BPS NEAR-HORIZON GEOMETRY OF 5D BLACK HOLES AND RINGS

Bernard de Wit

Utrecht University

Nikhef Amsterdam

IHÉS, Bures-sur Yvette

13 April 2011

The general context:

What is the statistical interpretation of black hole entropy ?

or, what is the relation between:

microscopic/statistical entropy ⟷
macroscopic/field-theoretic entropy

- microstate counting ⟶ entropy $S_{\rm micro} = \ln d(q,p)$
- supergravity: Noether surface charge *Wald, 1993*
  first law of black hole mechanics *(BH thermodynamics)*

STRING THEORY and SUPERSYMMETRY *Strominger, Vafa, 1996*

Ideal testing ground: *supergravities with 8 supercharges*

two obvious theories

- *D=4 space-time dimensions with N=2 supersymmetry*
- *D=5 space-time dimensions with N=1* supersymmetry

here: *near-horizon analysis in D=5 dimensions*

To clarify previous results and to further the understanding of the connection with four-dimensional results.
New features:

- *Chern-Simons terms*
- *both black holes and black rings*

Castro, Davis, Kraus, Larsen, 2008
dW, Katmadas, 2009

Further work in progress with Nabamita Banerjee and Stefanos Katmadas

BPS black holes and rings in five space-time dimensions

two different supersymmetric horizon topologies !

- S^3 **(SPINNING) BLACK HOLE** *Breckenridge, Myers, Peet, Vafa, 1996*
- $S^1 \times S^2$ **BLACK RING** *Elvang, Emparan, Mateos, Reall, 2004*

with near-horizon geometry: $\mathrm{AdS}_2 \times S^3$ *or* $\mathrm{AdS}_3 \times S^2$

(this result does not *depend on the specific Lagrangian)*

superconformal multiplet calculus

- off-shell irreducible supermultiplets
  in superconformal gravity background
- extra superconformal gauge invariances
- gauge equivalence (compensating supermultiplets)

dW, van Holten, Van Proeyen, et al., 1980-85
Bergshoeff, Vandoren, Van Proeyen, et al., 2001-04
Fujita, Ohashi, 2001
Hanaki, Ohashi, Tachikawa, 2006

vector supermultiplets contain
- scalar fields: σ^I
- vector fields: $W_\mu{}^I$

abelian field strengths $F_{\mu\nu}{}^I = \partial_\mu W_\nu{}^I - \partial_\nu W_\mu{}^I$

supergravity (Weyl) multiplet contains
(auxiliary) tensor $T_{\mu\nu} \;\to\; T_{ab} \begin{cases} T_{01} \\ T_{23} \end{cases}$

$$v^2 \equiv (T_{01})^2 + (T_{23})^2$$

supersymmetry + partial gauge choice

$\sigma^I = \text{constant}$ (remain subject to residual (constant) scale transformations!)

$T_{\mu\nu}$ *conformal Killing-Yano tensor*

$$\mathcal{D}_\rho T_{\mu\nu} = \tfrac{1}{2} g_{\rho[\mu}\,\xi_{\nu]} \qquad \xi^\mu T_{\mu\nu} = 0$$

Killing vector associated with the fifth dimension ψ

$$\xi^\mu \propto e^{-1}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,T_{\nu\rho}\,T_{\sigma\tau}$$

$\mathrm{AdS}_2 \times S^2$

$$ds^2 = \frac{1}{16\,v^2}\Big(-r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\Big) + \mathrm{e}^{2g}\big(\mathrm{d}\psi + \sigma\big)^2$$

$$\sigma = -\frac{1}{4\,v^2}\,\mathrm{e}^{-g}\,\big(T_{23}\,r\,\mathrm{d}t - T_{01}\,\cos\theta\,\mathrm{d}\varphi\big)$$

Horizon area $A_3 = \int_{\Sigma_{\rm hor}} = \pi^2 v^{-2}\,\mathrm{e}^g$

Horizon bi-normal $\epsilon_{01} = \pm 1$

Two distinct cases:

- $T_{01} \neq 0$ **SPINNING BLACK HOLE** *Breckenridge, Myers, Peet, Vafa, 1996*

  angular momentum $\propto \dfrac{T_{23}}{T_{01}}$

- $T_{01} = 0$ **BLACK RING** *Elvang, Emparan, Mateos, Reall, 2004*

Additional horizon condition and 'magnetic' charges

$$F_{\mu\nu}{}^{I} = 4\,\sigma^{I}\,T_{\mu\nu} \qquad \mathcal{Q}_{\mu\nu} = \partial_\mu\sigma_\nu - \partial_\nu\sigma_\mu$$

$$F_{\theta\varphi} \longrightarrow p^{I} = \frac{\sigma^{I}}{4\,v^2}\,T_{23}$$

$$\mathcal{Q}_{\theta\varphi} \longrightarrow p^{0} = \frac{\mathrm{e}^{-g}}{4\,v^2}\,T_{01}$$

Scale invariance (residue of conformal invariance)

