Generalized spin representations

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Outline of talk

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- Damour et al./Henneaux et al. description of E_{10} GSR
- GSR's for arbitrary simply laced diagrams

Part 2: 'Maximal compact' subgroups of simply laced Kac–Moody groups as amalgams of Lie groups

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Part 3: Spin covers

- lifting of presentation
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Part 1:

Generalized spin representations of ‘maximal compact’ subalgebras of simply laced Kac–Moody algebras

(joint with Hainke)

Simply laced real Kac–Moody algebras

Let $\mathfrak{g}$ be a simply laced real Kac–Moody algebra, presented by Gabber–Kac using Serre's relations:

The Kac–Moody algebra $\mathfrak{g}$ is the quotient of the free Lie algebra over $\mathbb{R}$ generated by e_i, f_i, h_i, $i = 1, \dots, n$, subject to the relations

$$[h_i, h_j] = 0,\ [h_i, e_j] = a_{ij} e_j,\ [h_i, f_j] = -a_{ij} f_j,$$

$$[e_i, f_j] = 0,\ [e_i, f_i] = h_i,$$

$$(\mathrm{ad} e_i)^{-a_{ij}+1}(e_j) = 0,\ (\mathrm{ad} f_i)^{-a_{ij}+1}(f_j) = 0 \text{ for } i \neq j$$

with $a_{ii} = 2$ and $a_{ij} \in \{0, -1\}$ for $i \neq j$.

‘Maximal compact’ subalgebras of Kac–Moody algebras

Let $\omega \in \mathrm{Aut}(\mathfrak{g})$ be the Cartan–Chevalley involution:

$$\omega(e_i) = -f_i,\ \omega(f_i) = -e_i,\ \omega(h_i) = -h_i.$$

The ‘maximal compact’ subalgebra is defined as

$$\mathfrak{k} := \{X \in \mathfrak{g} \mid \omega(X) = X\}.$$

Theorem 1 (Berman 1989)

The ‘maximal compact’ subalgebra $\mathfrak{k}$ is isomorphic to the quotient of the free Lie algebra over $\mathbb{R}$ generated by $X_1, \ldots, X_n$ subject to the relations

$$\begin{aligned} [X_i,[X_i,X_j]] &= -X_j, && \text{if the simple roots } \alpha_i,\ \alpha_j \text{ form an edge,} \\ [X_i,X_j] &= 0, && \text{otherwise,} \end{aligned}$$

via the map $X_i \mapsto e_i - f_i$.

The X_i are called *Berman generators*.

Generalized spin representations of $\mathfrak{k}$

A representation $\rho\colon \mathfrak{k} \to \mathrm{End}(\mathbb{C}^s)$ is called a *generalized spin representation* if the images of the Berman generators satisfy

$$\rho(X_i)^2 = -\frac{1}{4}\mathrm{id}_s \text{ for } i = 1, \ldots, n.$$

Put $A := \rho(X_i)$, $B := \rho(X_j)$.

If α_i, α_j do not form an edge:

$$[A, B] \overset{1}{=} 0 \Longleftrightarrow AB = BA.$$

If α_i, α_j form an edge:

$$-B \overset{1}{=} [A, [A, B]] = [A, AB - BA] = A^2 B - 2ABA + BA^2 = -\frac{1}{2}B - 2ABA$$

Left-multiplying with $-4A = A^{-1}$ ($\Longleftrightarrow A^2 = -\frac{1}{4}\mathrm{id}_s$) yields:

$$4AB = 2AB - 2BA \Longleftrightarrow AB = -BA$$

How to construct generalized spin representations?

Conversely, suppose that there are matrices $A_i \in \mathbb{C}^{s\times s}$ satisfying

(i) $A_i^2 = -\frac{1}{4} \cdot \mathrm{id}_s$,

(ii) $A_i A_j = A_j A_i$, if α_i, α_j do not form an edge,

(iii) $A_i A_j = -A_j A_i$, if α_i, α_j form an edge.

Then, by reversing the argument on the previous slide, the assignment $X_i \mapsto A_i$ gives rise to a representation of $\mathfrak{k}$.

A motivating example (Damour et al., Henneaux et al.)

