BPS NEAR-HORIZON GEOMETRY OF 5D BLACK HOLES AND RINGS

IHÉS, Bures-sur Yvette
13 April 2011

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Tuesday 12 April 2011
The general context:
What is the statistical interpretation of black hole entropy?

or, what is the relation between:

- microscopic/statistical entropy
- macroscopic/field-theoretic entropy

- microstate counting \( \longrightarrow \) entropy \( S_{\text{micro}} = \ln d(q, p) \)
- supergravity: Noether surface charge
  - first law of black hole mechanics \( (BH \text{ thermodynamics}) \)

STRING THEORY and SUPERSYMMETRY

Strominger, Vafa, 1996
Ideal testing ground: supergravities with 8 supercharges

two obvious theories

- $D=4$ space-time dimensions with $N=2$ supersymmetry
- $D=5$ space-time dimensions with $N=1$ supersymmetry

here: near-horizon analysis in $D=5$ dimensions

To clarify previous results and to further the understanding of the connection with four-dimensional results.

New features:

- Chern-Simons terms
- both black holes and black rings

Castro, Davis, Kraus, Larsen, 2008
$dW$, Katmadas, 2009

Further work in progress with Nabamita Banerjee and Stefanos Katmadas
BPS black holes and rings in five space-time dimensions

two different supersymmetric horizon topologies!

✧ \( S^3 \) (SPINNING) BLACK HOLE \hspace{1cm} Breckenridge, Myers, Peet, Vafa, 1996
✧ \( S^1 \times S^2 \) BLACK RING \hspace{1cm} Elvang, Emparan, Mateos, Reall, 2004

with near-horizon geometry: \( \text{AdS}_2 \times S^3 \) or \( \text{AdS}_3 \times S^2 \)

(this result does not depend on the specific Lagrangian)
superconformal multiplet calculus

- off-shell irreducible supermultiplets in superconformal gravity background
- extra superconformal gauge invariances
- gauge equivalence (compensating supermultiplets)

\[ F_{\mu\nu}^I = \partial_{\mu} W_{\nu}^I - \partial_{\nu} W_{\mu}^I \]

**vector supermultiplets** contain
- scalar fields: \( \sigma^I \)
- vector fields: \( W_{\mu}^I \)
- abelian field strengths

**supergravity (Weyl) multiplet** contains
- (auxiliary) tensor \( T_{\mu\nu} \rightarrow T_{ab} \)

\[ v^2 \equiv (T_{01})^2 + (T_{23})^2 \]
supersymmetry + partial gauge choice

\( \sigma^I = \text{constant} \)  
(remain subject to residual (constant) scale transformations!)

\[ T_{\mu\nu} \quad \text{conformal Killing-Yano tensor} \]

\[ \mathcal{D}_\rho T_{\mu\nu} = \frac{1}{2} g_{\rho[\mu} \xi_{\nu]} \]
\[ \xi^\mu T_{\mu\nu} = 0 \]

Killing vector associated with the fifth dimension \( \psi \)

\[ \xi^\mu \propto e^{-1} \varepsilon^{\mu\nu\rho\sigma\tau} T_{\nu\rho} T_{\sigma\tau} \]

\[ ds^2 = \frac{1}{16 v^2} \left( -r^2 dt^2 + \frac{dr^2}{r^2} + d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) + e^{2g} (d\psi + \sigma)^2 \]

\[ \sigma = -\frac{1}{4 v^2} e^{-g} \left( T_{23} r \, dt - T_{01} \cos \theta \, d\varphi \right) \]

Horizon area \( A_3 = \int_{\Sigma_{\text{hor}}} = \pi^2 v^{-2} e^g \)

Horizon bi-normal \( \epsilon_{01} = \pm 1 \)
Two distinct cases:

