Infrared Divergences
in
Quantum Gravity

General reference:

"Quantum Gravitation" (Springer Tracts in Modern Physics, 2009).

Herbert W. Hamber

IHES, Sep.15 2011

Outline

- *QFT Treatment of Gravity (PT and non-PT)*
- *QFT/RG Motivations for a Running G(k)*
- *Effective Covariant Field Equations with G(Box)*

With R.M. Williams and R. Toriumi

Diagrams

Dimensional Considerations

Coupling dimensions in gravity different from electrodynamics …

$$\Box\, h_{\mu\nu} \;=\; G\, T_{\mu\nu}$$

$$\Box\, A_{\mu} \;=\; e\, j_{\mu}$$

… as can be seen already from Poisson's equation :

$$\Delta\,\phi \;=\; 4\pi\, e\rho \qquad\qquad \rho \sim 1/l^3$$

$$\Delta\,\phi \;=\; 4\pi\, G\rho \qquad\qquad \rho \sim m/l^3$$

Simple Consequences of Lorentz Invariance

$$L_{Max} \sim -\partial_\mu A_\nu \, \partial^\mu A^\nu$$
$$\sim +(\dot{\mathbf{A}})^2 - (\nabla \times \mathbf{A})^2$$

$$L_{grav} \sim +(\dot{h_{ij}})^2 - (\nabla h)^2 + \ldots$$

In Maxwell's theory like charges *repel* :

$$\int \rho \cdot \frac{1}{-\nabla^2} \cdot \rho = E_{int} > 0$$

… whereas in gravity like charges always *attract :*

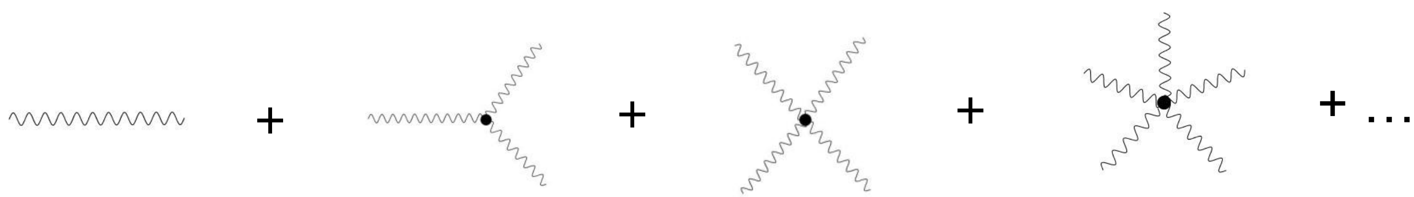
$$- \int T_{00} \cdot \frac{1}{-\nabla^2} \cdot T^{00} = E_{int} < 0$$

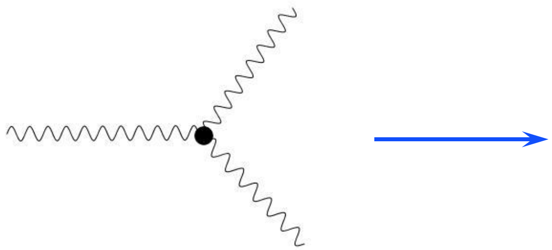
Vertices

Infinitely many interaction terms in *L* (unlike QED)

$$L \; = \; \left(\eta_{\mu\nu} + G\, h_{\mu\nu} + G^2 (h^2)_{\mu\nu} + \ldots\right) \cdot \partial h \cdot \partial h$$

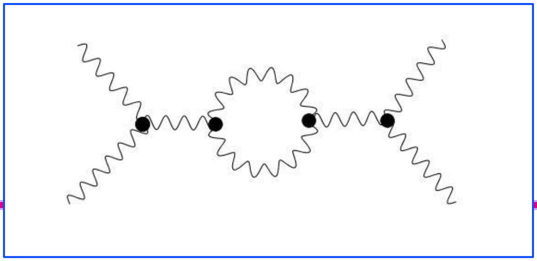
$$L \; = \; \sum_{n=0}^{\infty} (G^n\, h^n) \cdot \partial h\, \partial h$$

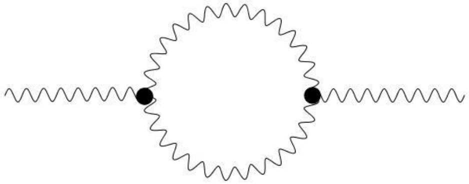




$$V(p) \;\sim\; h \cdot \partial h \cdot \partial h \;\sim\; p^2$$

One Loop





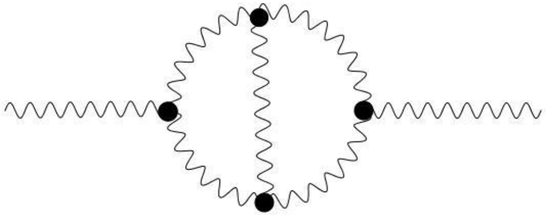
$$\sim \; G^2 \int \frac{d^4 p}{p^2\,(p+k)^2} \cdot V(p+k) \cdot V(p+k)$$

$$\sim \; G^2 \int^{\infty} \frac{d^4 p}{p^4} \cdot p^2 \cdot p^2 \;\sim\; G^2\, \infty^4$$

One loop diagram quartically divergent in d=4.

Electrodynamics : $\sim \; e^2 \int d^4 p \, \frac{1}{p^2\,(p+k)^2}$ *log* divergence

Two Loops



$$\sim \; G^4 \int \frac{d^4p_1 \, d^4p_2}{(p^2)^5} \cdot (p^2)^4 \; \sim \; G^4 \, \infty^6$$

First required counterterm :

$$R^2_{\mu\nu\rho\sigma} \; \sim \; (\partial \, h)^4 \; \sim \; l^{-8} \quad \text{and also} \quad \left(\partial^2 h\right)^2 \; \sim \; l^{-6}$$

$$L \; = \; R + \infty \, R^2 + \infty \, R^3 + \ldots$$

Perturbation th. badly divergent … wrong ground state ?

Counterterms

$$I \;=\; \lambda \int d^d x \, \sqrt{g} - \frac{1}{16\,\pi\,G} \int d^d x \, \sqrt{g}\, R$$

Radiative corrections generate lots of new interactions …

$$\Gamma^{(1)}_{div} \;=\; \frac{1}{4-d}\,\frac{\hbar}{16\pi^2} \int d^4x \sqrt{g} \left(\frac{7}{20} R_{\mu\nu} R^{\mu\nu} + \frac{1}{120} R^2 \right)$$

$$\frac{G(k^2)}{G} \;=\; 1 \,+\, \text{const.}\, G\, k^2 \,+\, \ldots$$

$$\Gamma^{(2)}_{div} \;=\; \frac{1}{4-d}\,\frac{209}{2880}\,\frac{\hbar^2\, G}{(16\pi^2)^2} \int d^4x \sqrt{g}\, R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\kappa\lambda} R_{\kappa\lambda}{}^{\mu\nu}$$

$$I \;\longrightarrow\; \lambda \int d^d x \, \sqrt{g} - \frac{1}{16\,\pi\,G} \int d^d x \, \sqrt{g}\, R + \frac{\alpha_0}{\Lambda^{4-d}} \int d^d x \, \sqrt{g}\, R_{\mu\nu} R^{\mu\nu} + \frac{\beta_0}{\Lambda^{4-d}} \int d^d x \, \sqrt{g}\, R^2 + \cdots$$

Λ = (string) UV cutoff

Perturbative renormalization in 4d requires the introduction of higher derivative terms …

High momentum behavior dominated by R2 terms [DeWitt & Utiyama 1962] …

Issues with unitarity ?

- Higher Derivative Quantum Gravity (pert. renormalizable)

$$I = \int d^4x \sqrt{g} \left[\lambda_0 + kR + aR_{\mu\nu}R^{\mu\nu} - \tfrac{1}{3}(b+a)R^2 \right]$$

$$\Delta \mathscr{L} = \frac{\sqrt{g}}{16\pi^2(4-d)} \left\{ \beta_2 \left(R_{\mu\nu}^2 - \tfrac{1}{3}R^2 \right) + \beta_3 R^2 + \beta_4 R + \beta_5 \right\} \ ,$$

with the coefficients for the divergent parts given by

$$\beta_2 = \frac{133}{10}$$

$$\beta_3 = \frac{10}{9}\omega^2 + \frac{5}{3}\omega + \frac{5}{36}$$

$$\beta_4 = \frac{1}{a\kappa^2}\left(\frac{10}{3}\omega - \frac{13}{6} - \frac{1}{4\omega} \right)$$

$$\beta_5 = \frac{1}{a^2\kappa^4}\left(\frac{5}{2} + \frac{1}{8\omega^2} \right) + \frac{\tilde{\lambda}}{a\kappa^4}\left(\frac{56}{3} + \frac{2}{3\omega} \right) \ .$$

Asymptotically free … PT no good in IR

Here $\omega \equiv b/a$ and $\tilde{\lambda}$ is the dimensionless combination of the cosmolog
Newton's constant $\tilde{\lambda} \equiv \frac{1}{2}\lambda_0\kappa^4$ with $\kappa^2 = 16\pi G$.

[Fradkin and Tseylin;
Avramiddy and Barvinky]

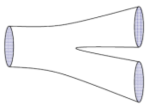
Evaluate Options…

(1) Denial : Gravity should *not* be quantized, only matter fields.
… goes against rules of QM & QFT.

(2) Keep gravity, but resort to "*other*" methods: Nontrivial RG fixed point (a.k.a. asymptotic safety), Lattice Gravity, Truncations, LQG…

(3) Add more fields so as to reduce (or *eliminate*) divergences:
N=8 Sugra ⟶ 70 massless scalars…

(4) Embed gravity in a larger non-local (finite) theory : e.g. Superstrings.
Gravity then emerges as an effective theory.



$N = 8$ Supergravity

- Restore applicability of P.T. in d=4 - by adding suitably tuned additional multiplets …

E.g. One-loop *G*-coupling beta function of N=*8* Sugra:

$$\beta_0 = -\frac{47}{3}\cdot 1 + \frac{26}{3}\cdot 8 - \frac{11}{3}\cdot 28 + \frac{2}{3}\cdot 56 + \frac{1}{3}\cdot 35 = 0$$

Should be a finite theory (correspondence to N=*4 SYM*) … at least to *six loops*
… but then we might not know for a few more years.

One also would like to understand, eventually, when and how SUSY is broken *…*

A Second Look at Non-Ren.Theories

An often repeated statement …

“ A non-renormalizable theory needs new counter-terms added at every new order of the perturbative expansion.

“ This implies an infinite number of experiments to fix all their values, and an infinite number of “physical’’ parameters.”

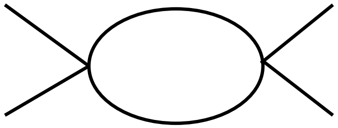
“ This is at the core of the lack of predictive power of non-renormalizable theories, such as quantum gravity.”

It can’t get any more hopeless than this …

Infinite or Zero ?

The QFT beta-function at one loop for the ϕ^4 theory in d = 4 is :

$$\beta(g) = \frac{3}{16\pi^2} g^2 + O(g^3)$$



So the sign suggests that the coupling constant might go to zero at low energies (Symanzik, Wilson 1973). If this behavior persists at large couplings, it would indicate quantum triviality.

The question can ultimately only be answered non-perturbatively (lattice), since it involves strong coupling.

Also, in d > 4 this is a non-renormalizable theory !

Proof of Triviality of (Non-Ren.) $\lambda\phi^4$

PHYSICAL REVIEW LETTERS

VOLUME 47 | 6 JULY 1981 | NUMBER 1

Proof of the Triviality of φ_d^4 Field Theory and Some Mean-Field Features of Ising Models for $d > 4$

Michael Aizenman
Department of Physics, Princeton University, Princeton, New Jersey 08544
(Received 23 April 1981)

It is rigorously proved that the continuum limits of Euclidean φ_d^4 lattice fields are free fields in $d > 4$. An exact geometric characterization of criticality in Ising models is introduced, and used to prove other mean-field features for $d > 4$ and hyperscaling in $d = 2$.

PACS numbers: 03.50.– z, 05.50.+q, 11.10.– z.

(1) *The main result.*—A constructive approach to the Euclidean φ_d^4 field (in R^d) is to define it as a continuum limit of lattice fields, with the distribution

$$\prod_x (d\varphi_x) \exp\Big[-\sum_x (\lambda_0 \varphi_x^4 + B_0 \varphi_x^2) + \sum_{|x-y|=1} \tfrac{1}{2} J \varphi_x \varphi_y\Big]/\text{norm}. \qquad (1)$$

A FREE field theory … situation seems rather confusing, to say the least.