This example extends the spin representation of $\mathfrak{so}(10)$.

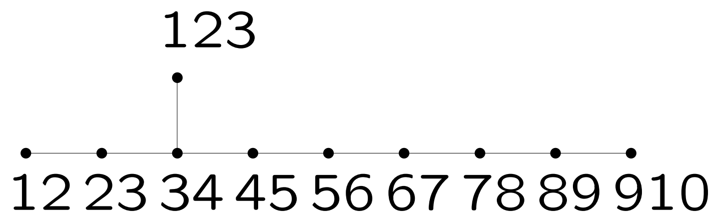
Let
- $V = \mathbb{R}^{10}$ with standard basis vectors v_i,
- $q : V \to \mathbb{R} : x \mapsto x_1^2 + \cdots + x_{10}^2$,
- $b : V \times V \to \mathbb{R} : (x, y) \mapsto 2(x_1 y_1 + \cdots + x_{10} y_{10})$ associated bilinear form,
- $T(V)$ the tensor algebra of V,
- $C := C(V, q) := T(V)/\langle vw + wv - b(v, w)\rangle$ the *Clifford algebra*.

In C we have $v_i^2 = 1$ and $v_i v_j = -v_j v_i$ for $i \neq j$.

Since C is associative, it becomes a Lie algebra by setting

$$[A, B] := AB - BA.$$

Let the diagram of E_{10} be labelled as



and define a Lie algebra homomorphism $\rho : \mathfrak{k}(E_{10}) \to C$ using these labels, i.e., via

$$
\begin{aligned}
&X_1 \mapsto \frac{1}{2} v_1 v_2, \quad X_2 \mapsto \tfrac{1}{2} v_1 v_2 v_3, \quad X_3 \mapsto \frac{1}{2} v_2 v_3, \\
&X_4 \mapsto \frac{1}{2} v_3 v_4, \quad X_5 \mapsto \tfrac{1}{2} v_4 v_5, \quad X_6 \mapsto \frac{1}{2} v_5 v_6, \\
&X_7 \mapsto \frac{1}{2} v_6 v_7, \quad X_8 \mapsto \tfrac{1}{2} v_7 v_8, \quad X_9 \mapsto \frac{1}{2} v_8 v_9, \\
&X_{10} \mapsto \tfrac{1}{2} v_9 v_{10},
\end{aligned}
$$

where X_i denotes the Berman generator corresponding to the root α_i, enumerated in Bourbaki style.

Observe that each $A_i := \rho(X_i)$ satisfies $A_i^2 = -\frac{1}{4}\mathrm{id}$.

Note that $(v_1v_2v_3)^2 = (v_2v_3)^2 = -1$ depends on $v_i^2 = 1$; for parity reasons, this would not be true in the Clifford algebra $C(V, -q)$, as then $(v_1v_2v_3)^2 = -(v_2v_3)^2 = 1$.

Using the criterion established above, one checks that ρ indeed is a Lie algebra homomorphism, i.e., that the defining relations of $\mathfrak{k}$ from Theorem 1 are respected.

One needs to establish
(i) $A_i^2 = -\frac{1}{4} \cdot \mathrm{id}_s$,
(ii) $A_iA_j = A_jA_i$, if α_i, α_j do not form an edge,
(iii) $A_iA_j = -A_jA_i$, if α_i, α_j form an edge.

We have already observed (i).

Assertions (ii) and (iii) are obvious for $i, j \in \{1, 3, 4, 5, 6, 7, 8, 9, 10\}$ (spin representation).

Moreover, one computes

$$(v_1 v_2 v_3)(v_3 v_4) = -(v_3 v_4)(v_1 v_2 v_3)$$

and

$$(v_1 v_2 v_3)(v_{k_1} v_{k_2}) = (v_{k_1} v_{k_2})(v_1 v_2 v_3),$$

if $\{k_1, k_2\}$ is a set of two elements that is either a subset of $\{1, 2, 3\}$ or disjoint from $\{1, 2, 3\}$.

The extension theorem for generalized spin representations (GSR)

Theorem 2 (Hainke, K.)

