theta, \ell, g, \theta)$$

The coordinates depend only on the Keplerian ellipses: Λ depends only on their semi-major axes, Γ depends also on the eccentricities. Equations are:

$$\begin{cases} \dot{\ell} = \partial_\Lambda H \ , & \dot{g} = \partial_\Gamma H \ , & \dot{\theta} = \partial_\Theta H \ ; \\ \dot{\Lambda} = -\partial_\ell H \ ; & \dot{\Gamma} = -\partial_g H \ ; & \dot{\Theta} = -\partial_\theta H \end{cases}$$

For H_K, all variables (except ℓ) are constant:

$$(\Lambda(t), \Gamma(t), \Theta(t)) = (\Lambda(0), \Gamma(0), \Theta(0)) \quad \forall t \in \mathbb{R}$$

<u>Question</u>: For H, study the variation

$$|(\Lambda(t), \Gamma(t), \Theta(t)) - (\Lambda(0), \Gamma(0), \Theta(0))|:$$

- small variation $(\sim \varepsilon)$: stability
- large variation $(\gtrsim 1)$: instability

(4) Abstract mathematical model

$B \subseteq \mathbb{R}^n$ open set, $H : B \times \mathbb{T}^n \to \mathbb{R}$ real-analytic

$$H(\theta, I) = h(I) + \varepsilon f(\theta, I) \ , \qquad \varepsilon > 0$$

Equations: $\dot{\theta} = \partial_I H \ , \ \dot{I} = -\partial_\theta H$

For $\varepsilon > 0$, $I(t) = I(0) \ \forall t \in \mathbb{R}$.

<u>Question</u>: For $\varepsilon > 0$, study the variation $|I(t) - I(0)|$

Planetary problem: $n = 3N$, $h = H_K$, $f = H_P$ but h depends only on part of the variables I ("proper degeneracy").

(5) Nekhoroshev theorem

Theorem (Nekhoroshev, ~70): Assume h is convex.
Then there exist positive constants ε_0, c_1, c_2 such that if $\varepsilon \leq \varepsilon_0$, for all solutions $(\theta(t), I(t))$ of H

$$|I(t) - I(0)| \leq r(\varepsilon), \quad |t| \leq T(\varepsilon)$$

with $r(\varepsilon) = c_1 \varepsilon^{\frac{1}{2n}}$, $T(\varepsilon) = \exp\left(\left(\frac{c_2}{\varepsilon}\right)^{\frac{1}{2n}}\right)$

"stability over an exponentially long interval of time"

- Kolmogorov (~50): under similar assumptions, "most" solutions are stable for all time ("less physical")
- Arnold (~60): unstable solutions exist

Constants ε_0, c_1, c_2 depend on n, the diameter of B, the convexity of h and the width of analyticity of f.

(6) Applicability of Nekhoroshev theorem

The goal is to apply Nekhoroshev theorem to the planetary problem with:

$$\varepsilon \sim 10^{-3}, \quad T(\varepsilon) \sim \text{age of the universe}$$

Several difficulties:

* Proper degeneracy : the theorem cannot be applied directly because $h = H_K$ is degenerate, fortunately one can use the specific $f = H_p$ to "remove" the degeneracy and apply the theorem

* Dependence of ε_0 on n : because of degeneracy, one has $n = N$ (and even $n = N-1$). It can be proved that :

$$\varepsilon_0 < e^{-n} \quad \text{(Bourgain-Kaloshin, 05)}$$

$$\varepsilon_0 < e^{-n(\ln(n \ln n))} \quad (11)$$

Solution : just take $N = 2$ (3BP)

* Smallness of ε_0 : for the problem Sun-Jupiter-Saturn, one can prove (Niederman, 96)

$\varepsilon \sim 10^{-24}$, $T(\varepsilon) \sim$ age of the universe

(7) Variants of Nekhoroshev theorem

Theorem (12) : Assume h is convex, and let $\rho > 0$. There exist positive constants $\varepsilon_0(\rho)$ and c st if $\varepsilon \leq \varepsilon_0(\rho)$, for all solutions $(\theta(t), I(t))$ of H

$$|I(t) - I(0)| \leq \rho, \quad |t| \leq T(\varepsilon, \rho) = \exp\left(\left(\frac{c\rho^2}{\varepsilon}\right)^{\frac{1}{2(n-1)}}\right)$$

__Theorem__ : Assume h is convex, and let $\rho > 0$ and $k \geq 1$ an integer. There exist positive constants $\varepsilon_0(\rho, k)$ and c such that if $\varepsilon \leq \varepsilon_0(\rho, k)$, for all solutions $(\theta(t), I(t))$ of H

$$|I(t) - I(0)| \leq \rho, \quad |t| \leq T(\varepsilon, \rho, k) = c\left(\frac{\rho^2}{\varepsilon}\right)^{\frac{k-2}{2(n-1)}}$$

The second statement is a variation of the first, the first statement is a more general and flexible version of Nekhoroshev theorem.

Point : when $\rho \nearrow$, $\varepsilon_0(\rho) \nearrow$ in the first statement, when $\rho \nearrow$ and $k \searrow$, $\varepsilon_0(\rho, k) \nearrow$ in the second statement.

Goal : choose ρ (or ρ and k) such that $\varepsilon_0(\rho)$ (or $\varepsilon_0(\rho, k)$) is $\sim 10^{-3}$, with $T(\varepsilon, \rho)$ (or $T(\varepsilon, \rho, k)$) as large as possible.