Affine Kac-Moody symmetric spaces

(IHES, 13.06.2013, Walter Freyn)

Aim: Introduce affine KM-sym. spaces.

Plan:

- Introduction (What are KM-sym. spaces
- Functionalanalysis (Algebras & groups of holomorphic loops)
- Geometry (Classification, structure of flats)

Old world: finite dimensional

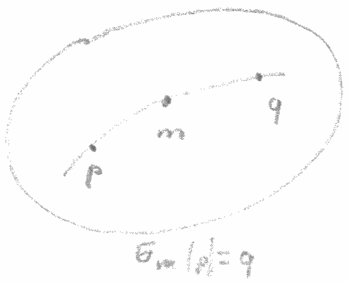
E. Cartan: Riemannian symmetric spaces:

Definition

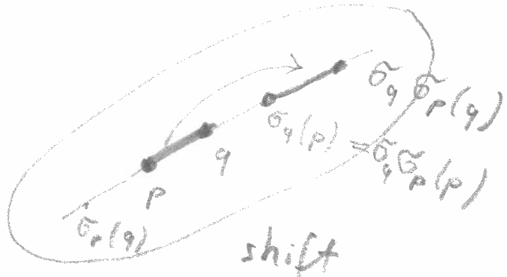
(M, g) is symmetric $\Leftrightarrow$ $\forall p \in M\ \exists \sigma_p : M \to M$ s.t. $d\sigma_p|_{T_pM} = -Id.$

Consequences:

- homogeneous



- complete



Classification: introduce algebra!

(M, g) symmetric $\Rightarrow$ Define $G = Isom(M)$, $p \in M \Rightarrow K_p := \{f \in G \mid f(p) = p\}$

$$\Rightarrow M = G/K \text{ and } K \text{ is compact.}$$

Let $\mathfrak{g} = Lie(G)$, $\mathfrak{k} = Lie(K) \subset \mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{p} \simeq T_pM$

Definition: (OSLA)

$(\mathfrak{g}, s)$ is OSLA iff
- $\mathfrak{g}$ is a real Lie algebra
- $s : \mathfrak{g} \to \mathfrak{g}$ is involution
- $\mathfrak{k} = Fix(s)$ is compact subalgebra

Classification of simply connected Riemannian symmetric spaces
$\hat{=}$ Classification of OSLA's

$\mathbb{R}$

Compact type	non-compact type
K compact simple type II	$G_{\mathbb{C}}/K$ $G_{\mathbb{C}}$ complex type IV
$K/K_p = Fix(\sigma)$ type I	$G_{nc}/K(G_{nc}) \equiv K_p$ $= Fix(\omega)$, ω: Cartan involution type III
$sect(X,Y) \geq 0$	$sect(X,Y) \leq 0$ diffeomorphic to a VS.

Examples:

type II: $M_1' = SU(2) = S^3$

type I: $M_2' = SU(2)/SO(3) = S^2$

type IV: $M_1 = SL(2,\mathbb{C})/SU(2) = H^3$

type III: $M_2 = SL(2,\mathbb{R})/SO(2) = H^2$

$M_3 = SL(n,\mathbb{R})/SO(n)$

$M_4 = E_8(\mathbb{R})/Spin(16)$

<u>New world</u>: infinite dimensional

Key idea: Kac 1967 semisimple Lie groups $\longrightarrow$ Kac-Moody algebras

Algebraic properties generalize

- (Cartans are conjugate, KM-groups have Bruhat, Iwasawa-dec.
- Kac-Peterson topology

Geometry: Main challenge: in general. no explicit realisation.

But: affine KMA admit realizations as extensions of (twisted) loop algebras

Kac: Classification of Riem. sym spaces via KMA
$\hat{=}$ Interest of Differential geometers!

Terng, Palais, Heintze, Thorbergsson:

Study the geometry of polar action, isoparam submanifolds in Hilbert spaces, which is deduced from affine KMA

1990: <u>Conjecture</u> (Terng):

There are affine KM-sym spaces!

Constructed 2009 in my thesis.