- $T_{01} \neq 0$  **SPINNING BLACK HOLE**  
  
  angular momentum $\propto \frac{T_{23}}{T_{01}}$

- $T_{01} = 0$  **BLACK RING**  
  
  Breckenridge, Myers, Peet, Vafa, 1996  
  Elvang, Emparan, Mateos, Reall, 2004

Additional horizon condition and ‘magnetic’ charges

$$F_{\mu\nu}^I = 4\sigma^I T_{\mu\nu} \quad Q_{\mu\nu} = \partial_\mu \sigma_\nu - \partial_\nu \sigma_\mu$$

$$F_{\theta\varphi} \rightarrow p^I = \frac{\sigma^I}{4v^2} T_{23}$$

$$Q_{\theta\varphi} \rightarrow p^0 = \frac{e^{-g}}{4v^2} T_{01}$$
Scale invariance (residue of conformal invariance)

\[ \sigma^I, T_{ab}, v, e^{-g} \] scale uniformly

the metric is scale dependent

Action in 5 space-time dimensions consists of \textbf{two} cubic invariants, each containing a Chern-Simons terms.

\[
\mathcal{L} \propto C_{IJK} \varepsilon^{\mu\nu\rho\sigma\tau} W_\mu^I F_{\nu\rho}^J F_{\sigma\tau}^K \\
\mathcal{L} \propto c_I \varepsilon^{\mu\nu\rho\sigma\tau} W_\mu^I R_{\nu\rho}^{ab} R_{\sigma\tau}^{ab}
\]

Hanaki, Ohashi, Tachikawa, 2006

4D analogue

\[
F(Y, \Upsilon) = \frac{D_{IJK} Y^I Y^J Y^K + d_I Y^I \Upsilon}{Y^0}
\]

\[
\Upsilon = -64 \quad Y^I = \frac{1}{2}(\phi^I + ip^I)
\]
Gauge fields

on $S^3$ the gauge fields are globally defined

$$W_\mu^I \, dx^\mu = - \frac{\sigma^I}{4 \, v^2} \left( T_{01} \, r \, dt + T_{23} \left( \frac{d\psi}{p^0} + \cos \theta \, d\varphi \right) \right)$$

on $S^1 \times S^2$ the gauge fields are not globally defined

* hence we must describe the gauge fields in sections! 
  * this requires the use of *Dirac two-branes* 
    *(generalizations of the Dirac string)*

Bekaert, Gomberoff, 2003
Kalkkinen, Stelle, 2003
Sections for the ring

trivial Dirac brane

non-trivial Dirac brane

\[ W_\mu^I dx^\mu = -p^I \left[ \cos \theta \, d\varphi \pm d(\varphi + \frac{1}{2} \psi) \right] + a^I d\psi \]

black hole potential
gauge transformation
Wilson line modulus defined up to integer

in agreement with explicit black ring solutions

Gauntlett, Gutowski, 2005
Elvang, Emparan, Mateos, Reall, 2004
The Noether potential

For any continuous symmetry there exists a Noether current, which is conserved by virtue of the equations of motion:

\[ \partial_\mu J^\mu (\phi, \delta \xi \phi) \propto \text{equations of motion} \]

only when the Lagrangian is strictly invariant

For a continuous variety of solutions of the equations of motion, one may define a two-form, the so-called Noether potential, \( Q^{\mu \nu} (\phi, \xi) \), by

\[ J^\mu_{\text{Noether}} = \partial_\nu Q^{\mu \nu} \]

The Noether potential is ambiguous. The relevant expression is usually fixed by imposing a suitable requirement on the current. In the case at hand one requires

\[ J^\mu_{\text{Noether}} \propto \delta \xi \phi \], even when the Lagrangian is not strictly invariant.
for $\delta_x \phi = 0$
An example: 5D electromagnetism with CS term

\[ \mathcal{L}^{\text{total}} = \mathcal{L}^{\text{inv}} (F_{\mu\nu}, \nabla_\rho F_{\mu\nu}, \psi, \nabla_\mu \psi) + \varepsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau} \]

The Noether potential associated with the abelian gauge symmetry takes the form

\[ Q^{\mu\nu}_{\text{gauge}} (\phi, \xi) = 2 \mathcal{L}^{\mu\nu}_F \xi - 2 \nabla_\rho \mathcal{L}^{\rho;\mu\nu}_F \xi + 6 e^{-1} \varepsilon^{\mu\nu\rho\sigma\tau} \xi A_\rho F_{\sigma\tau} \]

where \[ \delta \mathcal{L}^{\text{inv}} = \mathcal{L}^{\mu\nu}_F \delta F_{\mu\nu} + \mathcal{L}^{\rho;\mu\nu}_F \delta(\nabla_\rho F_{\mu\nu}) + \mathcal{L}_\psi \delta \psi + \mathcal{L}_\psi^\mu \delta(\nabla_\mu \psi) \]

\[ \partial_\nu Q^{\mu\nu}_{\text{gauge}} = J^\mu_{\text{Noether}} = 0 \]