Let $1 \leq r < n$, $\mathfrak{k}_{\leq r} := \langle X_1, \ldots, X_r \rangle$,
$\rho : \mathfrak{k}_{\leq r} \to \mathrm{End}(\mathbb{C}^s)$ *a GSR.*

(i) If X_{r+1} *centralizes* $\mathfrak{k}_{\leq r}$*, then* ρ *extends to a GSR*
$\rho' : \mathfrak{k}_{\leq r+1} \to \mathrm{End}(\mathbb{C}^s)$ *via* $\rho'(X_{r+1}) := \frac{1}{2} i \cdot \mathrm{id}_s$.

(ii) If X_{r+1} *does not centralize* $\mathfrak{k}_{\leq r}$*, then* ρ *extends to a GSR*
$\rho' : \mathfrak{k}_{\leq r+1} \to \mathrm{End}(\mathbb{C}^s \oplus \mathbb{C}^s)$ *as follows. Define*

$$s_0(X_i) := \begin{cases} X_i, & \text{if } \alpha_i,\ \alpha_{r+1} \text{ do not form an edge}, \\ -X_i, & \text{if } \alpha_i,\ \alpha_{r+1} \text{ form an edge}, \end{cases}$$

and let

$$\rho'|_{\mathfrak{k}_{\leq r}} := \rho \oplus \rho \circ s_0 \qquad \text{and} \qquad \rho'(X_{r+1}) := \frac{1}{2} i \cdot \mathrm{id}_s \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof

If X_{r+1} centralizes $\mathfrak{k}_{\leq r}$: $\rho'(X_{r+1})^2 = -\frac{1}{4}\mathrm{id}_s$ and $\rho'(X_{r+1})$ commutes with everything. The criterion above applies.

If X_{r+1} does not centralize $\mathfrak{k}_{\leq r}$: $\rho'|_{\mathfrak{k}_{\leq r}}$ is a GSR of $\mathfrak{k}_{\leq r}$ which extends ρ. (Multiplication with -1 does not change (anti)commutation relation.)

Moreover, $\rho'(X_i)$ commutes with $\rho'(X_{r+1})$, if α_i, α_{r+1} not an edge; and $\rho'(X_i)$ anticommutes with $\rho'(X_{r+1})$, if α_i, α_{r+1} an edge:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = - \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Again the criterion above applies.

Quotients

Corollary 3

$\mathfrak{k}$ admits 'many' compact quotients.

Proof: Let ρ be a GSR as constructed in Theorem 2.

Considering $\mathbb{C} \cong \mathbb{R}^2$, multiplication by i can be realized via the skew-symmetric matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

If the representation of $\mathfrak{k}_{\leq r}$ is given by skew-symmetric matrices, then step (ii) can be made to involve skew-symmetric matrices only, as

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are $\mathbb{C}$-conjugate (minimum polynomial $x^2 + 1$).

Quotients, ii

Corollary 4

Assume the diagram does not admit any isolated nodes.

Then $\mathfrak{k}$ admits 'many' semisimple quotients.

Proof: Compact + perfect $\Longrightarrow$ semisimple.

Example: The GSR by Damour et al./Henneaux et al. leads to

$$\mathfrak{k}(E_{10}) \twoheadrightarrow \mathfrak{so}_{32}.$$

Part 2:

‘Maximal compact’ subgroups of simply laced Kac–Moody groups as amalgams of Lie groups

(Classical facts)

‘Maximal compact’ subgroups

Let

- G a simply connected simply laced split Kac–Moody group,
- T a maximal torus,
- ω a Cartan–Chevalley involution fixing T,
- $K := \mathrm{Fix}_G(\omega)$ ‘maximal compact’ subgroup.