II: Lie algebras of holomorphic maps

Define $\cdot\ M\mathfrak{g}_{\mathbb{C}} := \mathrm{Hol}(\mathbb{C}^*, \mathfrak{g}_{\mathbb{C}})$, $[f,g]_0(z) = [f(z), g(z)]$

$\mathfrak{g}_{\mathbb{C}}$ reductive

$\cdot\ \widehat{M\mathfrak{g}_{\mathbb{C}}} = M\mathfrak{g} \oplus \underset{\substack{\uparrow \\ \text{central}}}{c\,\mathbb{C}} + \underset{\substack{\uparrow \\ \text{derivation}}}{d\,\mathbb{C}}$

$[f,g](z) := [f,g]_0(z) + \omega(f,g)\,c$

$\omega(f,g) = \frac{1}{2\pi}\int_{S^1}\langle f(z), g'(z)\rangle\, d$; $\frac{1}{2\pi} = \mathrm{Res}(fg')$

$[d,f] = izf'$

Remark:

$\mathfrak{g}_{\mathbb{C}}$ simple $\Rightarrow \widehat{M_{alg}\mathfrak{g}_{\mathbb{C}}}$ is an affine KMA.

Functional analysis:

$\|f\|_n = \sup_{z\in A_n} |f(z)|$ $\qquad A_n := \{z \in \mathbb{C} \mid e^{-n} \leq |z| \leq e^n\}$

$\mathbb{C}^* = \cup A_n$

Theorem

$M\mathfrak{g}_{\mathbb{C}}$ and $\widehat{M\mathfrak{g}_{\mathbb{C}}}$ are tame, Fréchet spaces

Disgression: Fréchet spaces

Definition

A Fréchet space is a complete, Hausdorff, l.c TVS, whose top. is generated by a countable family of norms. $\| \ \|_n$, $n \in \mathbb{N}$

Examples:

- B Banach
- $\mathrm{Hol}(\mathbb{C}, \mathbb{C})$
- $C^\infty(M, \mathbb{R})$

Problems:

1. No inverse function theorem
2. Dual spaces are in general not Fréchet

Tame Fréchet spaces

$\hat{=}$ spaces which are sufficiently like $Hol(\mathbb{C}, \mathcal{B})$

Grading: $\|f\|_n \leq \|f\|_{n+1} \leq$

Definition: (linear maps!)

Tame map: $\varphi: \mathcal{F} \to G$ is tame $:\Leftrightarrow$ $\|\varphi(f)\|_n \leq C(n)\|f\|_{n+r}$ $\forall n \geq b$

Tame isomorphism: φ, φ^{-1} are tame

Tame direct summand: $\mathcal{F} \hookrightarrow G$ $:\Leftrightarrow$ $\exists \varphi: \mathcal{F} \to G$, $\psi: G \to \mathcal{F}$ tame s.t.
$\psi \circ \varphi = Id_{\mathcal{F}}$ ("one sided inverse")

Modell space & Tameness

- Let $\mathcal{B}$ Banach: $\Sigma(\mathcal{B}) = \{(b_k) \in \mathcal{B} \mid \sum_k |b_k| e^{nk} < \infty \ \forall n\}$
$= \{(b_k) \in \mathcal{B} \mid \sup_{k \in \mathbb{N}} |b_k| e^{nk} < \infty \ \forall n\}$
- $\mathcal{F}$ is tame $:\Leftrightarrow$ $\exists \mathcal{B}$ s.t. $\mathcal{F}$ is tame direct summand of $\Sigma(\mathcal{B})$

Lemma

$\mathcal{B}$ complex $\Rightarrow$ $\Sigma(\mathcal{B}) = Hol(\mathbb{C}, \mathbb{C})$

Proof:

"$\Rightarrow$" $(b_k) \Rightarrow f(z) = \sum b_k z^k$. Choose a ball $B(0, e^n)$

$$\|f\|_n \leq \|\sum_k b_k z^k\|_n \leq \sum_k |b_k| |z^k|_n = \sum_k |b_n| e^{kn} \underset{\text{assumption}}{\leq} \infty$$

"$\Leftarrow$" f holomorphic $\Rightarrow$ $b_n = \dfrac{f^{(n)}(0)}{n!}$ Let $k \in \mathbb{N}$

$$\sup_{k \in \mathbb{N}} |b_k| e^{kn} = \sup_{k \in \mathbb{N}} \frac{|f^{(k)}(0)|}{k!} e^{kn} \underset{\uparrow}{\leq} \sup_{k \in \mathbb{N}} \frac{\cancel{k!}\, e^{kn}}{e^{kn}\, \cancel{k!}} \sup_{z \in B(0,e^n)} |f(z)| =$$

Cauchy
$$|f^{(k)}(0)| \leq k! \frac{\sup_{z \in B(0,n)} |f(z)|}{e^{kn}}$$

$$= \sup_{z \in B(0,e^n)} |f(z)| \underset{\uparrow \text{ assumption}}{\leq} \infty$$

III: Lie groups of holomorphic maps

Define $MG_{\mathbb{C}} := Hol(\mathbb{C}^*, G_{\mathbb{C}})$, $G_{\mathbb{C}}$ complex reductive.