**definition** for \( \delta \xi \phi = 0 \)
An example: 5D electromagnetism with CS term

\[ \mathcal{L}^{\text{total}} = \mathcal{L}^{\text{inv}} (F_{\mu\nu}, \nabla_\rho F_{\mu\nu}, \psi, \nabla_\mu \psi) + \varepsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau} \]

The Noether potential associated with the abelian gauge symmetry takes the form

\[ Q^{\mu\nu}_{\text{gauge}} (\phi, \xi) = 2 \mathcal{L}_{F}^{\mu\nu} \xi - 2 \nabla_\rho \mathcal{L}_{F}^{\rho, \mu\nu} \xi + 6 e^{-1} \varepsilon^{\mu\nu\rho\sigma\tau} \xi A_\rho F_{\sigma\tau} \]

where \( \delta \mathcal{L}^{\text{inv}} = \mathcal{L}_{F}^{\mu\nu} \delta F_{\mu\nu} + \mathcal{L}_{F}^{\rho, \mu\nu} \delta (\nabla_\rho F_{\mu\nu}) + \mathcal{L}_{\psi} \delta \psi + \mathcal{L}_{\psi}^{\mu} \delta (\nabla_\mu \psi) \)

\[ \partial_\nu Q^{\mu\nu}_{\text{gauge}} = J^\mu_{\text{Noether}} = 0 \]

\text{definition} \quad \text{for} \quad \delta \xi \phi = 0
Electric charge is defined as

\[ q = \int_{\Sigma_{\text{hor}}} \varepsilon_{\mu\nu} Q_{\text{gauge}}^{\mu\nu}(\phi, \xi) \]

This definition coincides with the definition based on the field equations!

Reconsider now the CS term. Obviously the electric charge now contains the integral over a 3-cycle of the CS term! This poses no difficulty for black holes for which the gauge fields are globally defined. However, the mixed CS term will lead to a 3-cycle of the gravitational CS term! And, as it turns out, that one is problematic.

\[ *Q_{\text{gauge}} \] is then a closed \((d-2)\)-form for symmetric configurations!

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The Noether potential for diffeomorphisms:

contribution from metric and tensor fields

\[ \hat{Q}_{\mu\nu}(\xi^\rho) = -2 \mathcal{L}^{\mu\nu\rho\sigma}_R \nabla_\rho \xi_\sigma + 4 \nabla_\rho \mathcal{L}^{\mu\nu\rho\sigma}_R \xi_\sigma \]

\[ + [\mathcal{L}^{\mu,\rho\sigma}_T T^\nu_\sigma + \mathcal{L}^{\rho,\mu\sigma}_T T^\nu_\sigma + \mathcal{L}^{\nu,\mu\sigma}_T T^\rho_\sigma - (\mu \leftrightarrow \nu)] \xi_\rho \]

adding gauge field contributions

\[ 8\pi^2 \varepsilon_{\mu\nu} Q_{\mu\nu}^{0} = \cdots + 2 \varepsilon_{01} \xi^\rho W^I_\rho T_{01} \left[ -6 C_{IJK} \sigma^J_\rho \sigma^K_\rho + 3 c_I (T_{23}^2 + 2 T_{01}^2) \right] \]

contributions from CS terms

\[ 8\pi^2 Q_{\mu\nu}^{CS} = \frac{1}{2} i e^{-1} \varepsilon^{\mu\nu\rho\sigma\tau} C_{IJK} \xi^\lambda W^I_\lambda W^J_\rho F^K_{\sigma\tau} \]

\[ + \frac{1}{32} i e^{-1} \varepsilon^{\mu\nu\rho\sigma\tau} c_I W^I_\rho \mathcal{R}^\kappa_\lambda \nabla_\kappa \xi_\lambda + \cdots \]