$\sigma^I\,, T_{ab}\,, v\,, \mathrm{e}^{-g}$ scale uniformly

the metric is scale dependent

Action in 5 space-time dimensions consists of two cubic invariants, each containing a Chern-Simons terms.

$$\mathcal{L} \propto C_{IJK}\, \varepsilon^{\mu\nu\rho\sigma\tau}\, W_\mu{}^I F_{\nu\rho}{}^J F_{\sigma\tau}{}^K$$

$$\mathcal{L} \propto c_I\, \varepsilon^{\mu\nu\rho\sigma\tau}\, W_\mu{}^I\, R_{\nu\rho}{}^{ab} R_{\sigma\tau\, ab}$$

Hanaki, Ohashi, Tachikawa, 2006

4D analogue

$$F(Y,\Upsilon) = \frac{D_{IJK} Y^I Y^J Y^K + d_I Y^I\, \Upsilon}{Y^0}$$

$$\Upsilon = -64 \qquad\qquad Y^I = \tfrac{1}{2}(\phi^I + \mathrm{i}p^I)$$

Gauge fields

on S^3 the gauge fields are globally defined

$$W_\mu{}^I \mathrm{d}x^\mu = -\frac{\sigma^I}{4\,v^2}\left(T_{01}\,r\,\mathrm{d}t + T_{23}\left(\frac{\mathrm{d}\psi}{p^0} + \cos\theta\,\mathrm{d}\varphi\right)\right)$$

on $S^1 \times S^2$ the gauge fields are not globally defined

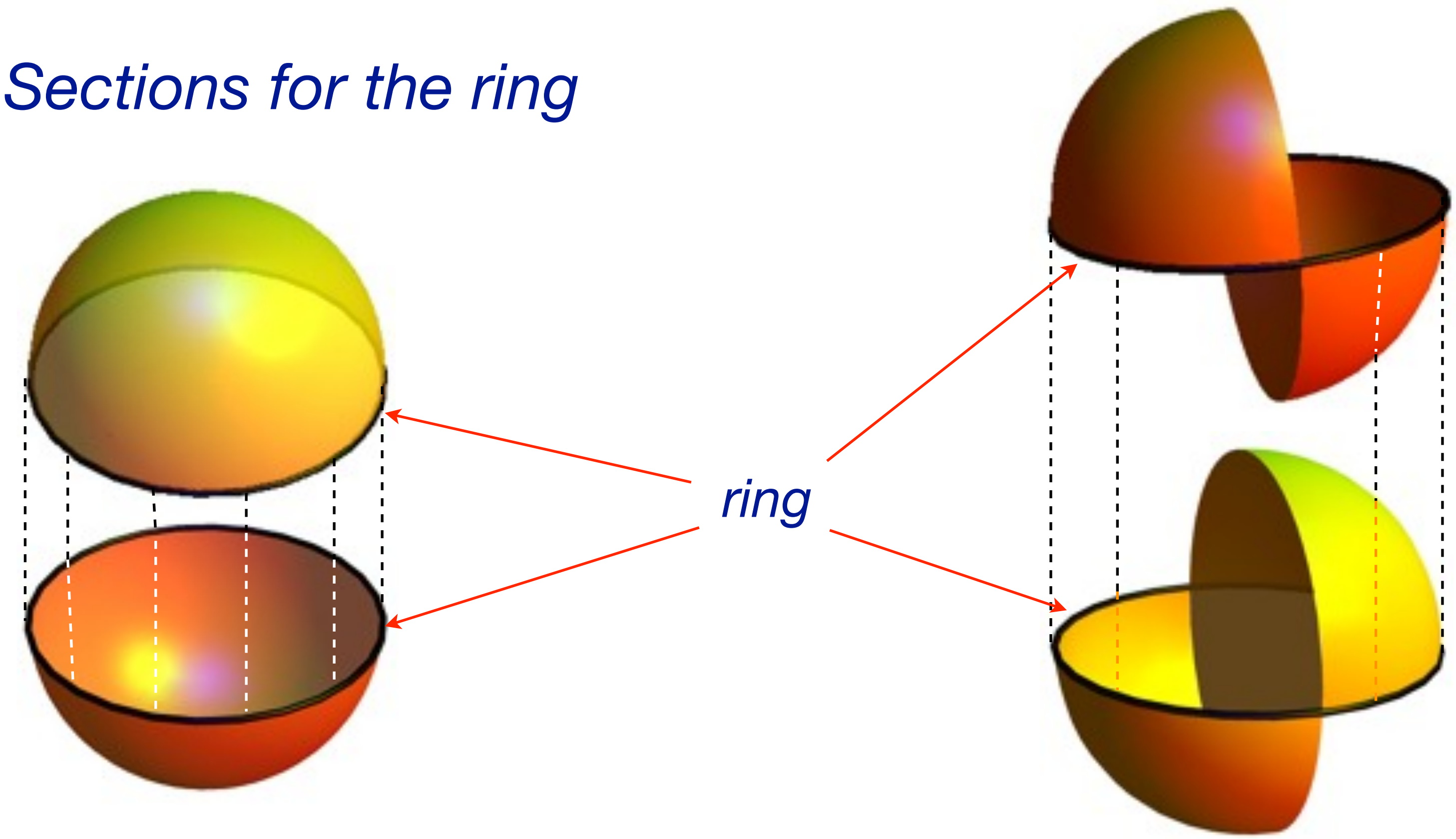
hence we must describe the gauge fields in sections!