Theorem 5 (Iwasawa decomposition; Kac–Peterson 1980ies)
Let B be a Borel subgroup of G containing the torus T. Then

$$G = KB.$$

Presentations arising from group actions on simply connected simplicial complexes

Theorem 6 (Simplicial geometric group theory)

Let

- Δ *simply connected finite-dim. coloured simpl. complex,*
- $G \to \mathrm{Aut}(\Delta)$ *colour-preserving simplicial rigid action, transitive on maximal simplices,*
- c *maximal simplex,*
- I *index set for vertices of* c*,*
- $(G_J)_{\emptyset \neq J \subseteq I}$ *family of pointwise stabilizers of non-empty sub-simplices of* c*,*
- $\phi_{J,J'} : G_J \hookrightarrow G_{J'}$ *canonical embedding for* $J \supseteq J'$*.*

Then

$$G \cong \left\langle \bigcup_{\emptyset \neq J \subseteq I} G_J \mid \begin{array}{c} \textit{all relations in the } G_J \textit{ plus} \\ \textit{all identifications via the } \phi_{J,J'} \end{array} \right\rangle .$$

Terminology: $(G_J)_{\emptyset \neq J \subseteq I}$ together with the connecting morphisms is a **diagram of groups**. The group G is called a **colimit**.

Theorem 7 (Non-simplicial version)

Let

- X *simply connected topological space,*
- $G \to \mathrm{Homeo}(X)$ *action,*
- U *an open path-connected weak fundamental domain (i.e., X = G.U),*
- $\Sigma = \{g \in G \mid U \cap g.U \neq \emptyset\}$,
- $R = \{xy = (xy) \mid x, y \in \Sigma, U \cap xU \cap xyU \neq \emptyset\}$.

Then

$$G \cong \langle \Sigma \mid R \rangle.$$

Theorem 7 implies Theorem 6:

Define U as an ϵ-neighbourhood of the maximal simplex c.

Example 8

Let Sym_4 act naturally on the barycentric subdivision of a 3-simplex considered as a 2-dimensional simplicial complex.

Let c be the maximal simplex consisting of the vertex 1, the barycentre of the edge $\{1, 2\}$, and the barycentre of the face $\{1, 2, 3\}$.

Then
$G_1 = \text{Sym}\{2, 3, 4\}$
$G_{\{1,2\}} = \text{Sym}\{1, 2\} \times \text{Sym}\{3, 4\}$
$G_{\{1,2,3\}} = \text{Sym}\{1, 2, 3\}$.
The other stabilizers arise as intersections.

Theorem 6 states that

$$\begin{aligned} \text{Sym}_4 &\cong \langle G_1 \cup G_{\{1,2\}} \cup G_{\{1,2,3\}} \mid \text{all relations in these groups} \rangle \\ &\cong \langle s_1, s_2, s_3 \mid s_i^2 = 1, (s_i s_{i+1})^3 = 1, s_1 s_3 = s_3 s_1 \rangle \end{aligned}$$

(Think $s_1 = (12)$, $s_2 = (23)$, $s_3 = (34)$.)

Note that the application of Theorem 6 can be iterated if the links of the simplicial complex are also simply connected:

Example 9

$$\begin{aligned}
\mathrm{Sym}_5 &\overset{6}{\cong} \langle G_1 \cup G_{\{1,2\}} \cup G_{\{1,2,3\}} \cup G_{\{1,2,3,4\}} \mid \text{their relations}\rangle \\
&\overset{6}{\cong} \langle G_{1,\{1,2\}} \cup G_{1,\{1,2,3\}} \cup \cdots \cup G_{\{1,2,3\},\{1,2,3,4\}} \mid \text{relations}\rangle \\
&\cong \langle \mathrm{Sym}\{3,4,5\} \cup \mathrm{Sym}\{2,3\} \times \mathrm{Sym}\{4,5\} \cup \cdots \\
&\quad \cdots \cup \mathrm{Sym}\{1,2,3\} \mid \text{their relations}\rangle.
\end{aligned}$$

A simplicial structure on G/B

Let
- G a simply connected simply laced split Kac–Moody group,
- T a maximal torus,
- B a Borel subgroup of G containing the torus T.

Theorem 10 (Tits 1987)

Let n be the rank of of the torus T as an algebraic group, i.e., the cardinality of the underlying Dynkin diagram.

Then G admits n maximal subgroups

$$P_i, \qquad 1 \leq i \leq n$$

that contain B, the maximal parabolic subgroups*.*

The *building* of G is the simplicial complex with
- the G-conjugates of the P_i as vertices, and
- the G-conjugates of B as maximal simplices.