NOTE: also here one has to insist on a certain form of the current, so that the (relevant) ambiguities are under control!
The Noether potential for diffeomorphisms:

\[ \hat{Q}^{\mu \nu}(\xi^\rho) = -2 \mathbb{L}^{\mu \nu \rho \sigma}_R \nabla_\rho \xi_\sigma + 4 \nabla_\rho \mathbb{L}^{\mu \nu \rho \sigma}_R \xi_\sigma + \left[ \mathbb{L}^{\mu \rho \sigma}_T T^\nu_\sigma + \mathbb{L}^{\rho \mu \sigma}_T T^\nu_\sigma + \mathbb{L}^{\nu \mu \sigma}_T T^\rho_\sigma - (\mu \leftrightarrow \nu) \right] \xi_\rho \]

adding gauge field contributions

\[ 8 \pi^2 \epsilon_{\mu \nu} Q^{\mu \nu} = \cdots + 2 \epsilon_{01} \xi^\rho W^I_\rho T_{01} \left[ -6 C_{IJK} \sigma^J \sigma^K + 3 c_I (T_{23}^2 + 2 T_{01}^2) \right] \]

contributions from CS terms

\[ 8 \pi^2 Q^{\mu \nu}_{\text{CS}} = \frac{1}{2} i e^{-1} \epsilon^{\mu \nu \rho \sigma \tau} C_{IJK} \xi^\lambda W^I_\lambda W^J_\rho F^{K}_{\sigma \tau} + \frac{1}{32} i e^{-1} \epsilon^{\mu \nu \rho \sigma \tau} c_I W^I_\rho R^{\kappa \lambda}_{\sigma \tau} \nabla_\kappa \xi_\lambda + \cdots \]

NOTE: also here one has to insist on a certain form of the current, so that the (relevant) ambiguities are under control!
Entropy and angular momentum

Entropy (based on first law of black hole mechanics)

\[ S_{\text{macro}} = -\pi \int_{\Sigma_{\text{hor}}} \varepsilon_{\mu\nu} Q^{\mu\nu}(\xi) \]

\[ \nabla_{[\mu} \xi_{\nu]} = \varepsilon_{\mu\nu}; \xi^\mu = 0 \]

\[ \xi^\mu \partial_\mu = \partial / \partial t \quad \text{timelike Killing vector} \]

Angular momenta

\[ J(\xi) = \int_{\Sigma_{\text{hor}}} \varepsilon_{\mu\nu} Q^{\mu\nu}(\xi) \]

\[ \xi^\mu \quad \text{periodic Killing vector} \]
Black ring:

**gauge fields not globally defined:**
- integrals over gauge fields defined in patches
- contributions from Chern-Simons terms subtle

Indeed: blindly using the results from the black hole analysis leads to incorrect results for the entropy, the electric charges and the angular momenta!

For the mixed Chern-Simons term one makes use of two forms which differ by a total derivative

for the black hole:

$$\mathcal{L}_{CS} \sim \varepsilon^{\mu\nu\rho\sigma\tau} c_I \ W_\mu^\ I \ R_{\nu\rho}^{\ ab} \ R_{\sigma\tau \ ab}$$

for the black ring:

$$\mathcal{L}_{CS} \sim 4 \varepsilon^{\mu\nu\rho\sigma\tau} c_I \ F_{\mu\nu}^\ I \ \omega_\rho^{\ ab} \left( \partial_\sigma \omega_{\tau \ ab} - \frac{2}{3} \omega_{\sigma \ ac} \ \omega_{\tau \ c \ b} \right)$$

In this case one has to deal with both diffeomorphisms and local Lorentz transformations!

_Tachikawa, 2007_
Identical expressions for the black hole and the black ring:

\[ S_{\text{BH/BR}}^{\text{macro}} = \frac{\pi e^g}{4v^2} \left[ C_{IJK} \sigma^I \sigma^J \sigma^K + 4c_I \sigma^I T_{23}^2 \right] \text{ universal!} \]

To obtain the same result for BH and BR is non-trivial and depends on how one deals with the mixed CS term!

For the ring, we have \( T_{01} = 0 \) \( \rightarrow \) \( v^2 = T_{23}^2 \)

The integration of the contributions from the CS terms to the electric charges and the angular momenta, is very subtle. This aspect is crucial for the results.