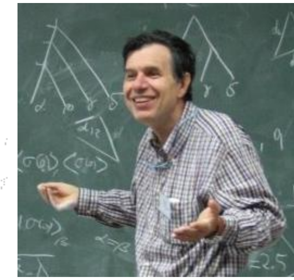
Perturbatively Non-Renorm. Interactions

Some early work :

- K.G. Wilson, *Quantum Field Theory Models in d < 4,* PRD 1973.
- G. Parisi, *Renormalizability of Not Renormalizable Theories,* LNC 1973.
- G. Parisi, *Theory of Non-renormalizable Interactions - Large N,* NPB 1975.
- K. Symanzik, *Renormalization of Nonrenormalizable Massless* φ^4 *Theory,* CMP 1975.
- E. Brézin and J. Zinn-Justin, *Nonlinear* σ *Model in* $2+\varepsilon$ *Dimensions,* PRL 1976; PRD 1976.
- D. Gross and A. Neveu, PRD 1974 …

(Un) fortunately not very relevant for *particle physics* …

ON NON-RENORMALIZABLE INTERACTIONS



G. PARISI

(I.H.E.S. - BURES-sur-YVETTE)

[Cargese 1976]

I. Introduction

Nonrenormalizable interactions have always been the black sheep of field theory. Long time ago (1) it was supposed that non-renormalizable interactions are characterized by having Green functions which are not C^{∞} in the coupling constant : if this interpretation is correct, the ultraviolet divergences found in the perturbative expansion arise from the non existence of the quantities which are computed in the standard approach (i.e., the coefficients of the Taylor expansion for zero coupling constant).

The first attempts in this direction were done using or the ξ-limiting procedure (2) or the peratization technique (3). However they were mainly inconclusive ; the full understanding of the problem required a better non perturbative knowledge of quantum field theory which is now given by the modern theory of second order phase transitions (4).

The purpose of these lectures is to study the existence and the properties of non-renormalizable interactions at the light of the knowledge gathered in the study of critical phenomena. We do

Les Houches 1977

COURSE 4

APPLICATIONS OF THE RENORMALIZATION GROUP TO HIGH-ENERGY PHYSICS

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subtractions will be required than in the two-dimensional theory. It therefore appears that, if perturbed in powers of $1/N$, $(\bar{\psi}\psi)^2$ is renormalizable in three dimensions.

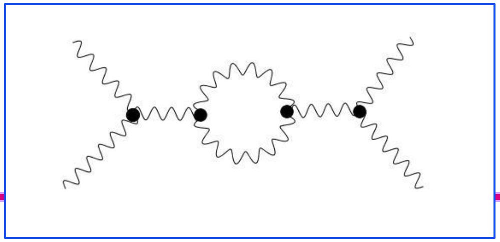
Further insight into this theory is gained by calculating $\beta(G)$,

$$\beta(G) = \tfrac{1}{2}G - \tfrac{1}{16}G^3 N\,. \tag{9.19}$$

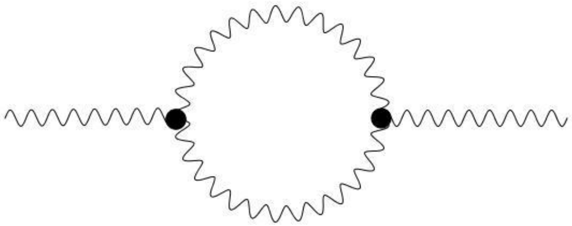
Depending on the value of G two qualitatively different theories emerge. For $G^2 < G_0^2 = 8/N$, β is positive, the σ propagator has no space-like singularities, and the theory is infrared free. For $G^2 > G_0^2$, β is negative, the effective coupling diverges for finite momenta and the σ propagator has a tachyon pole. Spontaneous symmetry breaking then occurs as before, the fermions develop a mass, and the tachyons disappear. In both cases the ultraviolet behavior of the theories is controlled by the fixed point of β, G_0. It is the existence of such an ultraviolet fixed point in the $1/N$ expansion that produces the non-perturbative improvement of perturbation theory.

One can speculate that this phenomena could occur in other "non-renormalizable" theories. One merely has to find a method of resumming ordinary perturbation theory, like the $1/N$ expansion, that produces an ultraviolet stable fixed point of the renormalization group. This is in general non-trivial, and I know of no realistic four-dimensional examples. For example, in the case of $(\bar{\psi}\psi)^2$ one finds that G_0 diverges as one approaches four dimensions.

Change Dimension : D<4



Simplest graviton loop :



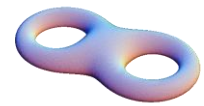
$$\sim \; G^2 \int \frac{d^D p}{p^2 \, (p+k)^2} \cdot V(p+k) \cdot V(p+k)$$

By lowering the dimension, Feynman diagrams can be made to converge … Wilson's $d = 2 + \varepsilon$ (double) expansion.

Back to evaluating diagrams !

Gravity in 2.000001 Dimensions

- Wilson expansion: Formulate in 2+ε dimensions…



$$\chi = \frac{1}{4\pi}\int d^2x\sqrt{g}\,R$$

G is dim-less, so theory is now *perturbatively renormalizable*

Wilson 1973
Weinberg 1977 …
Kawai, Ninomiya 1995
Kitazawa, Aida 1998

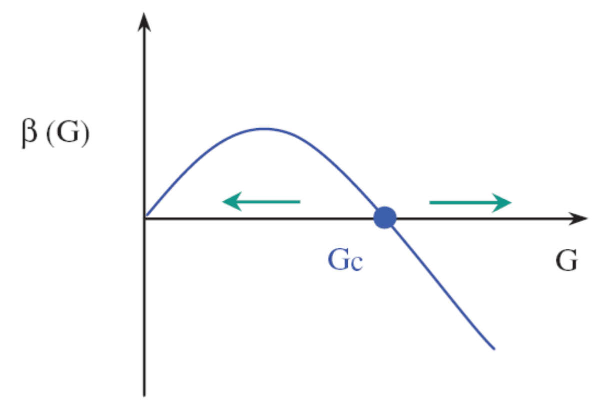
$$\mu\frac{\partial}{\partial\mu}G(\mu) \;=\; \beta(G(\mu))$$

$$\beta(G) \;=\; (d-2)\,G - \frac{2}{3}(25-n_s)\,G^2 - \frac{20}{3}(25-n_s)\,G^3 + \dots$$

(pure gravity : $n_s = 0$)

with non-trivial QFT UV fixed point :

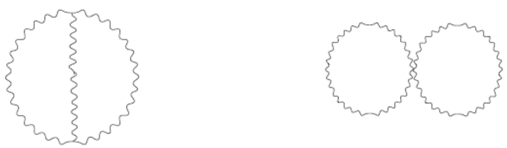
$$\begin{cases} G_c = \dfrac{3}{2\,(25-n_s)}(d-2) - \dfrac{45}{2(25-n_s)^2}(d-2)^2 + \dots \\ \nu^{-1} = -\beta'(G_c) = (d-2) + \dfrac{15}{25-n_s}(d-2)^2 + \dots \end{cases}$$



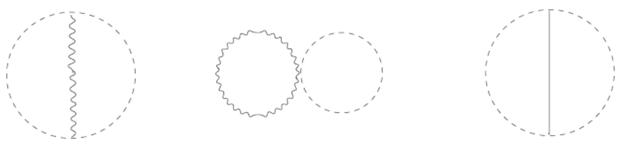
(two loops, manifestly covariant, gauge independent …)

Two phases of Gravity

2.000001 dimensions – cont'd



a Graviton loops b



a b c

Graviton-ghost loops

- Analytical control of UV fixed point at Gc ;
- Lorentzian = Euclidean to all orders in G ;
- Nontrivial scaling, determined by UV FP :

- New Scale, Strong IR divergence :

$$G(k^2) \;=\; \frac{G_c}{1 \pm (m^2/k^2)^{(d-2)/2}}$$

$$\xi^{-1} \equiv \; m \sim \Lambda \exp\left(- \int^{G} \frac{dG'}{\beta(G')} \right) \underset{G \to G_c}{\sim} \Lambda \, | \, G - G_c |^{-1/\beta'(G_c)}$$

$$\lambda_0 \to \lambda_0 \left[1 - \left(\frac{a_1}{\varepsilon} + \frac{a_2}{\varepsilon^2}\right) G\right]$$

$$\frac{\mu^\varepsilon}{16\pi G} \to \frac{\mu^\varepsilon}{16\pi G}\left(1 - \frac{b}{\varepsilon} G\right)$$

$$a_1 = -\frac{8}{\alpha} + 8\frac{(\beta-1)^2}{(\beta-2)^2} + 4\frac{(\beta-1)(\beta-3)}{\alpha(\beta-2)^2}$$

$$a_2 = 8\frac{(\beta-1)^2}{(\beta-2)^2} \ .$$

$$b = \frac{2}{3}\cdot 19 + \frac{4(\beta-1)^2}{(\beta-2)^2}$$

α, β gauge parameters,
answer gauge dependent

Rescale metric :

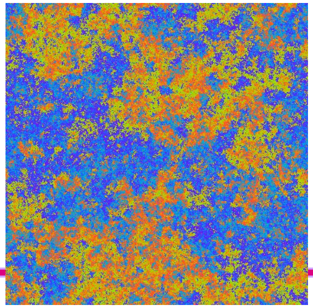
$$g_{\mu\nu} = \left[1 - \left(\frac{a_1}{\varepsilon} + \frac{a_2}{\varepsilon^2}\right) G\right]^{-2/d} g'_{\mu\nu}$$

$$\mathscr{L} \to -\frac{\mu^\varepsilon}{16\pi G}\left[1 - \frac{1}{\varepsilon}(b - \tfrac{1}{2}a_2)G\right]\sqrt{g'}R' + \lambda_0\sqrt{g'}$$

$$\frac{1}{G} \to \frac{1}{G}\left[1 - \frac{1}{\varepsilon}(b - \tfrac{1}{2}a_2)\, G\right]$$

G renormalization,
answer gauge independent

Detour : Non-linear Sigma model

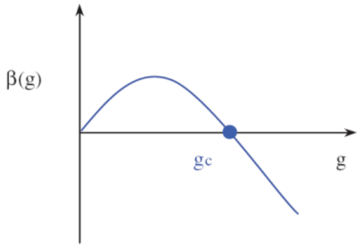


- Field theory description of $O(N)$ Heisenberg model :

$$Z \;=\; \int [d\sigma]\,\prod_x \delta\left(\sigma^a(x)\,\sigma^a(x)\,-\,1\right)\,\exp\left(-\frac{\Lambda^{d-2}}{g^2}\int d^dx\,\partial_\mu\sigma^a(x)\,\partial_\mu\sigma^a(x)\,+\,\int d^dx\,j^a(x)\sigma^a(x)\right)$$

E. Brezin J. Zinn-Justin 1975
F. Wegner, 1989
E. Brezin and S. Hikami, 1996

Coupling becomes *dimensionless* in *d = 2.* For *d > 2* theory is <u>not</u> perturbatively renormalizable, yet in the 2+ ε expansion one finds:



$$\Lambda\,\frac{\partial\,g^2}{\partial\,\Lambda}\;\equiv\;\beta(g^2)\;=\;(d-2)\,g^2\;-\;\frac{N-2}{2\pi}\,g^4\;+\;O\left(g^6,(d-2)g^4\right)$$

Phase Transition = non-trivial UV fixed point; new non-perturbative mass scale :

$$\nu^{-1}(\epsilon) = \epsilon + \frac{\epsilon^2}{n-2} + \frac{\epsilon^3}{2(n-2)} - \left[30 - 14 + n^2 + (54-18n)\zeta(3)\right]\frac{\epsilon^4}{4(n-2)^3} + \dots$$

$$\xi(g^2)\;\equiv\;m^{-1}(g^2)\;\simeq\;c_d\,\Lambda\left(\frac{1}{g_c^2}-\frac{1}{g^2}\right)^{\nu}$$

… But are the QFT predictions correct ?

Experimental test: *O(2)* non-linear sigma model describes
the phase transition of *superfluid Helium*

Space Shuttle experiment (2003)

High precision measurement of specific heat of superfluid Helium He4
(zero momentum energy-energy correlation at UV FP) yields ν

LIPA *et al.*

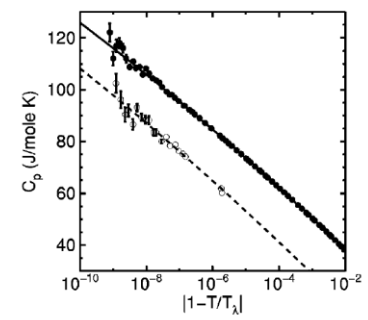


FIG. 15. Semilogarithmic plot of the specific heat vs reduced temperature over the full range measured. Below the transition the data (closed symbols) were binned with a density of 10 bins per decade, and above (open symbols) with a density of 8 bins per decade. Lines show best fits to the data.