this requires the use of Dirac two-branes

(generalizations of the Dirac string)

Bekaert, Gomberoff, 2003
Kalkkinen, Stelle, 2003

Sections for the ring



trivial Dirac brane

non-trivial Dirac brane

$$W_\mu{}^I \mathrm{d}x^\mu = -p^I \left[\cos\theta\,\mathrm{d}\varphi \pm \mathrm{d}(\varphi + \tfrac{1}{2}\psi)\right] + a^I \mathrm{d}\psi$$

black hole potential

gauge transformation

Wilson line modulus defined up to integer

in agreement with explicit black ring solutions

Gauntlett, Gutowski, 2005
Elvang, Emparan, Mateos, Reall, 2004

The Noether potential

For any continuous symmetry there exists a Noether current, which is conserved by virtue of the equations of motion:

$$\partial_\mu J^\mu(\phi, \delta_\xi \phi) \propto \text{ equations of motion}$$

only when the Lagrangian is strictly invariant

For a continuous variety of solutions of the equations of motion, one may define a two-form, the so-called Noether potential, $\mathcal{Q}^{\mu\nu}(\phi, \xi)$, by

$$J^\mu_{\mathrm{Noether}} = \partial_\nu \mathcal{Q}^{\mu\nu}_{\mathrm{gauge}}$$

The Noether potential is ambiguous. The relevant expression is usually fixed by imposing a *suitable* requirement on the current. In the case at hand one requires

$J^\mu_{\mathrm{Noether}} \propto \delta_\xi \phi$, even when the Lagrangian is not strictly invariant.

for $\delta_\xi \phi = 0$

An example: *5D electromagnetism with CS term*

$$\mathcal{L}^{\text{total}} = \mathcal{L}^{\text{inv}}(F_{\mu\nu}, \nabla_\rho F_{\mu\nu}, \psi, \nabla_\mu \psi) + \varepsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau}$$

The Noether potential associated with the abelian gauge symmetry takes the form

$$\mathcal{Q}^{\mu\nu}_{\text{gauge}}(\phi, \xi) = 2\,\mathcal{L}_F^{\mu\nu}\,\xi - 2\,\nabla_\rho \mathcal{L}_F^{\rho,\mu\nu}\,\xi + 6\,e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\,\xi A_\rho F_{\sigma\tau}$$

local gauge parameter

where $\delta\mathcal{L}^{\text{inv}} = \mathcal{L}_F^{\mu\nu}\,\delta F_{\mu\nu} + \mathcal{L}_F^{\rho,\mu\nu}\,\delta(\nabla_\rho F_{\mu\nu}) + \mathcal{L}_\psi\,\delta\psi + \mathcal{L}_\psi^\mu\,\delta(\nabla_\mu\psi)$

$$\partial_\nu\,\mathcal{Q}^{\mu\nu}_{\text{gauge}} = J^\mu_{\text{Noether}} = 0$$

definition

for $\delta_\xi \phi = 0$

An example: *5D electromagnetism with CS term*

$$\mathcal{L}^{\text{total}} = \mathcal{L}^{\text{inv}}(F_{\mu\nu}, \nabla_\rho F_{\mu\nu}, \psi, \nabla_\mu \psi) + \varepsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau}$$

The Noether potential associated with the abelian gauge symmetry takes the form

$$\mathcal{Q}^{\mu\nu}_{\text{gauge}}(\phi, \xi) = 2\,\mathcal{L}_F^{\mu\nu}\,\xi - 2\,\nabla_\rho \mathcal{L}_F^{\rho,\mu\nu}\,\xi + 6\,e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\,\xi A_\rho F_{\sigma\tau}$$

local gauge parameter

where $\delta \mathcal{L}^{\text{inv}} = \mathcal{L}_F^{\mu\nu}\,\delta F_{\mu\nu} + \mathcal{L}_F^{\rho,\mu\nu}\,\delta(\nabla_\rho F_{\mu\nu}) + \mathcal{L}_\psi\,\delta\psi + \mathcal{L}_\psi^{\mu}\,\delta(\nabla_\mu \psi)$

$$\partial_\nu\, \mathcal{Q}^{\mu\nu}_{\text{gauge}} = J^\mu_{\text{Noether}} = 0$$

definition **for** $\delta_\xi \phi = 0$

$^{*}\mathcal{Q}_{\rm gauge}$ is then a closed (*d*-2)-form for symmetric configurations!

Electric charge is defined as

$$q = \int_{\Sigma_{\rm hor}} \varepsilon_{\mu\nu}\, \mathcal{Q}^{\mu\nu}_{\rm gauge}(\phi, \xi)$$

bi-normal

This definition coincides with the definition based on the field equations!