Hanaki, Ohashi, Tachikawa, 2007
Evaluating the CS terms for black rings

The correct evaluation of the CS term for the ring geometry yields

$$Q^\text{CS}_I \propto \int \sum C_{IJK} W^J \wedge F^K \propto C_{IJK} \alpha^J p^K$$

Hence, integer shifts of the Wilson line moduli induce a shift in the integrated CS term.

For concentric rings, one finds

$$Q^\text{CS}_I - 6 C_{IJK} P^J P^K = -12 C_{IJK} \sum_i (\alpha^J + \frac{1}{2} p^J)_i p^K_i$$

with $$P^I = \sum_i p^J_i$$

The integrated CS terms are not additive!  

Confirmed by explicit results for global solutions.  

Additive charges take the following form (upon solving the Wilson line moduli in terms of the charges)

$$q_I = 6 C_{IJK} p^J p^K$$
The evaluation of the integrals with terms quadratic in the gauge fields, which appear for the angular momenta, is much more involved. Here there is a global feature that has to be taken into account in establishing the correct result.

\[
J_\varphi = -12 C_{IJ} p^I (a^J + \frac{1}{6} p^J)
\]

\[
J_\psi - J_\varphi = -\frac{e^{2g}}{2 T_{23}} \left[ C(\sigma) + 4 c_I \sigma^I T_{23}^2 \right] + 6 C_{IJ} (a^I + \frac{1}{2} p^I) (a^J + \frac{1}{2} p^J)
\]

\(\text{Hanaki, Ohashi, Tachikawa, 2007}\)

Introducing a scale invariant variable: \(\phi^0 = \frac{e^{-g}}{4 T_{23}}\)

and making use of the definitions: \(C_{IJ} = C_{IJK} p^K\)

\(C^{IJ} = [C_{IJK} p^K]^{-1}\)

the results can be summarized as follows.
5D black ring versus 4D black hole:

\[ S_{\text{micro}}^{\text{BR}} = \frac{4\pi}{\phi^0} \left[ C_{IJK} p^I p^J p^K + \frac{1}{4} c_I p^I \right] \]

\[ q_I = -12 C_{IJK} p^J a^K \]

\[ J_\psi - J_\phi - \frac{1}{24} C^{IJ} (q_I - 6 C_{IK} p^K) (q_J - 6 C_{JLP} p^L) = \frac{2}{\phi^{02}} \left[ C_{IJK} p^I p^J p^K + \frac{1}{4} c_I p^I \right] \]

\( D = 5 \)
5D black ring versus 4D black hole:

\[ S_{\text{macro}}^{BR} = \frac{4\pi}{\phi_0} \left[ C_{IJK} p^I p^J p^K + \frac{1}{4} c_I p^I \right] \]

\[ q_I = -12 C_{IJK} p^J a^K \]

\[ J_\psi - J_\phi - \frac{1}{24} C^{IJ} (q_I - 6 C_{IK} p^K) (q_J - 6 C_{JLP} p^L) = \frac{2}{\phi_0^2} \left[ C_{IJK} p^I p^J p^K + \frac{1}{4} c_I p^I \right] \]

\[ S_{4D}^{BH} = -\frac{2\pi}{\phi_0} \left[ D_{IJK} p^I p^J p^K + 256 d_I p^I \right] \]

\[ q_I^{4D} = \frac{6}{\phi_0} D_{IJK} p^J \phi^K \]

\[ \hat{q}_0^{4D} \equiv q_0^{4D} + \frac{1}{12} D^{IJ} q_I q_J = \frac{1}{\phi_0^2} \left[ D_{IJK} p^I p^J p^K + 256 d_I p^I \right] \]

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5D black ring versus 4D black hole:

\[
S_{\text{macro}}^{\text{BR}} = \frac{4\pi}{\phi_0} \left[ C_{IJK} p^I p^J p^K + \frac{1}{4} c_I p^I \right]
\]

\[
q_I = -12 C_{IJK} p^J a^K
\]

\[
J_\psi - J_\phi - \frac{1}{24} C^{IJ}(q_I - 6 C_{IK} p^K)(q_J - 6 C_{JL} p^L) = \frac{2}{\phi_0^2} \left[ C_{IJK} p^I p^J p^K + \frac{1}{4} c_I p^I \right]
\]