J.A. Lipa et al, Phys Rev 2003: $\alpha = 2 - 3\nu = -0.0127(3)$

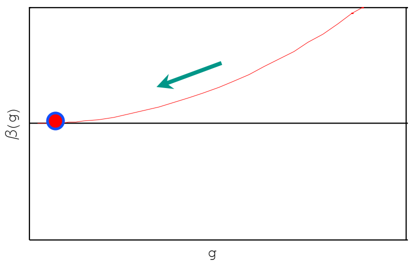
MC, HT, 4-ε exp. to 4 loops, & to 6 loops in d=3: $\alpha = 2 - 3\nu \approx -0.0125(4)$

Second most accurate predictions of QFT, after g-2

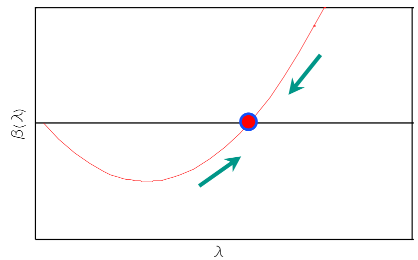
Running G(k)

In QFT, generally, *coupling constants* are not constant. *They run*.

RG Running Scenarios



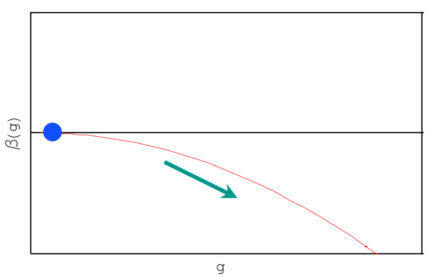
“Triviality” of lambda phi 4



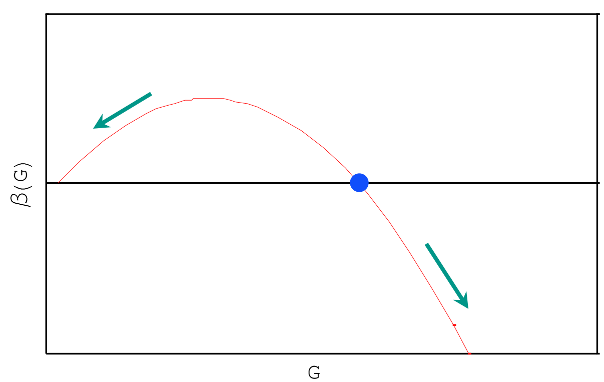
Wilson-Fisher FP in d<4

Callan-Symanzik. beta function(s):

$$\mu \frac{\partial}{\partial \mu} G(\mu) = \beta(G(\mu))$$



Asymptotic freedom of YM



Ising model, σ-model, Gravity (2+ε, lattice)

Running Coupling - QED

It is possible, of course, to avoid the use of the small $\mathbf{k}^2$ approximation. A static charge located at the origin will induce a modified Coulomb potential to order α given by

$$\frac{e}{4\pi r} \to V(r) = \frac{e}{4\pi r} Q(r)$$

$$Q(r) = 1 + \frac{2\alpha}{3\pi}\int_1^\infty du\, e^{-2mru}\left(1 + \frac{1}{2u^2}\right)\frac{(u^2-1)^{1/2}}{u^2} \tag{7-24}$$

$$Q(r) = \begin{cases} 1 + \frac{\alpha}{3\pi}\left[\ln\frac{1}{(mr)^2} - 2\gamma - \frac{5}{3} + \cdots\right] & mr \ll 1 \\ 1 + \frac{\alpha}{4\pi^{1/2}(mr)^{3/2}} e^{-2mr} + \cdots & mr \gg 1 \end{cases}$$

with γ equal to Euler's constant $-\int_0^\infty du \ln u\, e^{-u} = 0.5772\ldots$. According to the definition of charge $Q(\infty) = 1$. We see again that as r decreases $Q(r)$ increases, even becoming infinite as r tends to zero. The approximation used in Eq. (7-23) is only valid in the mean.

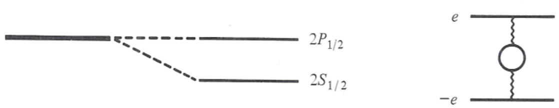
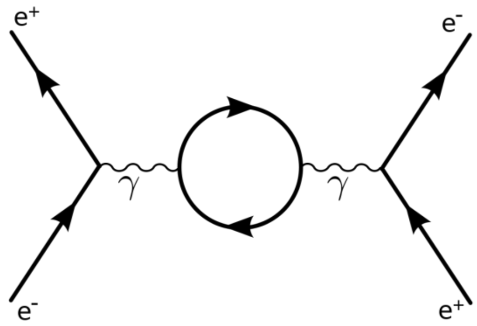


Figure 7-6 The vacuum polarization contribution to the $2S_{1/2} - 2P_{1/2}$ hydrogen splitting.

Itzykson & Zuber QFT, p. 328



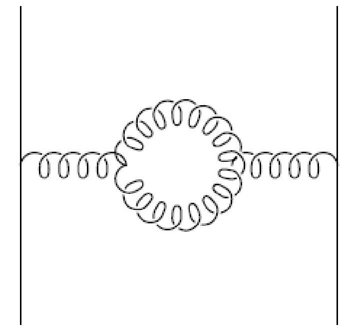
$$Q \to Q(r) = Q\left(1 + \frac{\alpha}{3\pi}\ln\frac{1}{m^2 r^2} + \ldots\right)$$

<u>IR divergence </u>… but electron Compton wavelength 10^-10 cm.

… What would happen if the electron mass was *much* smaller ??

Running Coupling – QCD

$$V_{QCD}(r) = -\frac{4}{3}\cdot\frac{4\pi}{11-\frac{2}{3}n_f}\cdot\frac{1}{r\,\log\frac{1}{r^2\,m^2}} + \ldots$$



The mass of the gluon is zero to all orders ... Strong IR divergences.

But *SU(N)* Yang Mills cures its own infrared divergences: $m^2 = \xi^{-2} \propto \Lambda^2_{\overline{MS}}$ $\Lambda^{-1}_{\overline{MS}} \simeq 1\,Fermi$

$$\langle 0|\, F^a_{\mu\nu}\, F^{\mu\nu a}\,|0\rangle \simeq \frac{1}{\xi^4}$$

$$\langle 0|\, \bar\psi\,\psi\,|0\rangle \simeq \frac{1}{\xi^3}$$

$$\xi^{-1} \equiv m \simeq \exp\left(-\frac{2\pi}{\beta_0\, g}\right)$$

A new scale

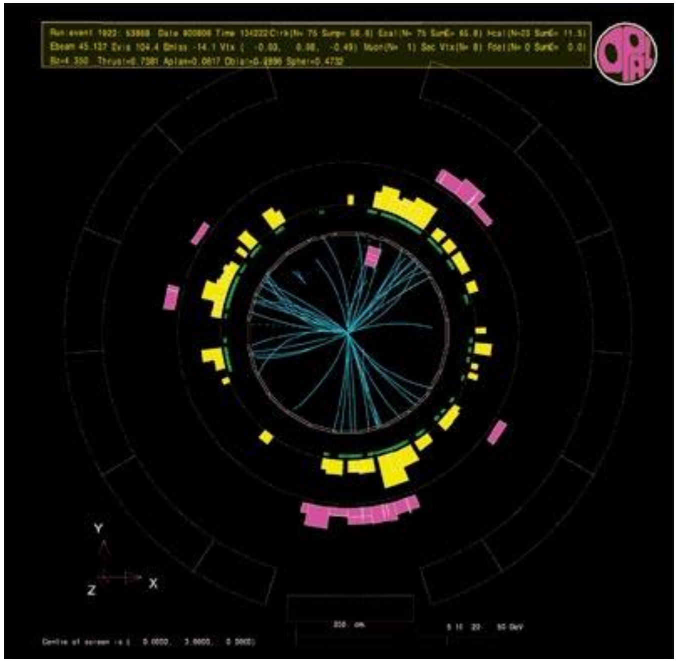
Non-perturbative condensates are non-analytic in g … phase change.

What would happen if *m* was very small ??

Mass without Mass

Is the gluon massless or massive ?

It <u>depends on the scale</u> … at short distances it certainly remains effectively massless – and weakly coupled.



Three jet event, Opal detector
Source: Cern Courier

Summary

Key ingredients of the previous QED & QCD RG results:

(a) A $log(E)$ – By dimensional arguments & power counting this can only happen for theories with *dimensionless* couplings. Log unlikely for gravity.

(b) Reference scale is *smallest* mass in problem (IR divergence).

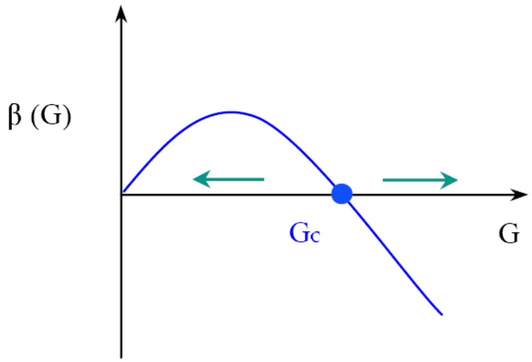
(c) Magnitude of new physical scale can only be *fixed by experiment:*

There is only so much QFT can predict…

Back to QFT Gravity…

Running of Newton's G(k) in 2+ε is of the form:

$$G(k^2) \simeq G_0 \left[1 \pm c_0 \left(\frac{1}{\xi^2 k^2} \right)^{1/2\nu} + \ldots \right]$$



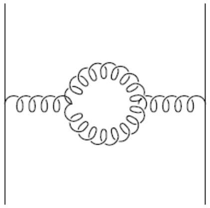
$$\nu^{-1} = -\beta'(G_c) = (d-2) + \frac{15}{25 - n_s}(d-2)^2 + \ldots$$

(Plus or minus, depending on which side of the FP one resides …)

Key quantities : i) the exponent, ii) the scale ξ .

What is left of the above QFT scenario in 4 dimensions ?

Running Coupling – Gravity



Proposed form :

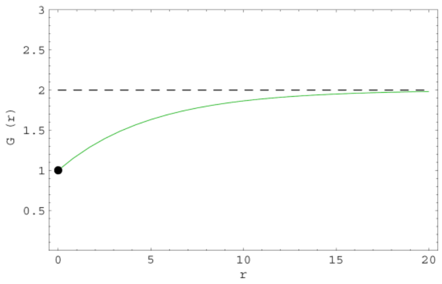
(obtained here from $G(\Box)$ static isotropic solution)

$$G \;\to\; G(r) \;=\; G_0 \left(1 \,+\, \frac{c_0}{3\,\pi}\, m^3\, r^3 \,\ln\, \frac{1}{m^2\, r^2} \,+\, \dots \right)$$

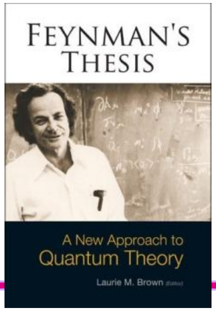
$$c_0 = O(1)$$

HH & R.M.Williams
NPB ‘95, PRD ‘00, ‘06,’07

$$m \;=\; \xi^{-1} = A_\xi \cdot \Lambda \, \exp \left(- \int^{G(\Lambda)} \frac{dG'}{\beta(G')} \right) \qquad\qquad \frac{1}{\xi^2} \;\simeq\; \frac{\lambda}{3}$$



New scale ξ has to be fixed by observation –
everything else is in principle fixed/calculable.