Let $\mathbb{C}^* \longrightarrow \widetilde{MG}_{\mathbb{C}} \longrightarrow MG_{\mathbb{C}}$ a central extension.

Then $\widehat{MG}_{\mathbb{C}} = \underset{f(z)}{\widetilde{MG}_{\mathbb{C}}} \ltimes \underset{w}{\mathbb{C}^*}$, $w \cdot f(z) = f(zw)$ is an affine KM-grou

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Compact real forms:

$(M\mathfrak{g})_c = \{ f \in M\mathfrak{g}_{\mathbb{C}} \mid f(S_1) \subset \mathfrak{g}_c \}$, $\mathfrak{g}_c$ compact form of $\mathfrak{g}_{\mathbb{C}}$

$(MG)_c = \{ f \in MG_{\mathbb{C}} \mid f(S^1) \subset G_c \}$

Let $S^1 \longrightarrow \widetilde{MG}_c \longrightarrow MG$ a central extension.

Then $(\widehat{MG})_c = \widetilde{MG}_c \ltimes S^1$

Exponential map:

$M\exp : M\mathfrak{g} \longrightarrow MG$, $f \mapsto \exp \circ f$.

Manifold structures:

Usual idea for loop groups: There are: $U \subset \mathfrak{g}$, $V \subset G$ open, s.t.

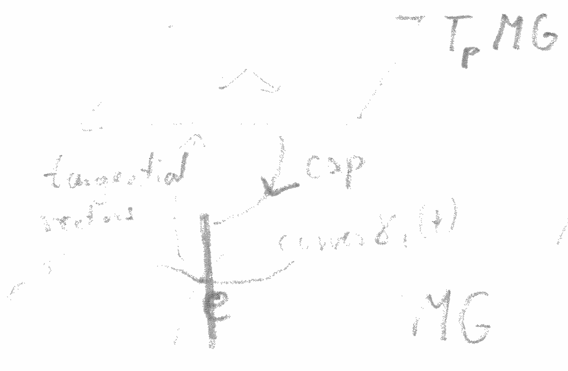
$\Rightarrow$ $\exp : U \to V$ is a diffeomorphism.

Define $LU := \{ f : S^1 \to U \}$, $LV = \{ f : S^1 \to V \}$

$\Rightarrow$ $\exp : LU \to LV$ is a diffeo, charts via left translation.

This does not work, as bounded holomorphic functions are constant

We have:



Theorem:

$M\mathfrak{g} = T_e MG$

$\widehat{M\mathfrak{g}} = T_e \widehat{MG}$

Theorem:

Mexp is no local diffeo!

Problem: Functions "close to the identity may not be in the image of exp
'Close' is characterized via the following

Proposition:

The compact-open topology and the tame Fréchet topology coincide.

Example:

$G = SL(2,\mathbb{C}) \Rightarrow \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \notin \operatorname{Im} \exp(sl(2,\mathbb{C}))$

Choose: $f_n : \mathbb{C}^* \longrightarrow SL(2,\mathbb{C})$, $f(z) = \begin{pmatrix} e^{i\pi \frac{z}{n}} & b\frac{z}{n} \\ 0 & e^{-i\pi\frac{z}{n}} \end{pmatrix}$

- Then $f_n(n) = \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \Rightarrow f_n(z) \notin M\exp(M sL(2,\mathbb{C}))$
- On any compact set $K \subset \mathbb{C}$: $\lim_{n\to\infty} f_n(z)|_K \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Theorem:

MG is a tame Fréchet manifold.

Proof:

$$MG \xrightarrow{\varphi} M\mathfrak{g} \times G \xrightarrow{\pi_2} G \quad \text{surjective} \rightarrow$$

$$f \longmapsto (\underbrace{f^{-1}f'}_{\alpha}, f(1))$$

$$M\mathfrak{g} \times G \xrightarrow{\pi_1} M\mathfrak{g} \xrightarrow{Mon} \mathfrak{g}$$

$$\alpha \longmapsto e^{\int_{S_1}\alpha}$$

- φ is injective: assume $\varphi(f) = \varphi(g)$
  $\Rightarrow f^{-1}f' = \alpha = g^{-1}g'$
  $\Rightarrow f' = f\alpha\ ;\ g' = g\alpha$
  $\Rightarrow$ linear DGL $\Rightarrow f(1) = g(1) \Rightarrow f \equiv g$ ▣

- as $\mathbb{C}^*$ is 1-$\mathbb{C}$-dimensional, there is no local integrability condition
  $\Rightarrow \pi_1 \circ \varphi(MG) = Mon^{-1}(0)$

- Adapt the result, that inverse images of regular values are submfg to this setting.