\[
S_{4D}^{\text{BH}} = -\frac{2\pi}{\phi_0} \left[ D_{IJK} p^I p^J p^K + 256 d_I p^I \right]
\]

\[
q_I^{4D} = \frac{6}{\phi_0} D_{IJK} p^J \phi^K
\]

\[
\hat{q}_0^{4D} \equiv q_0^{4D} + \frac{1}{12} D^{IJ} q_I q_J = \frac{1}{\phi_0^2} \left[ D_{IJK} p^I p^J p^K + 256 d_I p^I \right]
\]
This is the first confirmation from near-horizon analysis in the presence of higher-derivative couplings. Partial results were already known (but somewhat disputed at the time).

\[ q_I \rightarrow q_I - 12 C_{IJK} p^J k^K \]
\[ J_\phi \rightarrow J_\phi - 12 C_{IJK} p^I p^J k^K \]
\[ J_\psi \rightarrow J_\psi - q_I k^I - 6 C_{IJK} p^I p^J k^K + 6 C_{IJK} p^I k^J k^K \]

These transformations are in agreement with the corresponding 4D black holes where the above transformations correspond to a duality invariance!
Spinning black hole:

\[ S_{\text{macro}}^{BH} = \frac{\pi e^g}{4 v^2} \left[ C_{IJK} \sigma^I \sigma^J \sigma^K + 4 c_I \sigma^I T_{23}^2 \right] \]

\[ q_I = \frac{6 e^g}{4 T_{01}} \left[ C_{IJK} \sigma^J \sigma^K - c_I T_{01}^2 \right] \]

\[ p^0 = \frac{e^{-g}}{4 v^2} T_{01} \]

\[ J_{\psi} = \frac{T_{23} e^{2g}}{T_{01}^2} \left[ C_{IJK} \sigma^I \sigma^J \sigma^K - 4 c_I \sigma^I T_{01}^2 \right] \]

choose scale invariant variables

\[ \phi^I = \frac{\sigma^I}{4 T_{01}} \]

\[ \phi^0 = \frac{e^{-g} T_{23}}{4 v^2} = \frac{p^0 T_{23}}{T_{01}} \]
\[ S_{\text{BH macro}} = \frac{4\pi p^0}{\left(\phi^0{}^2 + p^0{}^2\right)^2} \left[ p^0{}^2 C_{IJK} \phi^I \phi^J \phi^K + \frac{1}{4} c_I \phi^I \phi^0{}^2 \right] \]

\[ q_I = \frac{6 p^0}{\phi^0{}^2 + p^0{}^2} \left[ C_{IJK} \phi^J \phi^K - \frac{1}{16} c_I \right] \quad D = 5 \]

\[ J_\psi = \frac{4 \phi^0 p^0}{\left(\phi^0{}^2 + p^0{}^2\right)^2} \left[ C_{IJK} \phi^I \phi^J \phi^K - \frac{1}{4} c_I \phi^I \right] \]
\[ S_{\text{BH}}^\text{macro} = \frac{4\pi p^0}{(\phi^0)^2 + (p^0)^2} \left[ p^{02} C_{IJK} \phi^I \phi^J \phi^K + \frac{1}{4} c_I \phi^{02} \right] \]

\[ q_I = \frac{6 p^0}{(\phi^0)^2 + (p^0)^2} \left[ C_{IJK} \phi^J \phi^K - \frac{1}{16} c_I \right] \quad D = 5 \]

\[ J_{\psi} = \frac{4\phi^0 p^0}{(\phi^0)^2 + (p^0)^2} \left[ C_{IJK} \phi^I \phi^J \phi^K - \frac{1}{4} c_I \phi^I \right] \]

\[ S_{\text{BH}}^{4D} = \frac{2\pi p^0}{(\phi^0)^2 + (p^0)^2} \left[ p^{02} D_{IJK} \phi^I \phi^J \phi^K + 256 d_I \phi^I \phi^0 \right] \]

\[ q_I^{4D} = -\frac{3 p^0}{(\phi^0)^2 + (p^0)^2} \left[ D_{IJK} \phi^J \phi^K - \frac{256}{3} d_I \right] \quad D = 4 \]

\[ q_0^{4D} = \frac{2\phi^0 p^0}{(\phi^0)^2 + (p^0)^2} \left[ D_{IJK} \phi^I \phi^J \phi^K - 256 d_I \phi^I \right] \]
$S_{\text{BH macro}} = \frac{4\pi p^0}{(\phi^0^2 + p^0^2)^2} \left[ p^0^2 C_{IJK} \phi^I \phi^J \phi^K + \frac{1}{4} c_I \phi^I \phi^0^2 \right]$

