Gravity = Gauge Theory

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Disambiguation: What this talk is NOT about

- ~~Bulk gravity = Boundary gauge theory~~

  AdS/CFT correspondence

- ~~Gravity = (Gauge Theory)^2~~

  string-theory inspired KLT relations

  color-kinematic duality

- ~~Gravity as gauge theory of Poincare group~~

- Gravity = SU(2) gauge theory

Main message:

General Relativity (in 4 dimensions) can be reformulated as an SU(2) gauge theory (of a certain type)

$\Lambda \neq 0$ **KK PRL106:251103,2011**

results on zero scalar curvature in early 90's Capovilla, Dell, Jacobson

Why should one be interested in any reformulations?

There are many:

- Tetrad (first order) formulation
- Plebanski (Ashtekar) self-dual formulation
- Mac Dowell-Mansouri SO(2,3) gauge theoretic formulation
- ...

Have not helped. Gravity is still best understood in the original metric formulation. So is the problem of quantum gravity (non-renormalizability)

Some exceptional things happen in the new formulation!

General Relativity

$g_{\mu\nu}$ - spacetime metric

$$S_{\rm EH}[g] = -\frac{1}{16\pi G}\int (R - 2\Lambda)$$

Beautiful geometric theory that physicists study for already about a century!

$$R_{\mu\nu} \sim g_{\mu\nu}$$

Very "rigid" theory! Any modification messes it up

Several GR uniqueness theorems

GR is the unique theory of interacting massless spin 2 particles

But GR is also very much unlike all other theories!

the only theory that is not scale invariant (apart from the Higgs potential term)

non-polynomial Lagrangian (in terms of the metric); non-renormalizable

there is a scale M_p

Linearized description:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

$\kappa^2 = 32\pi G$

$$\mathcal{L}^{(2)} = -\frac{1}{2}(\partial_\mu h_{\rho\sigma})^2 + \frac{1}{2}(\partial_\mu h)^2 + (\partial^\mu h_{\mu\nu})^2 + h\partial^\mu\partial^\nu h_{\mu\nu}$$

(Euclidean) action
unbounded from below!

conformal mode problem

"wrong" sign

conformal mode

$h = h^\mu{}_\mu$

Count of propagating DOF:

$\dim(h_{\mu\nu}) = 10$ (per point) $\underbrace{-4-4}_{\text{diffeomorphisms}} \to 2$ propagating DOF

Spinor representation:

$TM = S_+ \otimes S_-$ $\qquad \mu \to AA'$

$h_{\mu\nu} \to h_{AA'BB'} \quad \in S^2_+ \otimes S^2_- \oplus (\text{trivial})$

$S_\pm$ unprimed/primed spinors

after (covariant) gauge-fixing all 10 metric components propagate

Einstein gravity perturbatively: Nasty mess...

Expansion around an arbitrary background $g_{\mu\nu}$

quadratic order (together with the gauge-fixing term)

$$L_{g.f.} = -\sqrt{-g}\left(h^{\mu\nu}{}_{;\nu} - \frac{1}{2}h_\nu{}^{\nu;\mu}\right)\left(h^\rho{}_{\mu;\rho} - \frac{1}{2}h^\rho{}_{\rho;\mu}\right)$$

$$L_2 = \sqrt{-g}\left\{-\frac{1}{2}h^{\alpha\beta}{}_{;\gamma}h_{\alpha\beta}{}^{;\gamma} + \frac{1}{4}h^\alpha{}_{\alpha;\gamma}h_\beta{}^{\beta;\gamma} + h_{\alpha\beta}h_{\gamma\delta}R^{\alpha\gamma\beta\delta} - h_{\alpha\beta}h^\beta{}_\gamma R^{\delta\alpha\gamma}{}_\delta \right.$$
$$\left. + h^\alpha{}_\alpha h_{\beta\gamma}R^{\beta\gamma} - \frac{1}{2}h_{\alpha\beta}h^{\alpha\beta}R + \frac{1}{4}h^\alpha{}_\alpha h^\beta{}_\beta R\right\}.$$