Feynman Path Integral

Path Integral for Quantum Gravitation

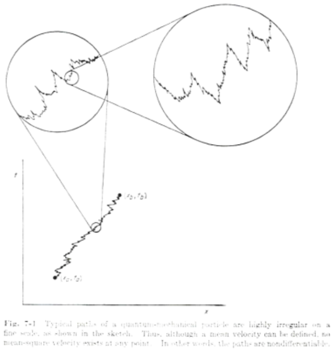
$$||\delta g||^2 \;\equiv\; \int d^d x \; G^{\mu\nu,\alpha\beta}[g(x)] \; \delta g_{\mu\nu}(x) \, \delta g_{\alpha\beta}(x)$$

DeWitt approach to measure :
introduce *super-metric*

$$G^{\mu\nu,\alpha\beta}[g(x)] \;=\; \tfrac{1}{2}\sqrt{g(x)}\,\left[g^{\mu\alpha}(x)g^{\nu\beta}(x) + g^{\mu\beta}(x)g^{\nu\alpha}(x) + \lambda\, g^{\mu\nu}(x)g^{\alpha\beta}(x)\right]$$

$$\int d\mu[g] \;=\; \int \prod_x \,[g(x)]^{(d-4)(d+1)/8} \prod_{\mu \ge \nu} dg_{\mu\nu}(x) \;\underset{d=4}{\rightarrow}\; \boxed{\int \prod_x \prod_{\mu \ge \nu} dg_{\mu\nu}(x)}$$

$$Z_{cont} \;=\; \int [d\, g_{\mu\nu}] \; e^{-\lambda \int dx \sqrt{g} + \frac{1}{16\pi G} \int dx \sqrt{g}\, R}$$



Definition of Path Integral requires a Lattice (Feynman &Hibbs, 1964).

Only One Coupling

Pure gravity path integral:

$$Z \; = \; \int [d\, g_{\mu\nu}] \; e^{-I_E[g]}$$

$$I_E[g] \; = \; \lambda_0 \, \Lambda^d \int dx \, \sqrt{g} \; - \; \frac{1}{16\pi \, G_0} \, \Lambda^{d-2} \int dx \, \sqrt{g} \, R$$

In the absence of matter, only one dim.less coupling:

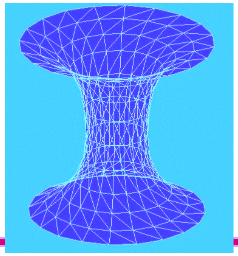
$$\tilde{G} \; \equiv \; G_0 \, \lambda_0^{(d-2)/d}$$

Rescale metric (edge lengths):

$$g'_{\mu\nu} \; = \; \lambda_0^{2/d} \, g_{\mu\nu} \qquad g'^{\mu\nu} \; = \; \lambda_0^{-2/d} \, g^{\mu\nu}$$

… similar to *g* of Y.M.

$$I_E[g] \; = \; \Lambda^d \int dx \, \sqrt{g'} \; - \; \frac{1}{16\pi \, G_0 \, \lambda_0^{\frac{d-2}{d}}} \, \Lambda^{d-2} \int dx \, \sqrt{g'} \, R'$$



Gravity on a Lattice

General aim :

i) Independently re-derive above scenario in d=4

ii) Determine phase structure

iii) Obtain exponent ν

iv) What is the scale ξ ?

Lattice is, at least in principle, non-perturbative and exact.

Lattice Theory of Gravity

“General Relativity Without Coordinates” (T.Regge , J.A. Wheeler)

[MTW, ch. 42]

- Based on a dynamical lattice.
- Incorporates continuous local invariance.
- Puts within the reach of computation problems which in practical terms are beyond the power of analytical methods.
- Affords any desired level of accuracy by a sufficiently fine subdivision of space-time.

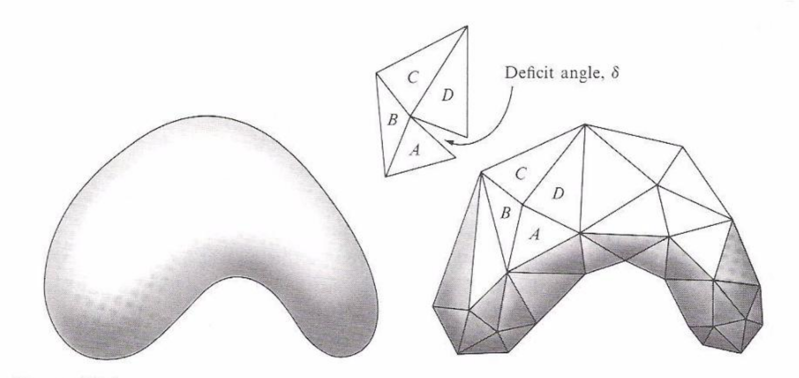
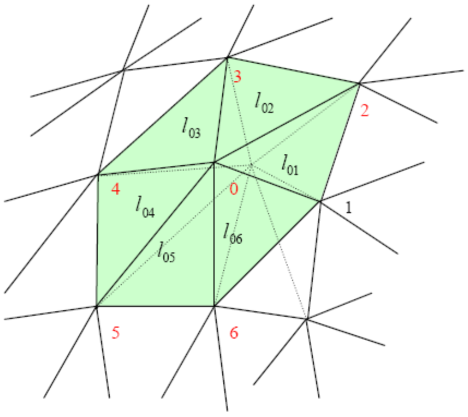


Figure 42.1.



Lattice *Gauge* Theory *Works*

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \, \ln(\mu^2/\Lambda^2)} \left[1 - \frac{2\beta_1}{\beta_0^2} \, \frac{\ln\left[\ln(\mu^2/\Lambda^2)\right]}{\ln(\mu^2/\Lambda^2)} + \frac{4\beta_1^2}{\beta_0^4 \ln^2(\mu^2/\Lambda^2)} \right.$$
$$\left. \times \left(\left(\ln\left[\ln(\mu^2/\Lambda^2)\right] - \frac{1}{2} \right)^2 + \frac{\beta_2\beta_0}{8\beta_1^2} - \frac{5}{4} \right) \right] .$$

Wilson's lattice gauge theory provides to this day the only convincing theoretical evidence for *confinement* and *chiral symmetry breaking* in QCD.

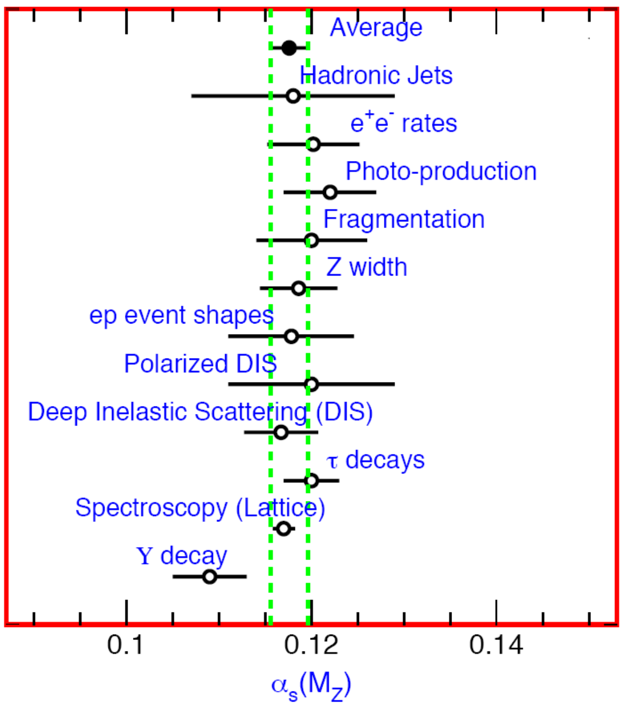
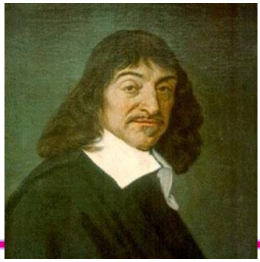
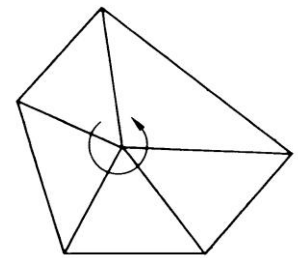


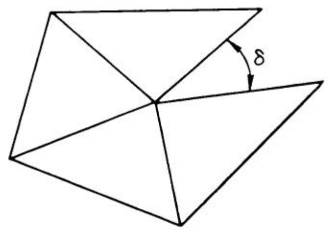
Figure 9.1: Summary of the value of $\alpha_s(M_Z)$ from various processes. The values shown indicate the process and the measured value of α_s extrapolated to $\mu = M_Z$. The error shown is the *total* error including theoretical uncertainties. The average quoted in this report which comes from these measurements is also shown. See text for discussion of errors.

[Particle Data Group LBL, 2010]

Curvature - Described by Angles







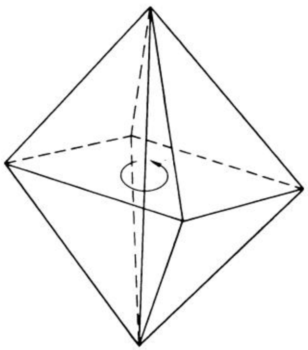
$d = 2$

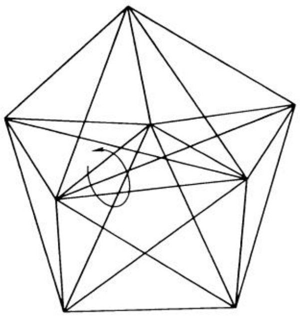
$$g_{ij} = \frac{1}{2}\left(l_{1,i+1}^2 + l_{1,j+1}^2 - l_{i+1,j+1}^2\right)$$

$$V_d = \frac{1}{d!}\sqrt{\det g_{ij}}$$

$$\sin\theta_d = \frac{d}{d-1}\frac{V_d\, V_{d-2}}{V_{d-1}\, V'_{d-1}}$$

$$\delta_h = 2\pi - \sum_{\substack{\text{d-simplices} \\ \text{meeting on h}}} \theta_d$$





$d = 3$ $d = 4$

Curvature determined by edge lengths

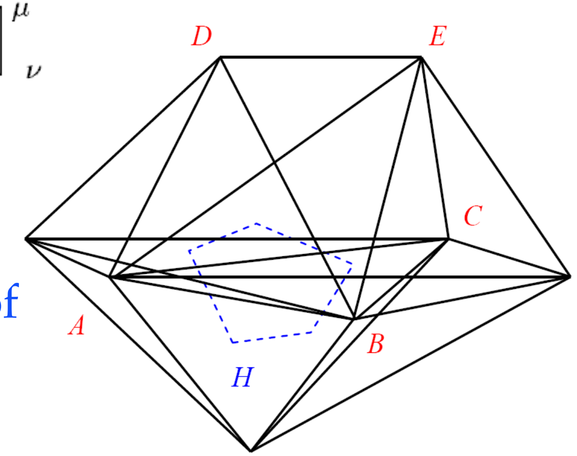
T. Regge 1961

J.A. Wheeler 1964

Lattice Rotations, Riemann tensor

$$\phi^\mu(s_{n+1}) = R^\mu{}_\nu(P)\,\phi^\nu(s_1) \qquad R^\mu{}_\nu = \left[P\, e^{\int_{\text{path between simplices}} \Gamma_\lambda dx^\lambda} \right]^\mu{}_\nu$$

$$\mathbf{R}(C) = \mathbf{R}(s_1, s_n) \cdots \mathbf{R}(s_2, s_1)$$



Elementary polygonal path around a hinge (triangle) in four dimensions.