$q_I = \frac{6 p^0}{\phi^0^2 + p^0^2} \left[ C_{IJK} \phi^J \phi^K - \frac{1}{16} c_I \right] \quad D = 5$

$J_\psi = \frac{4\phi^0 p^0}{(\phi^0^2 + p^0^2)^2} \left[ C_{IJK} \phi^I \phi^J \phi^K - \frac{1}{4} c_I \phi^I \right]$

looks similar but calibration is subtle

$S_{\text{BH 4D}} = \frac{2\pi p^0}{(\phi^0^2 + p^0^2)^2} \left[ p^0^2 D_{IJK} \phi^I \phi^J \phi^K + 256 d_I \phi^I \phi^0^2 \right]$

$q_I^{4D} = -\frac{3 p^0}{\phi^0^2 + p^0^2} \left[ D_{IJK} \phi^J \phi^K - \frac{256}{3} d_I \right] \quad D = 4$

$q_0^{4D} = \frac{2\phi^0 p^0}{(\phi^0^2 + p^0^2)^2} \left[ D_{IJK} \phi^I \phi^J \phi^K - 256 d_I \phi^I \right]$
looks similar but calibration is subtle

\[
S_{\text{BH}}^{\text{macro}} = \frac{4\pi p^0}{(\phi^0 + p^0)^2} \left[ p^0 \phi^I \phi^J \phi^K + \frac{1}{4} c_I \phi^I \phi^0 \right]
\]

\[
q_I = \frac{6 p^0}{\phi^0 + p^0} \left[ C_{IJK} \phi^J \phi^K - \frac{1}{16} c_I \right]
\]

\[
J_{\phi} = \frac{4 \phi^0 p^0}{(\phi^0 + p^0)^2} \left[ C_{IJK} \phi^I \phi^J \phi^K - \frac{1}{4} c_I \phi^I \right]
\]

\[
D = 5
\]

\[
S_{4D}^{\text{BH}} = \frac{2\pi p^0}{(\phi^0 + p^0)^2} \left[ p^0 \phi^I \phi^J \phi^K + 256 d_I \phi^I \phi^0 \right]
\]

\[
q_I^{4D} = -\frac{3 p^0}{\phi^0 + p^0} \left[ D_{IJK} \phi^J \phi^K - \frac{256}{3} d_I \right]
\]

\[
D = 4
\]

\[
q_0^{4D} = \frac{2 \phi^0 p^0}{(\phi^0 + p^0)^2} \left[ D_{IJK} \phi^I \phi^J \phi^K - 256 d_I \phi^I \right]
\]

\[
\text{relative factor } \frac{4}{3}
\]
$S_{\text{BH macro}} = \frac{4\pi p^0}{(\phi^0 + p^0)^2} \left[ p^{02} C_{IJK} \phi^I \phi^J \phi^K + \frac{1}{4} c_I \phi^I \phi^0 \phi^0 \right]$

$q_I = \frac{6 p^0}{\phi^0 + p^0} \left[ C_{IJK} \phi^J \phi^K - \frac{1}{16} c_I \right]$

$J_\psi = \frac{4\phi^0 p^0}{(\phi^0 + p^0)^2} \left[ C_{IJK} \phi^I \phi^J \phi^K - \frac{1}{4} c_I \phi^I \right]$

looks similar but calibration is subtle

$S_{4D} = \frac{2\pi p^0}{(\phi^0 + p^0)^2} \left[ p^{02} D_{IJK} \phi^I \phi^J \phi^K + 256 d_I \phi^I \phi^0 \phi^0 \right]$

$q_I^{4D} = -\frac{3 p^0}{\phi^0 + p^0} \left[ D_{IJK} \phi^J \phi^K - \frac{256}{3} d_I \right]$

$q_0^{4D} = \frac{2\phi^0 p^0}{(\phi^0 + p^0)^2} \left[ D_{IJK} \phi^I \phi^J \phi^K - 256 d_I \phi^I \right]$

relative factor $\frac{4}{3}$

agrees!
Comments:

Agrees with microstate counting for $J_\psi = 0$

Gaiotto, Strominger, Yin, 2005
Huang, Klemm, Mariño, Tavanfar, 2007

The 4D and 5D results are rather similar!
4D/5D connection?