Reconsider now the CS term. Obviously the electric charge now contains the integral over a 3-cycle of the CS term! This poses no difficulty for black holes for which the gauge fields are globally defined. However, the mixed CS term will lead to a 3-cycle of the gravitational CS term! And, as it turns out, that one is problematic.

The Noether potential for diffeomorphisms:

contribution from metric and tensor fields

$$\begin{aligned}\hat{Q}^{\mu\nu}(\xi^\rho) \;=\; & -2\,\mathcal{L}_{\rm R}^{\mu\nu\rho\sigma}\,\nabla_\rho\xi_\sigma + 4\,\nabla_\rho\mathcal{L}_{\rm R}^{\mu\nu\rho\sigma}\,\xi_\sigma \\ & + [\mathcal{L}_{\rm T}^{\mu,\rho\sigma}\,T^\nu{}_\sigma + \mathcal{L}_{\rm T}^{\rho,\mu\sigma}\,T^\nu{}_\sigma + \mathcal{L}_{\rm T}^{\nu,\mu\sigma}\,T^\rho{}_\sigma - (\mu\leftrightarrow\nu)]\,\xi_\rho\end{aligned}$$

adding gauge field contributions

$$8\pi^2\,\varepsilon_{\mu\nu}\,\mathcal{Q}_0^{\mu\nu} = \cdots + 2\,\varepsilon_{01}\,\xi^\rho W_\rho{}^I\,T_{01}\,\left[-6\,C_{IJK}\sigma^J\sigma^K + 3\,c_I(T_{23}{}^2 + 2\,T_{01}{}^2)\right]$$

contributions from CS terms

$$\begin{aligned}8\pi^2\,\mathcal{Q}_{\rm CS}^{\mu\nu} \;=\; & \tfrac{1}{2}{\rm i}\,e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\,C_{IJK}\,\xi^\lambda W_\lambda{}^I W_\rho{}^J F_{\sigma\tau}{}^K \\ & + \tfrac{1}{32}{\rm i}\,e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}c_I W_\rho{}^I\,\mathcal{R}_{\sigma\tau}{}^{\kappa\lambda}\,\nabla_\kappa\xi_\lambda + \cdots\end{aligned}$$

NOTE: *also here one has to insist on a certain form of the current, so that the (relevant) ambiguities are under control !*

The Noether potential for diffeomorphisms:

contribution from metric and tensor fields

$$\hat{\mathcal{Q}}^{\mu\nu}(\xi^\rho) = -2\,\mathcal{L}_{\rm R}^{\mu\nu\rho\sigma}\,\nabla_\rho\xi_\sigma + 4\,\nabla_\rho\mathcal{L}_{\rm R}^{\mu\nu\rho\sigma}\,\xi_\sigma + \big[\mathcal{L}_{\rm T}^{\mu,\rho\sigma}\,T^\nu{}_\sigma + \mathcal{L}_{\rm T}^{\rho,\mu\sigma}\,T^\nu{}_\sigma + \mathcal{L}_{\rm T}^{\nu,\mu\sigma}\,T^\rho{}_\sigma - (\mu\leftrightarrow\nu)\big]\,\xi_\rho$$

adding gauge field contributions

$$8\pi^2\,\varepsilon_{\mu\nu}\,\mathcal{Q}_0^{\mu\nu} = \cdots + 2\,\varepsilon_{01}\,\xi^\rho W_\rho{}^I\,T_{01}\,\big[-6\,C_{IJK}\sigma^J\sigma^K + 3\,c_I(T_{23}{}^2 + 2\,T_{01}{}^2)\big]$$

contributions from CS terms

$$8\pi^2\,\mathcal{Q}_{\rm CS}^{\mu\nu} = \tfrac{1}{2}{\rm i}\,e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\,C_{IJK}\,\xi^\lambda W_\lambda{}^I W_\rho{}^J F_{\sigma\tau}{}^K + \tfrac{1}{32}{\rm i}\,e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}c_I W_\rho{}^I\,\mathcal{R}_{\sigma\tau}{}^{\kappa\lambda}\,\nabla_\kappa\xi_\lambda + \cdots$$

NOTE: *also here one has to insist on a certain form of the current, so that the (relevant) ambiguities are under control !*

Entropy and angular momentum

Entropy (based on first law of black hole mechanics

$$\mathcal{S}_{\rm macro} = -\pi \int_{\Sigma_{\rm hor}} \varepsilon_{\mu\nu}\, \mathcal{Q}^{\mu\nu}(\xi)\Big|_{\nabla_{[\mu}\xi_{\nu]}=\varepsilon_{\mu\nu}\,;\,\xi^\mu=0}$$

Wald, 1993

$\xi^\mu\partial_\mu = \partial/\partial t$ timelike Killing vector

bi-normal

Angular momenta

$$J(\xi) = \int_{\Sigma_{\rm hor}} \varepsilon_{\mu\nu}\, \mathcal{Q}^{\mu\nu}(\xi)$$

ξ^μ periodic Killing vector

Black ring:

gauge fields not globally defined:

- *integrals over gauge fields defined in patches*
- *contributions from Chern-Simons terms subtle*

Indeed: blindly using the results from the black hole analysis leads to incorrect results for the entropy, the electric charges and the angular momenta !