Due to the hinge's intrinsic orientation, only components of the vector in the plane *perpendicular to the hinge* are rotated:

$$U_{\mu\nu}(h) = \mathcal{N}\,\epsilon_{\mu\nu\alpha_1\alpha_{d-2}}\, l^{\alpha_1}_{(1)} \cdots l^{\alpha_{d-2}}_{(d-2)}$$

$$R^\mu{}_\nu(C) = \left(e^{\delta U} \right)^\mu{}_\nu$$

$$R_{\mu\nu\lambda\sigma}(h) = \frac{\delta(h)}{A_C(h)}\, U_{\mu\nu}(h)\, U_{\lambda\sigma}(h)$$

$$R(h) = 2\,\frac{\delta(h)}{A_C(h)}$$

Exact lattice Bianchi identity,

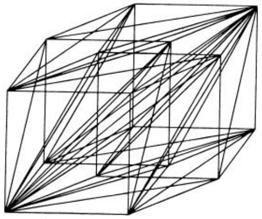
$$\prod_{\substack{\text{hinges h} \\ \text{meeting on edge p}}} \left[e^{\delta(h) U(h)} \right]^\mu{}_\nu = 1$$

Lattice Weak Field Expansion

Regge-Wheeler th. is the ***only*** lattice theory of gravity with correct degrees of freedom : *one m = 0, s = 2 degree of freedom*

$$- 2\,\kappa_0 \sum_h \delta_h(l^2)\, A_h(l^2)$$

… call small edge fluctuations "*e*" :



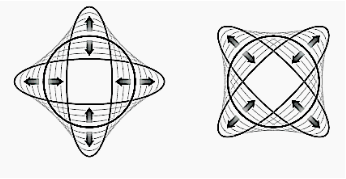
$$I_R = \frac{1}{2} \sum_{ij} e_i\, M_{ij}\, e_j$$

… then Fourier transform, and express result in terms of metric deformations :

$$\delta g_{ij}(l^2) \;=\; \tfrac{1}{2}\,(\delta l_{0i}^2 + \delta l_{0j}^2 - \delta l_{ij}^2)$$

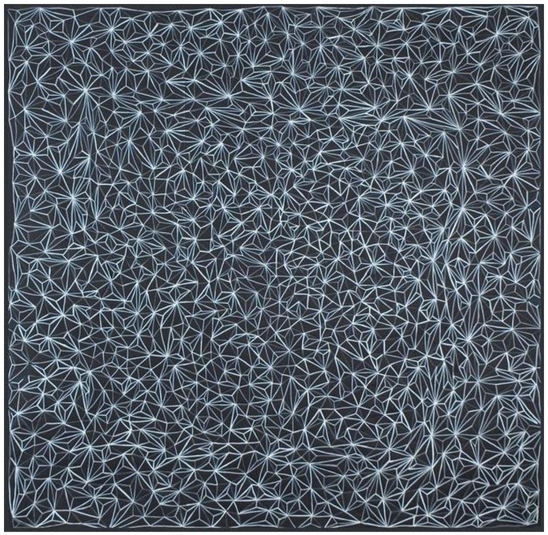
… obtaining in the vacuum gauge precisely the familiar *TT form* in k→0 limit:

$$\frac{1}{4}\mathbf{k}^2 \bar{h}_{ij}^{TT}(\mathbf{k})\; h_{ij}^{TT}(\mathbf{k})$$



R.M.Williams, M. Roceck, 1981

Choice of Lattice Structure

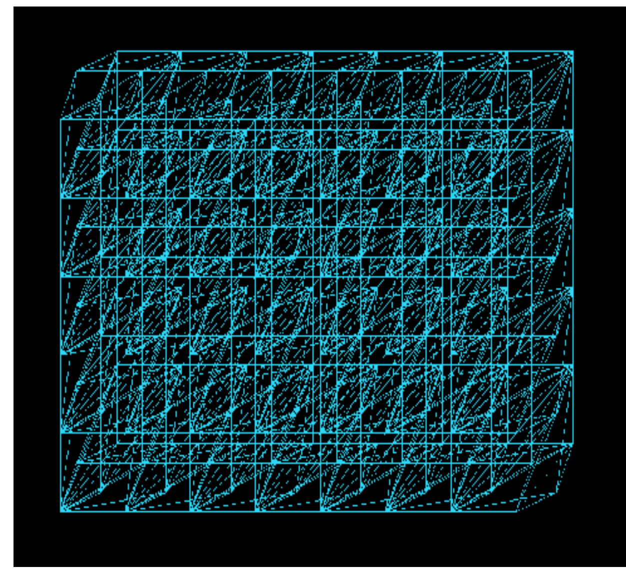


Timothy Nolan,

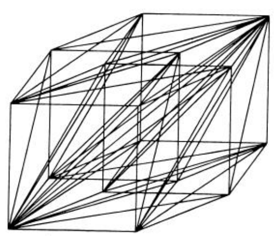
Carl Berg Gallery, Los Angeles

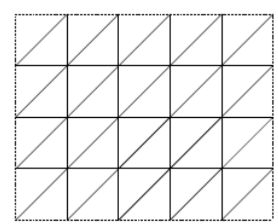
A not so regular lattice …

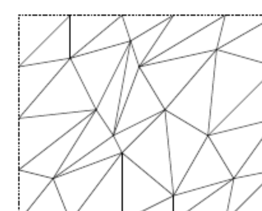
… and a more regular one:



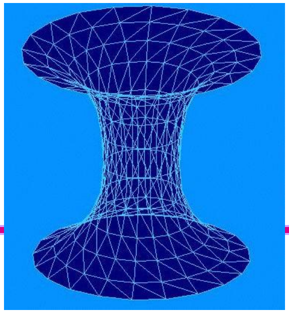
Regular geometric objects can be *stacked.*







Lattice Path Integral



Lattice path integral follows from edge assignments,

$$g_{ij} = \frac{1}{2}\left(l_{1,i+1}^2 + l_{1,j+1}^2 - l_{i+1,j+1}^2\right) \qquad V_d = \frac{1}{d!}\sqrt{\det g_{ij}}$$

$$I_E[g] = \lambda_0\,\Lambda^d \int dx\,\sqrt{g} - \frac{1}{16\pi\,G_0}\,\Lambda^{d-2}\int dx\,\sqrt{g}\,R \longrightarrow I_L = \lambda_0 \sum_h V_h(l^2) - 2\,\kappa_0 \sum_h \delta_h(l^2)\,A_h(l^2)$$

$$\int [d\,g_{\mu\nu}] = \int \prod_x\, [g(x)]^{\frac{(d-4)(d+1)}{8}} \prod_{\mu\geq\nu} dg_{\mu\nu}(x) \longrightarrow \int [d\,l^2] \equiv \int_0^\infty \prod_{ij} dl_{ij}^2 \prod_s\, [V_d(s)]^\sigma\;\Theta(l_{ij}^2)$$

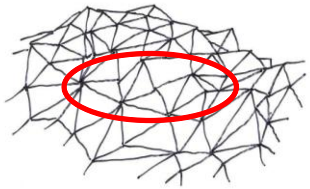
(Lattice analog of the DeWitt measure)

$$Z = \int [d\,g_{\mu\nu}]\; e^{-\lambda_0 \int d^dx\,\sqrt{g} + \frac{1}{16\pi G}\int d^dx\sqrt{g}\,R} \longrightarrow Z_L = \int [d\,l^2]\,e^{-I_L[l^2]}$$

Without loss of generality, one can set bare $\lambda_0 = 1$;

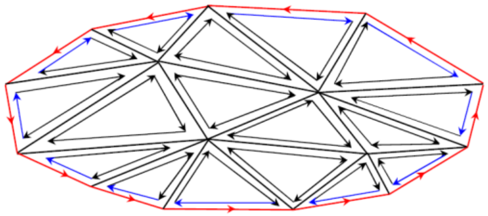
Besides the cutoff, the only relevant coupling is κ (or G).

Gravitational Wilson Loop



- Parallel transport of a vector done via lattice rotation matrix

$$R^{\alpha}{}_{\beta}(C) \;=\; \left[\, \mathcal{P} \exp \left\{ \oint_{\mathbf{path\ C}} \Gamma^{\cdot}_{\lambda\,\cdot} dx^{\lambda} \right\} \right]^{\alpha}{}_{\beta}$$



For a *large* closed circuit obtain *gravitational Wilson loop;* compute at *strong coupling (G large) …*

$$W(\Gamma) \sim \mathrm{Tr}\, \mathcal{P} \exp \left[\int_C \Gamma^{\lambda}{}_{..}\, dx_{\lambda} \right] \underset{A \to \infty}{\sim} \exp\left(-A_C/\xi^2\right)$$

"Minimal area law" follows from loop tiling.

… then compare to *semi-classical result* (from Stokes' theorem)

$$R^{\alpha}{}_{\beta}(C) \sim \left[\exp \left\{ \tfrac{1}{2} \int_{S(C)} R^{\cdot}{}_{\cdot\,\mu\nu} A^{\mu\nu}_{C} \right\} \right]^{\alpha}{}_{\beta} \qquad A^{\mu\nu}_{C} = \tfrac{1}{2} \oint dx^{\mu}\, x^{\nu}$$

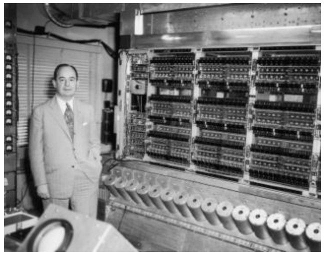
- suggests ξ related to curvature.
- argument predicts a positive cosmological constant.

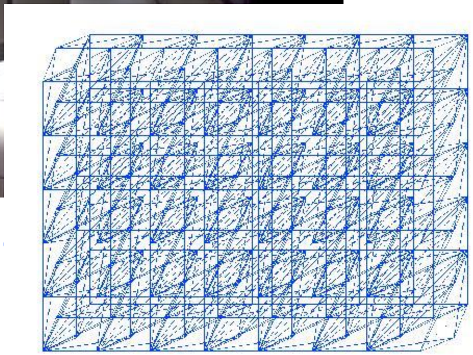
$$\lambda_{obs} \simeq +\frac{1}{\xi^2}$$

HH & R Williams,
Phys Rev D 76 (2007) ; D 81 (2010)

[Peskin and Schroeder, page 783]

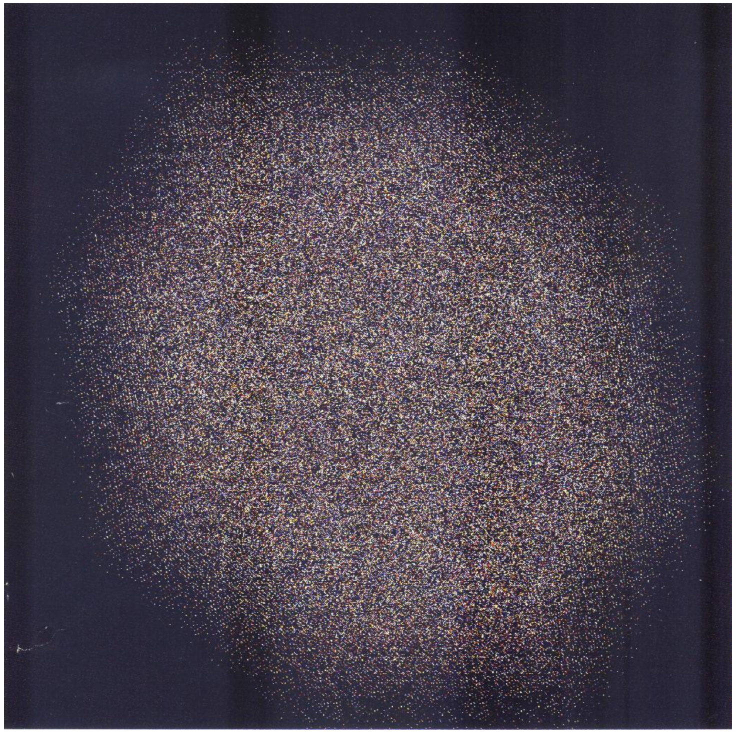
Numerical Evaluation of *Z*

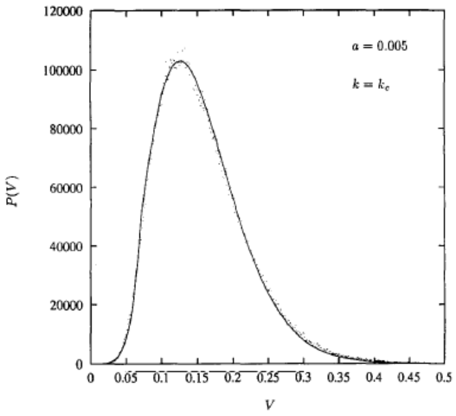


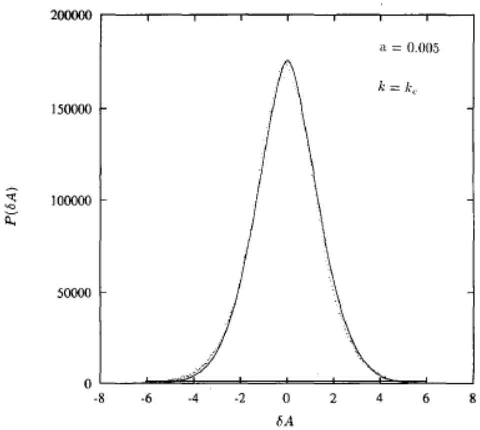


Edge length/metric distributions

- L=4 → 6,144 simplices
- L=8 → 98,304 simplices
- L=16 → 1,572,864 simplices
- L=32 → 25,165,824 simplices





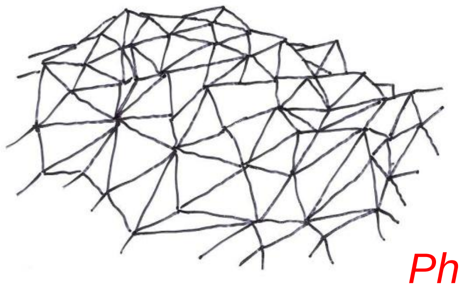


Phases of L. Quantum Gravity

(Euclidean) Lattice Quantum Gravity in d = 4
exhibits two phases :