Gaiotto, Strominger, Yin, 2005
Behrndt, Cardoso, Mahapatra, 2005

The most obvious discrepancy concerns the expression for the electric charges. Only for $J_\psi = 0$ there is agreement with the literature. The direct relation between $J_\psi$ and $q_0$ is impressive but not in agreement with alternative results.

Castro, Davis, Kraus, Larsen, 2007
The discrepancy is related to the mixed CS term, which contributes to the charges through a gravitational CS term integrated over the horizon. Upon reinvestigating this CS term, it turns out to that its evaluation is considerably more subtle than originally participated.

The discrepancy:

The alternative result takes the form:

\[
q_I = \frac{6 p^0}{\phi^0 \phi^0 + p^0 p^0} \left[ C_{IJK} \phi^J \phi^K - \frac{1}{12} c_I \right] + \frac{c_I}{8 p^0}
\]

\textit{Castro, Davis, Kraus, Larsen, 2007}

For zero angular momentum the two results coincide! If the second result is correct, then one can absorb the last term into the charge, so that the 4D result is obtained.
The CS term is not a covariant density! Let us evaluate it in a regular metric \((t, r, \theta, \varphi, \psi)\):

\[
g_{\mu\nu} = \frac{1}{16 \, v^2} \begin{pmatrix}
-(1 - \alpha)r^2 & \mp 1 & 0 & -r \cos \theta \sqrt{\alpha(1 - \alpha)} & -r \sqrt{\alpha(1 - \alpha)/p^0} \\
\mp 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
-r \cos \theta \sqrt{\alpha(1 - \alpha)} & 0 & 0 & 1 - \alpha \cos^2 \theta & (1 - \alpha) \cos \theta / p^0 \\
-r \sqrt{\alpha(1 - \alpha)/p^0} & 0 & 0 & (1 - \alpha) \cos \theta / p^0 & (1 - \alpha) / (p^0)^2
\end{pmatrix}
\]

where \(\alpha = \frac{T_{23}}{v}\)

The horizon is located at \(r = 0\).

\[
\int_{\text{horizon}} CS(\Gamma) \propto \frac{T_{01}^2 \, T_{23}^2}{p^0 (T_{01}^2 + T_{23}^2)^2}
\]

which vanishes for \(T_{01}\) or \(T_{23} \approx 0\)
The CS term is a composite 3-form potential $C_{\mu \nu \rho}$ satisfying

$$D_{[\mu} C_{\nu \rho \sigma]} \propto R_{[\mu \nu}^{ab} R_{\rho \sigma]}^{ab}$$

The right-hand side vanishes for $T_{01}$ or $T_{23} \approx 0$

**Consistent!**

The integral over the 3-form is ill defined and one must describe it in different patches. Upon dimensional reduction it is a priori unclear which patch to choose. With zero angular momentum the CS term does not contribute and this causes a discrepancy in the charge with the four-dimensional result. Accepting this fact, the corresponding difference for the case of non-vanishing angular momentum must depend non-trivially on the value of the momentum. The result shown earlier does exhibit this feature, unlike alternative results in the literature!
One explanation, at the time, was that the reduction of the $D=5$ theory with higher derivatives is not given by the usual $D=4$ action, because it contains an additional vector multiplet that originates from the $D=5$ Weyl multiplet, which is now expected to contain new higher-derivative couplings. Therefore, electric charges may be subject to change. This explanation seems no longer tenable in view of the fact that these new higher-derivative couplings are subject to a non-renormalization theorem.

$dW$, Katmadas, van Zalk, 2010

In general, it seems the open questions concern the case of black holes with non-zero angular momentum. Also the comparison with microstate counting for small black holes shows discrepancies. These questions are further pursued at the moment, as well as the more general problem of dimensional reduction in the presence of higher-derivative couplings.