For the mixed Chern-Simons term one makes use of two forms which differ by a total derivative

for the black hole:

$$\mathcal{L}_{\rm CS} \sim \varepsilon^{\mu\nu\rho\sigma\tau} c_I\, W_\mu{}^I \mathcal{R}_{\nu\rho}{}^{ab}\, \mathcal{R}_{\sigma\tau ab}$$

for the black ring:

$$\mathcal{L}_{\rm CS} \sim 4\,\varepsilon^{\mu\nu\rho\sigma\tau}\, c_I F_{\mu\nu}{}^I \omega_\rho{}^{ab} \left(\partial_\sigma \omega_{\tau ab} - \tfrac{2}{3}\omega_{\sigma ac}\, \omega_\tau{}^c{}_b\right)$$

In this case one has to deal with both diffeomorphisms and local Lorentz transformations!

Tachikawa, 2007

Identical expressions for the black hole and the black ring:

$$\mathcal{S}^{\mathrm{BH/BR}}_{\mathrm{macro}} = \frac{\pi\, \mathrm{e}^{g}}{4\, v^2} \left[C_{IJK}\sigma^I \sigma^J \sigma^K + 4\, c_I \sigma^I\, {T_{23}}^2 \right]$$ **universal !**

To obtain the same result for BH and BR is non-trivial and depends on how one deals with the mixed CS term!

For the ring, we have $T_{01} = 0 \longrightarrow v^2 = {T_{23}}^2$

The integration of the contributions from the CS terms to the electric charges and the angular momenta, is very subtle. This aspect is crucial for the results.

Hanaki, Ohashi, Tachikawa, 2007

Evaluating the CS terms for black rings

The correct evaluation of the CS term for the ring geometry yield

$$Q_I^{\rm CS} \propto \oint_\Sigma C_{IJK}\, W^J \wedge F^K \propto C_{IJK}\, a^J p^K$$

Hence, integer shifts of the Wilson line moduli induce a shift in the integrated CS term

For concentric rings, one finds

$$Q_I^{\rm CS} - 6\, C_{IJK} P^J P^K = -12\, C_{IJK} \sum_i (a^J + \tfrac{1}{2} p^J)_i\, p^K{}_i$$

with $P^I = \sum_i p^J{}_i$

The integrated CS terms are not additive! *Hanaki, Ohashi, Tachikawa, 2007*

Confirmed by explicit results for global solutions.

Gauntlett, Gutowski, 2004

Additive charges take the following form
(upon solving the Wilson line moduli in terms of the charges)

$$q_I - 6\, C_{IJK}\, p^J p^K$$

The evaluation of the integrals with terms quadratic in the gauge fields, which appear for the angular momenta, is much more involved. Here there is a global feature that has to be taken into account in establishing the correct result.

$$J_\varphi = -12\, C_{IJ} p^I (a^J + \tfrac{1}{6} p^J)$$

$$J_\psi - J_\varphi = -\frac{\mathrm{e}^{2g}}{2\, T_{23}} \left[C(\sigma) + 4\, c_I \sigma^I\, T_{23}^2\right] + 6\, C_{IJ} (a^I + \tfrac{1}{2} p^I)(a^J + \tfrac{1}{2} p^J)$$

Hanaki, Ohashi, Tachikawa, 2007

Introducing a scale invariant variable: $\phi^0 = \dfrac{\mathrm{e}^{-g}}{4\, T_{23}}$

and making use of the definitions: $C_{IJ} = C_{IJK} p^K$

$$C^{IJ} = [C_{IJK} p^K]^{-1}$$

the results can be summarized as follows.

5D black ring versus *4D* black hole:

$$\mathcal{S}^{\rm BR}_{\rm macro} = \frac{4\pi}{\phi^0}\left[C_{IJK}\,p^I p^J p^K + \frac{1}{4}c_I p^I\right] \qquad D = 5$$

$$q_I = -12\,C_{IJK}\,p^J a^K$$

$$J_\psi - J_\phi - \frac{1}{24}C^{IJ}(q_I - 6\,C_{IK}p^K)(q_J - 6\,C_{JL}p^L) = \frac{2}{{\phi^0}^2}\left[C_{IJK}\,p^I p^J p^K + \frac{1}{4}\,c_I p^I\right]$$

5D black ring versus *4D* black hole:

$$\mathcal{S}^{\text{BR}}_{\text{macro}} = \frac{4\pi}{\phi^0}\left[C_{IJK}\,p^I p^J p^K + \frac{1}{4}c_I p^I\right] \qquad D=5$$

$$q_I = -12\,C_{IJK}\,p^J a^K$$

$$J_\psi - J_\phi - \frac{1}{24}C^{IJ}(q_I - 6\,C_{IK}p^K)(q_J - 6\,C_{JL}p^L) = \frac{2}{{\phi^0}^2}\left[C_{IJK}\,p^I p^J p^K + \frac{1}{4}\,c_I p^I\right]$$

$$\mathcal{S}^{\text{BH}}_{\text{4D}} = -\frac{2\pi}{\phi^0}\left[D_{IJK}\,p^I p^J p^K + 256\,d_I p^I\right] \qquad D=4$$

$${q_I}^{\text{4D}} = \frac{6}{\phi^0}D_{IJK}\,p^J\phi^K$$

$$\hat{q}_0{}^{4D} \equiv q_0{}^{4D} + \tfrac{1}{12}D^{IJ}q_I q_J = \frac{1}{{\phi^0}^2}\left[D_{IJK}\,p^I p^J p^K + 256\,d_I p^I\right]$$