An amalgamation result

Theorem 11 (Kac–Peterson 1980ies)

Let

- G *a simply connected simply laced split Kac–Moody group,*
- K *a 'maximal compact' subgroup,*
- Π *a set of simple roots,*
- $K_\alpha \cong \mathrm{SO}(2)$, $\alpha \in \Pi$, fundamental rank 1 subgroups of K,
- $K_{\alpha,\beta} \cong \begin{cases} \mathrm{SO}(3), & \alpha,\beta \in \Pi \text{ edge}, \\ \mathrm{SO}(2)\times\mathrm{SO}(2), & \alpha,\beta \in \Pi \text{ non-edge}, \end{cases}$

fundamental rank 2 subgroups of K.

Then

$$K \cong \left\langle \bigcup_{\alpha,\beta\in\Pi} K_{\alpha,\beta} \mid \begin{array}{c} \text{all relations in the } K_{\alpha,\beta} \text{ plus} \\ \text{all identifications } K_\alpha \hookrightarrow K_{\alpha,\beta} \end{array} \right\rangle .$$

Proof (using geometric group theory)

Assume rank n of G satisfies $n \geq 3$

• building Δ of G is a
– simply connected (Tits 1974)
– finite-dimensional
– coloured simplicial complex (P_i are not conjugate under G)
• K acts
– colour-preservingly
– simplicially
– rigidly
– transitively on maximal simplices ($G = KB$, Theorem 5)
• inductive application of Theorem 6 yields

$$K \cong \left\langle \bigcup_{\alpha,\beta\in\Pi} K_{\alpha,\beta}T_K \mid \begin{matrix}\text{all relations in the } K_{\alpha,\beta}T_K \text{ plus} \\ \text{all identifications } K_\alpha T_K \hookrightarrow K_{\alpha,\beta}T_K\end{matrix} \right\rangle,$$

where $T_K := K \cap T$
• since G is simply connected, T_K can be omitted

Geometric proof of Theorem 5 (Iwasawa decomposition)

The common-face relation $\sim_\alpha$ of type $\alpha \in \Pi$ in Δ is given by:

$$gB \sim_\alpha hB \Longleftrightarrow \exists g' \in gB, h' \in hB : (g')^{-1}h' \in G_\alpha.$$

The $\sim_\alpha$-equivalence class of gB is isomorphic to $\mathbb{P}_1(\mathbb{R})$ with a natural transitive action of the group $gG_\alpha g^{-1} \cong \mathsf{SL}_2(\mathbb{R})$.

By the Iwasawa decomposition

$$\mathsf{SL}_2(\mathbb{R}) \cong G_\alpha = K_\alpha \cdot \text{“upper triangular matrices”}$$

the group $gK_\alpha g^{-1}$ also acts transitively on this equivalence class.

Induction on the “distance” from B yields a transitive action of K on G/B, i.e.,

$$G = KB.$$

Part 3:
Spin covers

(joint with Ghatei, Horn, Weiß)

Spin cover of this amalgam

Define

- $L_\alpha \cong \text{Spin}(2)$,
- $L_{\alpha,\beta} \cong \begin{cases} \text{Spin}(3), & \alpha,\beta \in \Pi \text{ edge}, \\ (\text{Spin}(2) \times \text{Spin}(2))/\langle(-1,-1)\rangle, & \alpha,\beta \in \Pi \text{ non-edge}. \end{cases}$

Consider the commutative diagram with exact lines:

$$\begin{CD}
1 @>>> \mathbb{Z}/2 @>>> L_\alpha @>>> K_\alpha @>>> 1 \\
@. @V{\cong}VV @V{\exists! \phi_\alpha^{\alpha,\beta}}VV @V{\text{inj}}VV @. \\
1 @>>> \mathbb{Z}/2 @>>> L_{\alpha,\beta} @>>> K_{\alpha,\beta} @>>> 1
\end{CD}$$

We conclude that a given SO(3) amalgam arising from K can be uniquely lifted to a Spin(3) amalgam.

Spin cover of the 'maximal compact subgroup'
(Ghatei, Horn, K., Weiß)

Spin(n) is obtained by integrating the spin representation of $\mathfrak{so}_n$. This can be used to define a double 'spin' cover of K as follows.