IV Symmetric spaces:

Theorem:

- $\widehat{M\mathfrak{g}}$ admits an Ad-invariant metric.

Given by Kac - for the algebraic one!

- Adjoint action: $Ad(\widehat{MG}) : \widehat{M\mathfrak{g}} \longrightarrow \widehat{M\mathfrak{g}}$

Let $x = (g, z, w) \in \widehat{MG}$

$\in MG$ — central extension — semidirect product

$$Ad(x)\,u = g w(u) g^{-1} + \langle g w(u) g^{-1}, g' g^{-1} \rangle c$$
$$Ad(x)\,c = c$$
$$Ad(x)\,d = d - g' g^{-1} + \langle g' g^{-1}, g' g^{-1} \rangle c$$

- Ad-invariant scalar product:

$$\langle u, v \rangle = \frac{1}{2\pi} \int_{S^1} \langle u(t), v(t) \rangle dt$$
$$\langle c, d \rangle = 1$$
$$\langle c, c \rangle = \langle d, d \rangle = \langle u, c \rangle = 0$$

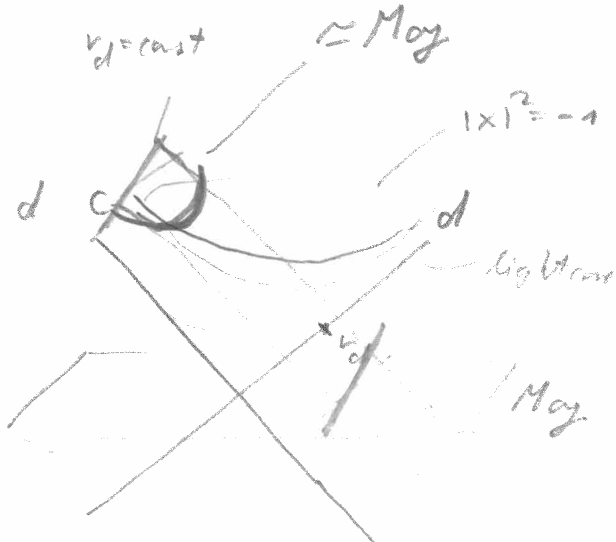
Remark:

preserved

$$Ad(x)(u + r_c c + r_d d) = v + r_c' c + r_d d$$

Proposition

$\langle\ ,\ \rangle|_{\widehat{MG}_c}$ is Lorentzian



Polar action of C.-L. Terng like

Theorem: (Popescu)

All tame Fréchet Lie groups admit a unique LC connection s.t. $\nabla_X Y = \frac{1}{2}[X, Y]$.

$\Rightarrow$ Algebra survives.

Definition:

A KM-symmetric space is a tame Fréchet Lorentz, symmetric space whose isometry group is an affine KM group

$\downarrow \cong$

finite dim: Riemannian.

Classification:

Definition (OSAKA):

$(\mathfrak{g}, s)$ s.t.
- $\mathfrak{g}$ is a real affine KMA
- s is an involutive automorphism
- Fix(s) is a 'compact loop' algebra.

Classification of irred KMS

compact type		$\widehat{M\mathbb{C}^*}$	$\widehat{M\mathbb{C}}$	noncompact type	
$\widehat{MG_c}$	II			$\widehat{MG_{\mathbb{C}}}/\widehat{MG_c}$	IV
$\widehat{MG_c}/Fix(s)$	I			$\widehat{MG_{nc}}/Fix(s)$	III

$\langle R(X,Y)X, Y\rangle \geq 0$

Why no sectional curvature?

$\langle R(X,Y)X, Y\rangle \leq 0$

diffeomorphic to a VS

$\downarrow$

'almost split' real form

'first type' (Rousseau)
'second kind' (Heintze)

$\downarrow$

Kulkarni:

(M, g) Lorentzian $\Rightarrow$ Equivalence of

i) $sect_p(X,Y) \leq a$ bove
ii) $sect_p(X,Y) \geq b$ elow
iii) $sect_p(X,Y) = c$ onstant