5D black ring versus *4D* black hole:

$$\mathcal{S}^{\text{BR}}_{\text{macro}} = \frac{4\pi}{\phi^0}\left[C_{IJK}\,p^I p^J p^K + \frac{1}{4}c_I p^I\right] \qquad D=5$$

$$q_I = -12\,C_{IJK}\,p^J a^K$$

$$J_\psi - J_\phi - \frac{1}{24}C^{IJ}(q_I - 6\,C_{IK}p^K)(q_J - 6\,C_{JL}p^L) = \frac{2}{\phi^{0\,2}}\left[C_{IJK}\,p^I p^J p^K + \frac{1}{4}\,c_I p^I\right]$$

$$\mathcal{S}^{\text{BH}}_{\text{4D}} = -\frac{2\pi}{\phi^0}\left[D_{IJK}\,p^I p^J p^K + 256\,d_I p^I\right] \qquad D=4$$

$$q_I{}^{\text{4D}} = \frac{6}{\phi^0}D_{IJK}\,p^J\phi^K$$

$$\hat{q}_0{}^{4D} \equiv q_0{}^{4D} + \tfrac{1}{12}D^{IJ}q_I q_J = \frac{1}{\phi^{0\,2}}\left[D_{IJK}\,p^I p^J p^K + 256\,d_I p^I\right]$$

COMMENTS:

This It is the first confirmation from near-horizon analysis in the presence of higher-derivative couplings. Partial results were already known (but somewhat disputed at the time).

Bena, Kraus, etc

The Wilson line moduli are defined up to integers. This implies that the electric charges and angular momenta are shifted under the large gauge transformations (spectral flow) induced by these integer shifts. Indeed, under

Bena, Kraus,Warner, Cheng, de Boer, etc

$$a^I \rightarrow a^I + k^I$$

one finds,

$$\begin{aligned} q_I &\rightarrow q_I - 12\, C_{IJK} p^J k^K \\ J_\varphi &\rightarrow J_\varphi - 12\, C_{IJK} p^I p^J k^K \\ J_\psi &\rightarrow J_\psi - q_I k^I - 6\, C_{IJK} p^I p^J k^K + 6\, C_{IJK} p^I k^J k^K \end{aligned}$$

These transformations are in agreement with the corresponding *4D* black holes where the above transformations correspond to a duality invariance!

Spinning black hole:

$$\mathcal{S}^{\rm BH}_{\rm macro} = \frac{\pi\, {\rm e}^{g}}{4\, v^2}\left[C_{IJK}\sigma^I\sigma^J\sigma^K + 4\, c_I\sigma^I\, {T_{23}}^2\right]$$

$$q_I = \frac{6\,{\rm e}^{g}}{4\, T_{01}}\left[C_{IJK}\sigma^J\sigma^K - c_I {T_{01}}^2\right]$$

$$p^0 = \frac{{\rm e}^{-g}}{4\, v^2}\, T_{01}$$

$$J_\psi = \frac{T_{23}\, {\rm e}^{2g}}{{T_{01}}^2}\left[C_{IJK}\sigma^I\sigma^J\sigma^K - 4\, c_I\sigma^I\, {T_{01}}^2\right]$$

choose scale invariant variables

$$\phi^I = \frac{\sigma^I}{4T_{01}}$$

$$\phi^0 = \frac{e^{-g}T_{23}}{4v^2} = \frac{p^0\, T_{23}}{T_{01}}$$

$$\mathcal{S}^{\rm BH}_{\rm macro} = \frac{4\pi p^0}{({\phi^0}^2 + {p^0}^2)^2}\left[{p^0}^2 C_{IJK}\phi^I\phi^J\phi^K + \frac{1}{4}\,c_I\phi^I\,{\phi^0}^2\right]$$

$$q_I = \frac{6\,p^0}{{\phi^0}^2 + {p^0}^2}\left[C_{IJK}\phi^J\phi^K - \frac{1}{16}\,c_I\right] \qquad D = 5$$

$$J_\psi = \frac{4\phi^0 p^0}{({\phi^0}^2 + {p^0}^2)^2}\left[C_{IJK}\phi^I\phi^J\phi^K - \frac{1}{4}\,c_I\phi^I\right]$$

$$\mathcal{S}^{\rm BH}_{\rm macro} = \frac{4\pi p^0}{({\phi^0}^2 + {p^0}^2)^2}\left[{p^0}^2 C_{IJK}\phi^I\phi^J\phi^K + \frac{1}{4}\,c_I\phi^I\,{\phi^0}^2\right]$$