$G > G_c$ Smooth phase: R ≈ 0

$$\langle g_{\mu\nu} \rangle \approx c\, \eta_{\mu\nu}$$

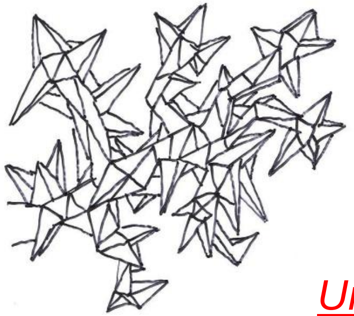


Physical

$$N(\tau) \underset{\tau \to \infty}{\sim} \tau^{d_v}$$

$G < G_c$ Rough phase :
branched polymer, d ≈ 2

$$\langle g_{\mu\nu} \rangle = 0$$

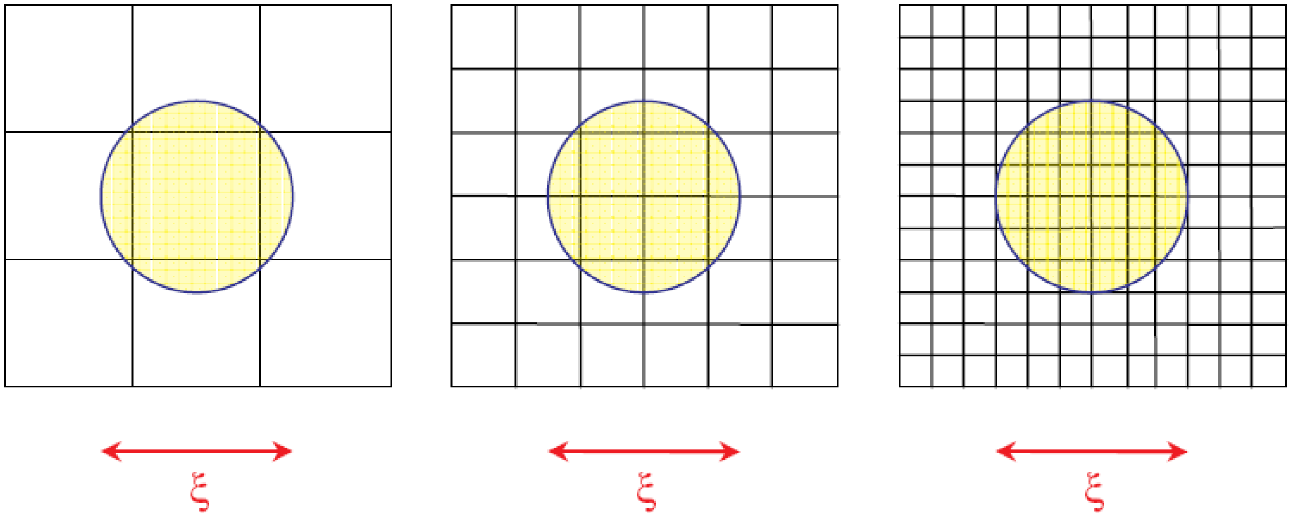


Unphysical

[HH & RMW, NPB, PLB 1984 ;
B. Berg 1985 , Beirl et al 1993, …]

(Lattice manifestation of conformal instability)

Lattice Continuum Limit



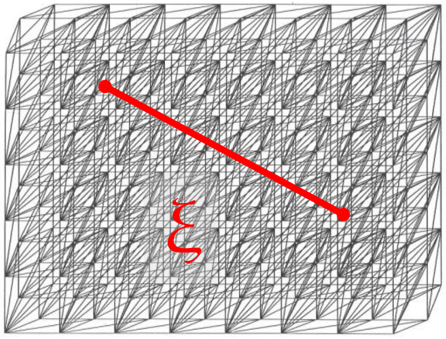
The lattice quantum continuum limit is gradually approached by considering sequences of lattices with increasingly larger correlation lengths ξ in lattice units. Such a limit requires the existence of an ultraviolet fixed point, where quantum field correlations extend over many lattice spacing.

Continuum limit <u>requires</u> the existence of an UV fixed point.

(Lattice) Continuum Limit $\Lambda \to \infty$

$$\frac{\partial G}{\partial \log \Lambda} = -\frac{1}{\nu}\,(G - G_c) + \ldots$$

integrated to give :



(Standard (Wilson) procedure in cutoff field theory)

$$\xi = 1/m \qquad m \underset{G(\Lambda)\to G_c}{\sim} \Lambda \left[\frac{G(\Lambda) - G_c}{a_0\, G_c}\right]^{\nu}$$

RG *invariant* correlation length ξ is kept fixed

UV cutoff $\Lambda \to \infty$
(average lattice spacing → 0)

Bare G *must* approach UV fixed point at G_c .

The *very same* relation gives the RG running of G(μ) close to the FP.

Determination of Scaling Exponents

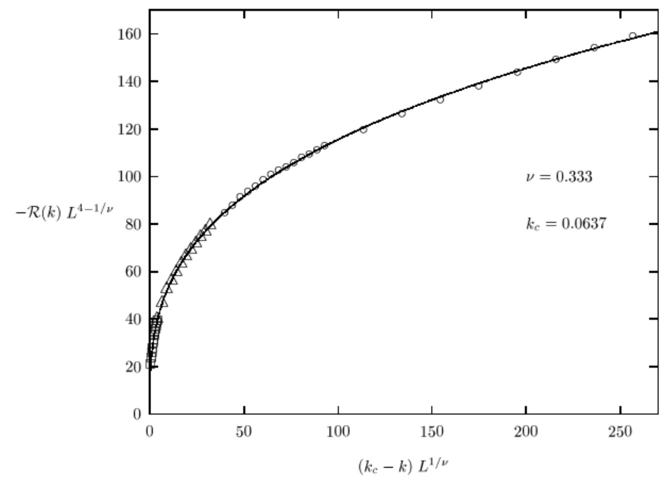
$$\mathscr{R}(k) \sim \frac{< \int dx \sqrt{g} R(x) >}{< \int dx \sqrt{g} >} \sim \frac{1}{V} \frac{\partial}{\partial k} \ln Z \underset{k \to k_c}{\sim} -A_{\mathcal{R}} \, (k_c - k)^{\delta} \qquad \nu = \frac{1+\delta}{d}$$

$$\chi_{\mathscr{R}}(k) \sim \frac{< (\int dx \sqrt{g} R)^2 > - < \int dx \sqrt{g} R >^2}{< \int dx \sqrt{g} >} \sim \frac{1}{V} \frac{\partial^2}{\partial k^2} \ln Z \underset{k \to k_c}{\sim} -A_{\mathcal{R}} \, (k_c - k)^{-(1-\delta)}$$

Scaling assumption:

$$F_{sing}(G) \sim \xi^{-d}$$

$$\xi(k) \equiv m(k)^{-1} \underset{k \to k_c}{\sim} A_{\xi} \, (k_c - k)^{-\nu}$$



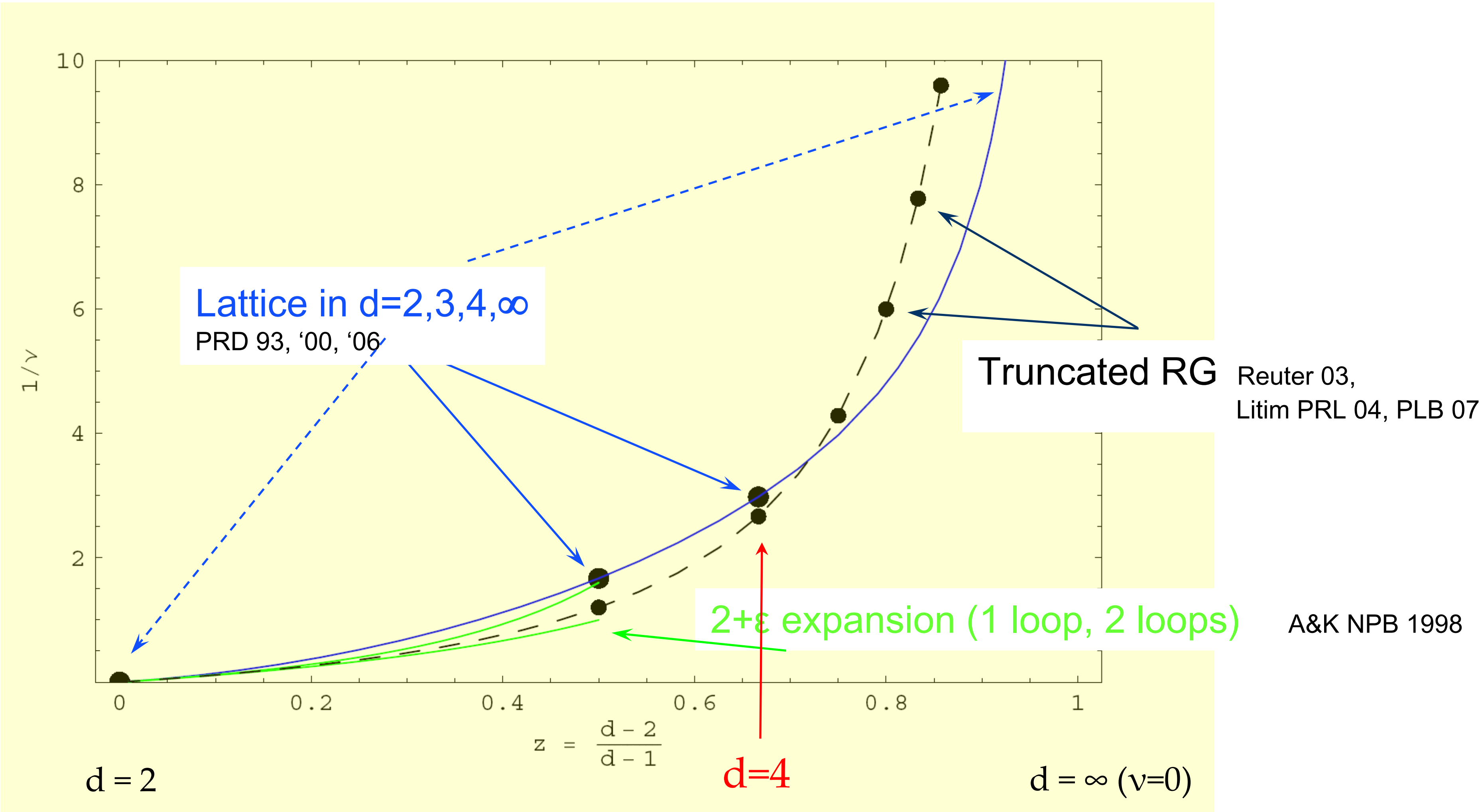
Find value for ν close to $1/3$:

$$k_c = 0.0636(11) \qquad \nu = 0.335(9)$$

$$\nu \approx 1/3$$

[Phys Rev D 1992, 1993, 2000]

Universal Gravitational Exponent ν



Back to the Continuum

Running Newton's Constant G

ξ is a new RG invariant scale of gravity $m \equiv \xi^{-1} = \Lambda F(G)$

Running of G determined largely by scale ξ and exponent ν :

$$G(k^2) = G_0 \left[1 + c_0 \left(\frac{1}{\xi^2 k^2} \right)^{1/2\nu} + \dots \right]$$

Almost identical to 2 + ε expansion result, but with 4 - d exponent ν = 1/3 and calculable coefficient c_0 … "Covariantize" : $k^2 \to -\Box$

$$G(\Box) = G_0 \left[1 + c_0 \left(\frac{1}{\xi^2 \Box} \right)^{1/2\nu} + \dots \right]$$

Three Theories Compared

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\partial^{\mu} F_{\mu\nu} + \mu^2 A_{\nu} = 4\pi e j_{\nu}$$

$$\partial^{\mu} \partial_{\mu} \phi + m^2 \phi = \frac{g}{3!} \phi^3$$

Suggests $\lambda_{phys} \simeq \frac{1}{\xi^2}$

$m = 1/\xi$

RG invariants

Running couplings

Vacuum Condensate Picture of QG

- Lattice Quantum Gravity: Curvature condensate *See also J.D.Bjorken, PRD '05*

$$\mathcal{R} \simeq (10^{-30} eV)^2 \sim \xi^{-2} \qquad \lambda_{phys} \simeq \frac{1}{\xi^2}$$

- Quantum Chromodynamics: Gluon and Fermion condensate

$$\alpha_S < F_{\mu\nu} \cdot F^{\mu\nu} > \simeq (250 MeV)^4 \sim \xi^{-4}$$

$$\xi_{QCD}^{-1} \sim \Lambda_{\overline{MS}}$$

$$(\alpha_S)^{4/\beta_0} < \bar{\psi} \psi > \simeq -(230 MeV)^3 \sim \xi^{-3}$$

- Electroweak Theory: Higgs condensate

Effective Theory

Effective Field Equations with G(□)