Define $\mathrm{Spin}(K) \cong \left\langle \bigcup_{\alpha,\beta\in\Pi} L_{\alpha,\beta} \mid \begin{array}{l}\text{all relations in the } L_{\alpha,\beta} \text{ plus} \\ \text{all identifications } L_\alpha \hookrightarrow L_{\alpha,\beta}\end{array} \right\rangle$.

By Theorem 11 there exists an epimorphism Spin(K) $\to$ K with kernel of order 1 or 2. (Group generated by $-1 \in L_{\alpha,\beta}$.)

Consider a generalized spin representation $\mathfrak{k} \to \mathrm{End}(\mathbb{C}^s)$.

Integrate locally to spin representations $L_{\alpha,\beta} \to \mathrm{GL}(\mathbb{C}^s)$.

Observe that this leads to a lift of the SO(3) amalgam of K to a defining Spin(3) amalgam as above.

By definition this extends to a representation $\mathrm{Spin}(K) \to \mathrm{GL}(\mathbb{C}^s)$.

$-1 \in L_{\alpha,\beta}$ acts non-trivially; kernel of Spin(K) $\to$ K has order 2.

An extended Weyl group inside Spin(K)

Consider elements (indexed by $\alpha \in \Pi$)

$$R_\alpha \text{ corresponding to } \frac{1}{\sqrt{2}}(1 - v_1 v_2) \text{ inside } L_\alpha \cong \mathrm{Spin}(2)$$

in such a way that inside $L_{\alpha,\beta} \cong \mathrm{Spin}(3)$

$$R_\alpha \text{ corresponds to } \frac{1}{\sqrt{2}}(1 - v_1 v_2),$$

$$R_\beta \text{ corresponds to } \frac{1}{\sqrt{2}}(1 - v_2 v_3).$$

Theorem 12 (Ghatei)

The subgroup $W^{\mathrm{Spin}(K)}$ *of* $\mathrm{Spin}(K)$ *generated by* $(R_\alpha)_{\alpha\in\Pi}$ *satisfies the relations*

- $(R_\alpha)^4 = -1$,
- $$\begin{array}{ll} (R_\alpha R_\beta)^3 = -1, & \text{if } \alpha,\beta\in\Pi \text{ form an edge}, \\ R_\alpha R_\beta = R_\beta R_\alpha, & \text{if } \alpha,\beta\in\Pi \text{ do not form an edge.} \end{array}$$

Moreover, the subgroup D *of* $W^{\mathrm{Spin}(K)}$ *generated by* $(R_\alpha^2)_{\alpha\in\Pi}$

- *is normal in* $W^{\mathrm{Spin}(K)}$,
- *has order* $2^{|\Pi|+1}$,
- *satisfies* $W^{\mathrm{Spin}(K)}/D = W(\Pi)$.

Proof (of first part)

$R_\alpha^2 = \left(\frac{1}{\sqrt{2}}(1 - v_1 v_2)\right)^2 = \frac{1}{2}(1 - 2v_1 v_2 - 1) = -v_1 v_2$;
squaring yields -1.

For adjacent α, β we have

$R_\alpha R_\beta = \dfrac{1}{2}(1-v_1v_2)(1-v_2v_3) = \dfrac{1}{2}(1-v_1v_2-v_2v_3+v_1v_3)$, and so

$$\begin{aligned}
(R_\alpha R_\beta)^2 &= \frac{1}{4}(1 - v_1v_2 - v_2v_3 + v_1v_3)^2 \\
&= \frac{1}{4}(1 - 1 - 1 - 1 - 2v_1v_2 - 2v_2v_3 + 2v_1v_3) \\
&= -\frac{1}{2}(1 + v_1v_2 + v_2v_3 - v_1v_3) \\
&= -\overline{R_\alpha R_\beta} \quad (\text{using Spin}(3) \cong U_1(\mathbb{H})) \\
\Longrightarrow &\quad (R_\alpha R_\beta)^3 = -R_\alpha R_\beta \overline{R_\alpha R_\beta} = -1
\end{aligned}$$

For non-adjacent α, β, clearly $R_\alpha R_\beta = R_\beta R_\alpha$.

Thank you!