$$q_I = \frac{6\,p^0}{{\phi^0}^2 + {p^0}^2}\left[C_{IJK}\phi^J\phi^K - \frac{1}{16}\,c_I\right] \qquad D=5$$

$$J_\psi = \frac{4\phi^0 p^0}{({\phi^0}^2 + {p^0}^2)^2}\left[C_{IJK}\phi^I\phi^J\phi^K - \frac{1}{4}\,c_I\phi^I\right]$$

$$\mathcal{S}^{\rm BH}_{4D} = \frac{2\pi p^0}{({\phi^0}^2 + {p^0}^2)^2}\left[{p^0}^2 D_{IJK}\phi^I\phi^J\phi^K + 256\,d_I\phi^I{\phi^0}^2\right]$$

$$q_I{}^{4D} = -\frac{3\,p^0}{{\phi^0}^2 + {p^0}^2}\left[D_{IJK}\phi^J\phi^K - \tfrac{256}{3}\,d_I\right] \qquad D=4$$

$$q_0{}^{4D} = \frac{2\phi^0 p^0}{({\phi^0}^2 + {p^0}^2)^2}\left[D_{IJK}\phi^I\phi^J\phi^K - 256\,d_I\phi^I\right]$$

$$\mathcal{S}^{\rm BH}_{\rm macro} = \frac{4\pi p^0}{(\phi^{0\,2}+p^{0\,2})^2}\left[p^{0\,2} C_{IJK}\phi^I\phi^J\phi^K + \frac{1}{4}\,c_I\phi^I\,\phi^{0\,2}\right]$$

$$q_I = \frac{6\,p^0}{\phi^{0\,2}+p^{0\,2}}\left[C_{IJK}\phi^J\phi^K - \frac{1}{16}\,c_I\right] \qquad D=5$$

$$J_\psi = \frac{4\phi^0 p^0}{(\phi^{0\,2}+p^{0\,2})^2}\left[C_{IJK}\phi^I\phi^J\phi^K - \frac{1}{4}\,c_I\phi^I\right]$$

looks similar but calibration is subtle

$$\mathcal{S}^{\rm BH}_{4D} = \frac{2\pi p^0}{(\phi^{0\,2}+p^{0\,2})^2}\left[p^{0\,2} D_{IJK}\phi^I\phi^J\phi^K + 256\,d_I\phi^I\phi^{0\,2}\right]$$

$${q_I}^{4D} = -\frac{3\,p^0}{\phi^{0\,2}+p^{0\,2}}\left[D_{IJK}\phi^J\phi^K - \tfrac{256}{3}\,d_I\right] \qquad D=4$$

$${q_0}^{4D} = \frac{2\phi^0 p^0}{(\phi^{0\,2}+p^{0\,2})^2}\left[D_{IJK}\phi^I\phi^J\phi^K - 256\,d_I\phi^I\right]$$

$$\mathcal{S}^{\rm BH}_{\rm macro} = \frac{4\pi p^0}{({\phi^0}^2+{p^0}^2)^2}\left[{p^0}^2 C_{IJK}\phi^I\phi^J\phi^K + \frac{1}{4}\,c_I\phi^I\,{\phi^0}^2\right]$$

$$q_I = \frac{6\,p^0}{{\phi^0}^2+{p^0}^2}\left[C_{IJK}\phi^J\phi^K - \frac{1}{16}\,c_I\right] \qquad D=5$$

$$J_\psi = \frac{4\phi^0 p^0}{({\phi^0}^2+{p^0}^2)^2}\left[C_{IJK}\phi^I\phi^J\phi^K - \frac{1}{4}\,c_I\phi^I\right]$$

looks similar but calibration is subtle

relative factor $\frac{4}{3}$

$$\mathcal{S}^{\rm BH}_{4D} = \frac{2\pi p^0}{({\phi^0}^2+{p^0}^2)^2}\left[{p^0}^2 D_{IJK}\phi^I\phi^J\phi^K + 256\,d_I\phi^I{\phi^0}^2\right]$$

$${q_I}^{4D} = -\frac{3\,p^0}{{\phi^0}^2+{p^0}^2}\left[D_{IJK}\phi^J\phi^K - \frac{256}{3}\,d_I\right] \qquad D=4$$

$${q_0}^{4D} = \frac{2\phi^0 p^0}{({\phi^0}^2+{p^0}^2)^2}\left[D_{IJK}\phi^I\phi^J\phi^K - 256\,d_I\phi^I\right]$$

$$\mathcal{S}^{\rm BH}_{\rm macro} = \frac{4\pi p^0}{(\phi^{0\,2} + p^{0\,2})^2}\left[p^{0\,2} C_{IJK}\phi^I\phi^J\phi^K + \frac{1}{4}\,c_I\phi^I\,\phi^{0\,2}\right]$$

$$q_I = \frac{6\,p^0}{\phi^{0\,2} + p^{0\,2}}\left[C_{IJK}\phi^J\phi^K - \frac{1}{16}\,c_I\right] \qquad D = 5$$

$$J_\psi = \frac{4\phi^0 p^0}{(\phi^{0\,2} + p^{0\,2})^2}\left[C_{IJK}\phi^I\phi^J\phi^K - \frac{1}{4}\,c_I\phi^I\right]$$

looks similar but calibration is subtle

relative factor $\frac{4}{3}$

agrees!