Explore *manifestly covariant,* non-local effective field equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G(\Box) T_{\mu\nu}$$

G.A. Vilkovisky …
G. Veneziano
HH & R Williams PRD 06,07
Deser et al. 2008

$$G(\Box) = G_0 \left(1 + \frac{\delta G(\Box)}{G_0}\right)$$

Consistency condition (Bianchi) on $T_{\mu\nu} + T^{vac}_{\mu\nu}$

$$\nabla^\mu \left(T_{\mu\nu} + T^{vac}_{\mu\nu}\right) = 0 \qquad T^{vac}_{\mu\nu} \equiv \frac{\delta G(\Box)}{G_0} T_{\mu\nu}$$

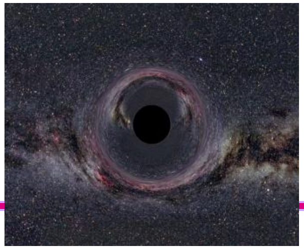
$$\frac{\delta G(\Box)}{G_0} \equiv c_0 \left(\frac{1}{\xi^2 \Box}\right)^{1/2\nu}$$

Form of d'Alembertian depends on nature of object it acts on,

$$\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu \qquad \Box\, T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots} = g^{\mu\nu} \nabla_\mu \left(\nabla_\nu T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots}\right)$$

[1820 terms for 2-nd rank tensor]

Static Isotropic Solution



Start from fully covariant effective field equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G \left(1 + A(\Box)\right) T_{\mu\nu}$$

Additional source term due to vacuum polarization contribution

General *static isotropic* metric

$$ds^2 = - B(r)\, dt^2 + A(r)\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

$$A(r)^{-1} = 1 - \frac{2\,M\,G}{r} + \frac{\sigma(r)}{r}$$

$$B(r) = 1 - \frac{2\,M\,G}{r} + \frac{\theta(r)}{r}$$

$$r \gg 2MG$$

Search solution for a point source, or *vacuum solution* for r ≠ 0

$$T_{\mu\nu} = \mathrm{diag}\left[\, B(r)\,\rho(r),\, A(r)\,p(r),\, r^2\,p(r),\, r^2 \sin^2\theta\, p(r)\,\right]$$

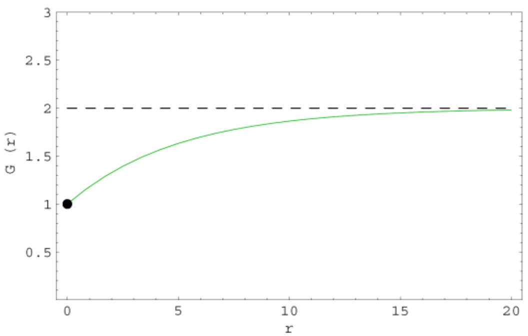
$$\rho_m(r) = \frac{1}{8\pi}\, c_\nu\, a_0\, M\, m^3\, (m\,r)^{-\frac{1}{2}(3-\frac{1}{\nu})}\, K_{\frac{1}{2}(3-\frac{1}{\nu})}(m\,r)$$

Static Isotropic Solution - cont'd

Solution of covariant equation (only for ν = 1/3)

$$B(r) = 1 - \frac{2\,M\,G}{r} + \frac{4\,a_0\,M\,G\,m^3}{3\,\pi}\,r^2\,\ln(m\,r) + \ldots$$

$$A^{-1}(r) = 1 - \frac{2\,M\,G}{r} + \frac{4\,a_0\,M\,G\,m^3}{3\,\pi}\,r^2\,\ln(m\,r) + \ldots$$



…which can be *consistently* interpreted as a ***G(r)*** **:**

$$G \;\rightarrow\; G(r) \;=\; G\left(1 + \frac{a_0}{3\,\pi}\,m^3\,r^3\,\ln\frac{1}{m^2\,r^2} + \ldots\right)$$

$$m = 1/\xi$$

$a_0 \simeq\ 33.$

Reminiscent of QED (Uehling) result :

H.H. & R. Williams, PLB '06; PRD '08

$$Q(r) \;=\; 1 + \frac{\alpha}{3\,\pi}\,\ln\frac{1}{m^2\,r^2} + \ldots \qquad m\,r \ll 1$$

Cosmological Solutions

Need to solve effective field equations :

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G(\Box) T_{\mu\nu}$$

$$G(\Box) = G_0 \left(1 + \frac{\delta G(\Box)}{G_0}\right)$$

$$\frac{\delta G(\Box)}{G_0} \equiv c_0 \left(\frac{1}{\xi^2 \Box}\right)^{1/2\nu}$$

… for standard FRW metric

$$d\tau^2 = dt^2 - a^2(t) \left\{ \frac{dr^2}{1 - k\, r^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \right\} \qquad k = 0, \pm 1$$

… and perfect fluid with $p(t) = 0$

E.g. Compute action of $\Box^n$, then analytically continue in $n = -1/2\nu$ (!)

Simplest treatment initially assumes power laws: $\rho(t) = \rho_0\, t^\beta$

Later include *perturbations*, e.g. : $d\tau^2 = dt^2 - a^2 \left(\delta_{ij} + h_{ij}\right) dx^i dx^j$

Running Cosmological Constant ?

Field equations with $\lambda(\Box)$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Write: $\lambda = \lambda_0 + \delta\lambda(k)$ with $\delta\lambda(k) \sim c_1 (k^2)^{-\sigma}$

or : $\delta\lambda(\Box) \sim (-\Box + \mu^2)^{-\sigma}$

IR regulate :

$$\left(\frac{1}{-\Box(g) + \mu^2} \right)^{\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^{\infty} d\alpha \, \alpha^{\sigma-1} \, e^{-\alpha(-\Box(g)+\mu^2)}$$

$\mu \to 0$

Then :

$$\delta\lambda(\Box) \cdot g_{\mu\nu} = c_1 \frac{1}{\Gamma(\sigma)} \int_0^{\infty} d\alpha \, \alpha^{\sigma-1} \, e^{-\alpha(-\Box(g)+\mu^2)} \cdot g_{\mu\nu} = c_1 (\mu^2)^{-\sigma} \cdot g_{\mu\nu}$$

using $\nabla_\lambda g_{\mu\nu} = 0$, so it cannot run.

Summary

(I) Matter density perturbation growth exponent γ

$$f(a=a_0) \equiv \left.\frac{\partial \ln \delta(a)}{\partial \ln a}\right|_{a=a_0} \equiv \Omega^\gamma$$

$$\gamma = 0.556 - 106.4\, c_t + O(c_t^2)$$

$q \to 0$

Correction always negative

(II) Gravitational slip function η :

$$\eta \equiv \frac{\psi - \phi}{\phi}$$

$$\eta(z) \;=\; -1.491\, c_t - 6.418\, c_t\, z + 30.074\, c_t\, z^2 + \cdots$$

$q \to 0$

Ratio :

$$\frac{\delta\,\gamma}{\delta\,\eta} \simeq \frac{-106.4\, c_t}{-1.491\, c_t} \simeq +71.4$$

Zeroth Order Field Equations

$$
\begin{aligned}
3\,\frac{\dot{a}^2(t)}{a^2(t)} &= 8\pi\, G_0 \left(1 + \frac{\delta G(t)}{G_0}\right) \bar{\rho}(t) + \lambda \\
\frac{\dot{a}^2(t)}{a^2(t)} + 2\,\frac{\ddot{a}(t)}{a(t)} &= -8\pi\, G_0 \left(w + w_{vac}\frac{\delta G(t)}{G_0}\right) \bar{\rho}(t) + \lambda
\end{aligned}
$$

$$
G(t) \equiv G_0 \left(1 + \frac{\delta G(t)}{G_0}\right) = G_0 \left[1 + c_t \left(\frac{t}{\xi}\right)^{1/\nu} + \ldots\right]
$$

To zeroth order, *vacuum fluid* has same equation of state as *radiation*.

$$
\rho_{vac}(t) = \frac{\delta G(t)}{G_0}\,\rho(t) \qquad p_{vac}(t) = \frac{1}{3}\,\frac{\delta G(t)}{G_0}\,\rho(t)
$$

$$
\bar{p}_{vac}(t) = w_{vac}\,\bar{\rho}_{vac}(t) \qquad w_{vac} = \tfrac{1}{3}
$$

First Order in the Fluctuations

$$d\tau^2 = dt^2 - a^2\,(\delta_{ij} + h_{ij})\,dx^i dx^j$$

Perturbed FRW metric

[PRD 2010,2011, with R. Toriumi]

Comoving frame, $q \to 0$,
focus on trace mode h

$$\begin{aligned} \delta\rho(\mathbf{x},t) &= \delta\rho_{\mathbf{q}}(t)\, e^{i\,\mathbf{q}\cdot\mathbf{x}} \qquad \delta p(\mathbf{x},t) = \delta p_{\mathbf{q}}(t)\, e^{i\,\mathbf{q}\cdot\mathbf{x}} \\ \delta\mathbf{v}(\mathbf{x},t) &= \delta\mathbf{v}_{\mathbf{q}}(t)\, e^{i\,\mathbf{q}\cdot\mathbf{x}} \qquad h_{ij}(\mathbf{x},t) = h_{\mathbf{q}\,ij}(t)\, e^{i\,\mathbf{q}\cdot\mathbf{x}} \end{aligned}$$

Standard GR result :

$$\begin{aligned} \frac{\dot a(t)}{a(t)}\,\dot h(t) &= 8\pi\, G_0\, \bar\rho(t)\, \delta(t) \\ \ddot h(t) + 3\,\frac{\dot a(t)}{a(t)}\,\dot h(t) &= -\,24\pi\, G_0\, w\, \bar\rho(t)\, \delta(t) \end{aligned}$$

$$-\tfrac{1}{2}\,(1+w)\;h(t) \;=\; \delta(t)$$

$$\delta(t) \equiv \delta\rho(t)/\bar\rho(t),\;\; h(t) \equiv h_{ii}(t)$$

Re-compute Pert. with G(Box)

$$\delta p_{\mathbf{q}}(t) = w \, \delta\rho_{\mathbf{q}}(t) \qquad \delta p_{\mathbf{q}\, vac}(t) = w_{vac} \, \delta\rho_{\mathbf{q}\, vac}(t)$$

$q \rightarrow 0$

Now Box contributes to the fluctuations as well, to O(h) :

$$\Box(g) \;=\; \Box^{(0)} + \Box^{(1)}(h) + O(h^2)$$

$$G(\Box) = G_0 \left[1 + \frac{c_0}{\xi^{1/\nu}} \left(\left(\frac{1}{\Box^{(0)}} \right)^{1/2\nu} - \frac{1}{2\,\nu} \frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h) \cdot \left(\frac{1}{\Box^{(0)}} \right)^{1/2\nu} + \ldots \right) \right]$$

$$\delta\rho_{vac}(t) \;=\; \frac{\delta G(t)}{G_0} \, \delta\rho(t) + \frac{1}{2\,\nu} \, c_h \, \frac{\delta G(t)}{G_0} \, h(t) \, \bar{\rho}(t) \qquad c_h \;\simeq\; +7.927$$

$$\delta p_{vac}(t) = w_{vac} \delta\rho_{vac}(t) \qquad w_{vac} = \tfrac{1}{3}$$

Need to assume background is *slowly varying* : $\dot{h}/h \gg \dot{a}/a$

Equation for density contrast

Single ODE for density perturbation, from cov. field equations with running G(Box) :

$$\ddot{\delta}(t) \quad + \left[\left(2\,\frac{\dot a(t)}{a(t)} - \frac{1}{3}\,\frac{\delta \dot G(t)}{G_0} \right) - \frac{1}{2\nu} \cdot 2\,c_h \cdot \left(\frac{\dot a(t)}{a(t)}\,\frac{\delta G(t)}{G_0} + 2\,\frac{\delta \dot G(t)}{G_0} \right) \right] \dot\delta(t)$$

$$+ \left[- 4\pi\, G_0 \left(1 + \frac{7}{3}\,\frac{\delta G(t)}{G_0} - \frac{1}{2\nu} \cdot 2\,c_h \cdot \frac{\delta G(t)}{G_0} \right) \bar\rho(t) \right.$$

$$\left. - \frac{1}{2\nu} \cdot 2\,c_h \cdot \left(\frac{\dot a^2(t)}{a^2(t)}\,\frac{\delta G(t)}{G_0} + 3\,\frac{\dot a(t)}{a(t)}\,\frac{\delta \dot G(t)}{G_0} + \frac{\ddot a(t)}{a(t)}\,\frac{\delta G(t)}{G_0} + \frac{\delta \ddot G(t)}{G_0} \right) \right] \delta(t) = 0 \ .$$

Classical GR result is much simpler (eg. Weinberg 1973, p. 588) :

$$\ddot\delta(t) + 2\,\frac{\dot a}{a}\,\dot\delta(t) - 4\pi\, G_0\, \bar\rho(t)\, \delta(t) = 0$$

Density Contrast in *a(t)* *cont'd*

Useful variable:

$$\theta \equiv \frac{\lambda}{8\,\pi\,G_0\,\bar{\rho}_0} = \frac{\Omega_\lambda}{\Omega} = \frac{1-\Omega}{\Omega}$$

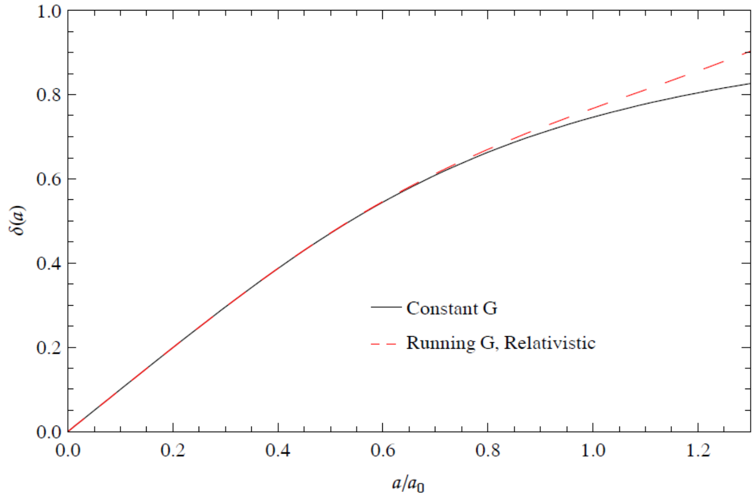
Standard GR result for density contrast : (eg Peebles 1993)

$$\delta_0(a) = \quad a \cdot {}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -a^3\,\theta\right)$$

Compute small correction due to running G(a) :

$$\delta(a) \propto \delta_0(a)\ [\,1 + c_a\,\mathcal{F}(a)\,]$$

$$G(a) = G_0\left(1 + \frac{\delta G(a)}{G_0}\right)$$



Structure Growth Indices

$$f(a) \equiv \frac{\partial \ln \delta(a)}{\partial \ln a}$$

$$f(a = a_0) \equiv \left.\frac{\partial \ln \delta(a)}{\partial \ln a}\right|_{a=a_0} \equiv \Omega^{\gamma}$$ with $\Omega \approx 0.25$

$$\gamma = 0.556 - 106.4\, c_t + O(c_t^2)$$

$q \rightarrow 0$

Classical GR result

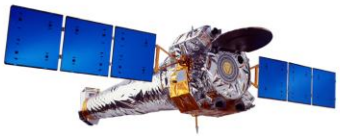
$c_t \lesssim 5 \times 10^{-4}$

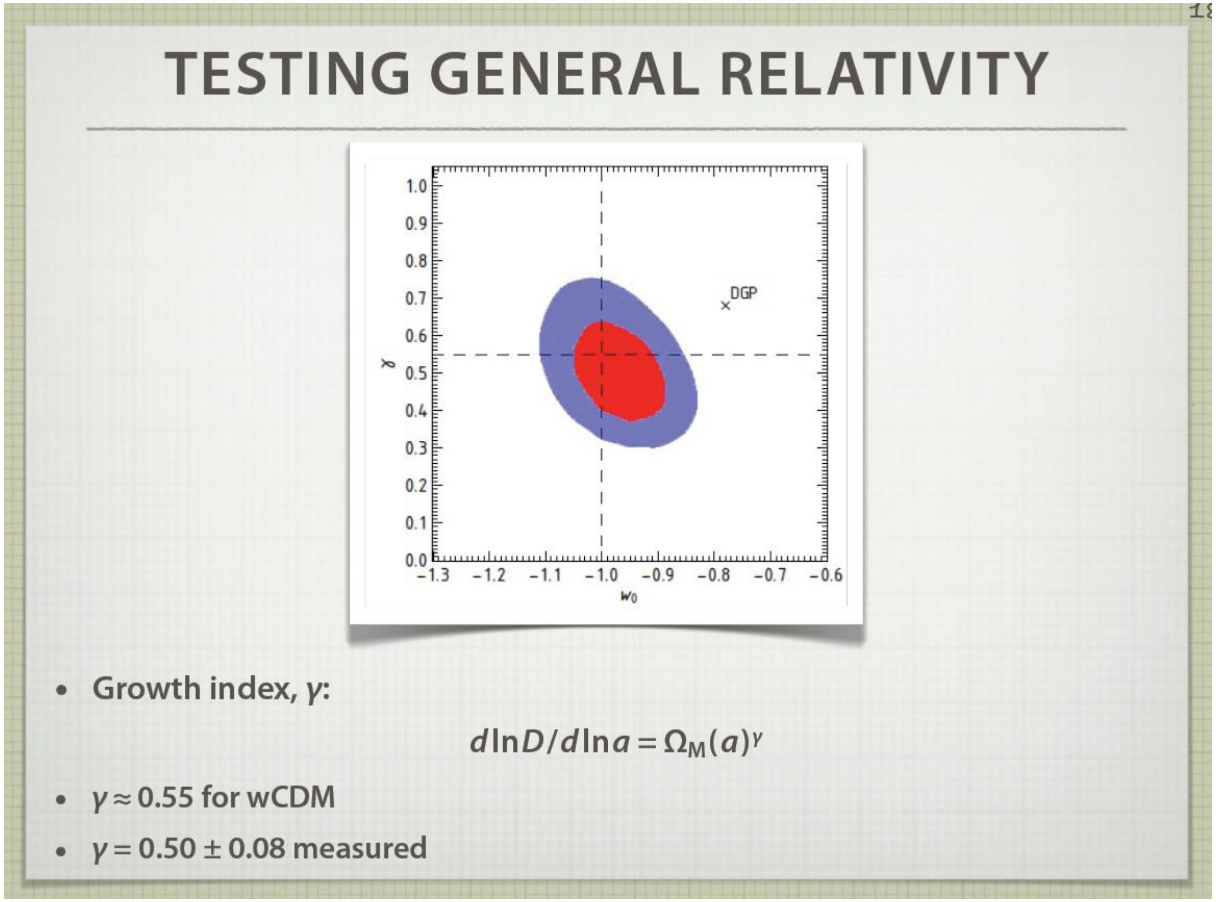
Correction always negative;

Significant uncertainty in magnitude of c_t coefficient

Newtonian (G(k)) result two orders of magnitude smaller …

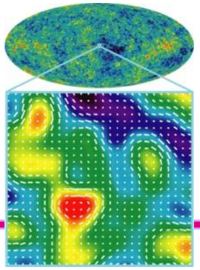
Measured growth parameter γ





Alexei Vikhlin et al., Rapetti et al.

Gravitational “Slip” with G(□)



Now use conformal Newtonian gauge :

$$ds^2 = a^2(\tau)\left\{-(1+2\psi)d\tau^2 + (1-2\phi)dx^i dx_i\right\}$$

In GR $\eta = \psi/\phi - 1 = 0$.

“Old” field equations :

$$\begin{aligned}
k^2\phi + 3\frac{\dot a}{a}\left(\dot\phi + \frac{\dot a}{a}\psi\right) &= 4\pi G a^2 \delta T^0{}_0 \\
k^2\left(\dot\phi + \frac{\dot a}{a}\psi\right) &= 4\pi G a^2 (\bar\rho + \bar P)\theta \\
\ddot\phi + \frac{\dot a}{a}(\dot\psi + 2\dot\phi) + \left(2\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}\right)\psi + \frac{k^2}{3}(\phi - \psi) &= \frac{4\pi}{3} G a^2 \delta T^i{}_i \\
k^2(\phi - \psi) &= 12\pi G a^2 (\bar\rho + \bar P)\sigma
\end{aligned}$$

Next, re-derive with : $G \rightarrow G(\Box)$

[HH & R. Toriumi 2011]

New Field Equations with G(Box)

Need to expand G(box) in the relevant perturbations:

$$G(\Box) = G_0 \left[1 + \frac{c_0}{\xi^{1/\nu}} \left(\left(\frac{1}{\Box^{(0)}} \right)^{1/2\nu} - \frac{1}{2\,\nu} \frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h) \cdot \left(\frac{1}{\Box^{(0)}} \right)^{1/2\nu} + \dots \right) \right]$$

"New" field equations :

$$k^2 \phi + 3 \frac{\dot{a}}{a} \left(\dot{\phi} + \frac{\dot{a}}{a} \psi \right) = -4\pi G_0 a^2 \left(1 + \frac{\delta G}{G_0} \right) \bar{\rho}\, \delta - 4\pi G_0 a^2 \frac{\delta G}{G_0} \frac{c_h}{2\nu} h \bar{\rho} + \mathcal{O}(k^2)$$

$$\left(\dot{\phi} + \frac{\dot{a}}{a} \psi \right) = 4\pi G_0 \frac{\delta G}{G_0} \left(-\frac{1}{2\nu} \right) \frac{2}{9} \frac{1}{i\omega} (h - 2\,s)\ \bar{\rho} + \mathcal{O}(k^2)$$

$$\begin{aligned} \ddot{\phi} + \frac{\dot{a}}{a} \left(\dot{\psi} + 2\,\dot{\phi} \right) + \left(2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) \psi + \frac{k^2}{3} (\phi - \psi) &= 4\pi G_0 a^2 \left(w + w_{vac} \frac{\delta G}{G_0} \right) \bar{\rho}\, \delta \\ &+ 4\pi G_0 a^2 \frac{\delta G}{G_0} w_{vac} \frac{c_h}{2\nu} h \bar{\rho} \\ &+ \mathcal{O}(k^2) \end{aligned}$$

$$k^2 (\phi - \psi) = +8\pi G_0 a^2 \frac{\delta G}{G_0} \frac{c_s}{2\nu} s \bar{\rho} + \mathcal{O}(k^2)\ .$$

Gravitational Slip Function

$$\eta \equiv \frac{\psi - \phi}{\phi} = -16\pi\, G_0 \frac{\delta G}{G_0} \frac{1}{2\nu} \frac{8}{3} \frac{\int s\, \mathrm{d}t}{\dot{s}} \bar{\rho}$$

$$\frac{\delta G(t)}{G_0} = c_t \left(\frac{t}{\xi}\right)^{\frac{1}{\nu}}$$

$$\eta(a) = \frac{16}{3\nu} \frac{\delta G(a)}{G_0} \log\left[\frac{a}{a_\xi}\right]$$

$$\left\{ \begin{array}{l} \ddot{s} + 3\frac{\dot{a}}{a}\dot{s} = 0 \\ s(a) \propto \frac{2}{3a^{3/2}}\sqrt{1 + a^3\,\theta} \\ H(a) = \sqrt{\frac{\lambda}{3} + \frac{4}{9\,a^3}} \end{array} \right.$$

$$a_\xi = \left(\frac{1}{\theta}\right)^{\frac{1}{3}} \mathrm{Sinh}^{\frac{2}{3}}\left[\frac{3}{2}\right]$$

$$\theta \equiv \frac{\lambda}{8\pi G_0 \bar{\rho}_0} = \frac{1 - \Omega}{\Omega}$$

Answer for GR "Slip" Function

Slip function η useful in parametrizing deviations from standard GR :

$$\eta(z) = -1.491\, c_t - 6.418\, c_t\, z + 30.074\, c_t\, z^2 + \cdots$$

$q \to 0$

(Classical GR result = 0)

CMB measurements give values around 0.09 ± 0.7

$$c_t \lesssim 0.3$$

$$\frac{\delta\,\gamma}{\delta\,\eta} \simeq \frac{-106.4\, c_t}{-1.491\, c_t} \simeq +71.4$$

Correction is always negative …

[IHES preprint Aug 2011]

… and much smaller than in growth exponent γ.

Testing of QFT G(Box) Scenario ?

In conclusion, maybe three possible astrophysical tests of QFT running of $G(Box)$:

(1) Growth exponent γ with $G(Box)$.

(2) cN gauge slip function ψ/ϕ with $\delta G(t)$.

(3) N-Body simulations with $G(r)$.

QFT generally predicts that G will run …

The End