$$\mathcal{S}^{\rm BH}_{4D} = \frac{2\pi p^0}{(\phi^{0\,2} + p^{0\,2})^2}\left[p^{0\,2} D_{IJK}\phi^I\phi^J\phi^K + 256\,d_I\phi^I\phi^{0\,2}\right]$$

$$q_I{}^{4D} = -\frac{3\,p^0}{\phi^{0\,2} + p^{0\,2}}\left[D_{IJK}\phi^J\phi^K - \frac{256}{3}\,d_I\right] \qquad D = 4$$

$$q_0{}^{4D} = \frac{2\phi^0 p^0}{(\phi^{0\,2} + p^{0\,2})^2}\left[D_{IJK}\phi^I\phi^J\phi^K - 256\,d_I\phi^I\right]$$

Comments:

Agrees with microstate counting for $J_\psi = 0$

Vafa, 1997
Huang, Klemm, Mariño, Tavanfar, 2007

The 4D and 5D results are rather similar!
4D/5D connection?

Gaiotto, Strominger, Yin, 2005
Behrndt, Cardoso, Mahapatra, 2005

The most obvious discrepancy concerns the expression for the electric charges. Only for $J_\psi = 0$ there is agreement with the literature. The direct relation between J_ψ and q_0 is impressive but not in agreement with alternative results.

Castro, Davis, Kraus, Larsen, 2007

The discrepancy is related to the mixed CS term, which contributes to the charges through a gravitational CS term integrated over the horizon. Upon reinvestigating this CS term, it turns out to that its evaluation is considerably more subtle than originally participated.

The discrepancy:

The alternative result takes the form:

$$q_I = \frac{6\,p^0}{{\phi^0}^2 + {p^0}^2}\Big[C_{IJK}\phi^J\phi^K - \frac{1}{12}\,c_I\Big] + \frac{c_I}{8\,p^0}$$

Castro, Davis, Kraus, Larsen, 2007

For zero angular momentum the two results coincide!
If the second result is correct, then one can absorb the last term into the charge, so that the 4D result is obtained.

The CS term is not a covariant density! Let us evaluate it in a regular metric $(t, r, \theta, \varphi, \psi)$:

$$g_{\mu\nu} = \frac{1}{16\, v^2} \begin{pmatrix} -(1-\alpha)r^2 & \mp 1 & 0 & -r\cos\theta\,\sqrt{\alpha(1-\alpha)} & -r\sqrt{\alpha(1-\alpha)}/p^0 \\ \mp 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -r\cos\theta\,\sqrt{\alpha(1-\alpha)} & 0 & 0 & 1-\alpha\cos^2\theta & (1-\alpha)\cos\theta/p^0 \\ -r\sqrt{\alpha(1-\alpha)}/p^0 & 0 & 0 & (1-\alpha)\cos\theta/p^0 & (1-\alpha)/(p^0)^2 \end{pmatrix}$$

where $\alpha = \dfrac{T_{23}}{v}$

The horizon is located at $r = 0$.

$$\int_{\text{horizon}} CS(\Gamma) \propto \frac{{T_{01}}^2\, T_{23}^2}{p^0({T_{01}}^2 + {T_{23}}^2)^2}$$

which vanishes for $T_{01}\,\text{or}\, T_{23} \approx 0$

The CS term is a composite 3-form potential $C_{\mu\nu\rho}$ satisfying

$$D_{[\mu}C_{\nu\rho\sigma]} \propto R^{ab}_{[\mu\nu}R_{\rho\sigma]\,ab}$$

The right-hand side vanishes for T_{01} or $T_{23} \approx 0$

Consistent!

The integral over the 3-form is ill defined and one must describe it in different patches. Upon dimensional reduction it is a priori unclear which patch to choose. With zero angular momentum the CS term does not contribute and this causes a discrepancy in the charge with the four-dimensional result. Accepting this fact, the corresponding difference for the case of non-vanishing angular momentum must depend non-trivially on the value of the momentum. The result shown earlier does exhibit this feature, unlike alternative results in the literature!

One explanation, at the time, was that the reduction of the *D=5* theory with higher derivatives is not given by the usual *D=4* action, because it contains an additional vector multiplet that originates from the *D=5* Weyl multiplet, which is now expected to contain new higher-derivative couplings. Therefore, electric charges may be subject to change. This explanation seems no longer tenable in view of the fact that these new higher-derivative couplings are subject to a non-renormalization theorem.

dW, Katmadas, van Zalk, 2010

In general, it seems the open questions concern the case of black holes with non-zero angular momentum. Also the comparison with microstate counting for small black holes shows discrepancies. These questions are further pursued at the moment, as well as the more general problem of dimensional reduction in the presence of higher-derivative couplings.