cubic order

from Goroff-Sagnotti "2-loop" paper

$$L_3 = \sqrt{-g}\left\{-\frac{1}{2}h^{\alpha\beta}h^{\gamma\delta}{}_{;\alpha}h_{\gamma\delta;\beta} + 2h^{\alpha\beta}h^{\gamma\delta}{}_{;\alpha}h_{\beta\gamma;\delta} - h^{\alpha\beta}h^\gamma{}_{\gamma;\alpha}h^\delta{}_{\beta;\delta} - \frac{1}{2}h^\alpha{}_\alpha h^{\beta\gamma;\delta}h_{\beta\delta;\gamma}\right.$$
$$+ \frac{1}{4}h^\alpha{}_\alpha h^{\beta\gamma;\delta}h_{\beta\gamma;\delta} - h^{\alpha\beta}h^\gamma{}_{\gamma;\delta}h^\delta{}_{\alpha;\beta} + \frac{1}{2}h^{\alpha\beta}h^\gamma{}_{\gamma;\alpha}h^\delta{}_{\delta;\beta} - h^{\alpha\beta}h_{\alpha\beta;\gamma}h^{\gamma\delta}{}_{;\delta}$$
$$+ \frac{1}{2}h^\alpha{}_\alpha h^\beta{}_{\beta;\gamma}h^{\gamma\delta}{}_{;\delta} + h^{\alpha\beta}h_{\alpha\beta;\gamma}h_\delta{}^{\delta;\gamma} + \frac{1}{4}h^\alpha{}_\alpha h^\beta{}_{\beta;\gamma}h_\delta{}^{\delta;\gamma} - h^{\alpha\beta}h^\gamma{}_{\alpha;\delta}h_{\beta\gamma}{}^{;\delta}$$
$$+ h^{\alpha\beta}h^\gamma{}_{\alpha;\delta}h^\delta{}_{\beta;\gamma} + R_{\alpha\beta}(2h^{\alpha\gamma}h_{\gamma\delta}h^{\beta\delta} - h^\gamma{}_\gamma h^{\alpha\delta}h^\beta{}_\delta - \frac{1}{2}h^{\alpha\beta}h^{\gamma\delta}h_{\gamma\delta}$$
$$\left. + \frac{1}{4}h^{\alpha\beta}h^\gamma{}_\gamma h^\delta{}_\delta) + R(-\frac{1}{3}h^{\alpha\beta}h_{\beta\gamma}h^\gamma{}_\alpha + \frac{1}{4}h^\alpha{}_\alpha h^{\beta\gamma}h_{\beta\gamma} - \frac{1}{24}h^\alpha{}_\alpha h^\beta{}_\beta h^\gamma{}_\gamma)\right\}$$

even in flat space, the corresponding vertex has about 100 terms!

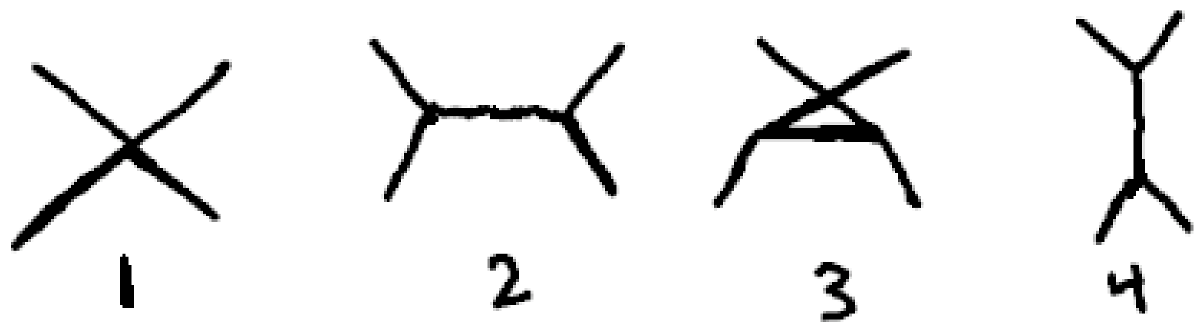
quartic order

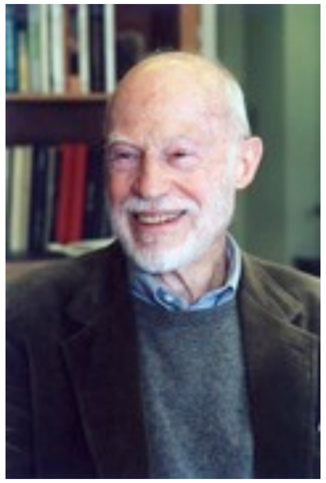
$$
\begin{aligned}
L_4 = \sqrt{-g}\Big\{ & (h^\alpha{}_\alpha h^\beta{}_\beta - 2h^{\alpha\beta}h_{\alpha\beta})\Big(\frac{1}{16}h^{\gamma\delta;\sigma}h_{\gamma\delta;\sigma} - \frac{1}{8}h^{\gamma\delta;\sigma}h_{\gamma\sigma;\delta} + \frac{1}{8}h^\gamma{}_{\gamma;\delta}h^{\delta\sigma}{}_{;\sigma} \\
& - \frac{1}{16}h^\gamma{}_{\gamma;\delta}h_\sigma{}^{\sigma;\delta}\Big) + h^\alpha{}_\alpha h^{\beta\gamma}\Big(-\frac{1}{2}h_{\beta\gamma;\delta}h^{\delta\sigma}{}_{;\sigma} + \frac{1}{2}h_{\beta\gamma;\delta}h_\sigma{}^{\sigma;\delta} - \frac{1}{2}h^\delta{}_{\delta;\beta}h^\sigma{}_{\sigma;\gamma} \\
& + \frac{1}{4}h^\delta{}_{\delta;\beta}h^\sigma{}_{\sigma;\gamma} + h^\delta{}_{\beta;\sigma}h^\sigma{}_{\delta;\gamma} - \frac{1}{4}h^{\delta\sigma}{}_{;\beta}h_{\delta\sigma;\gamma} - \frac{1}{2}h^\delta{}_{\beta;\sigma}h_{\delta\gamma}{}^{;\sigma} - \frac{1}{2}h^\delta{}_{\delta;\sigma}h^\sigma{}_{\beta;\gamma} \\
& + \frac{1}{2}h_{\beta\delta;\sigma}h_\gamma{}^{\sigma;\delta}\Big) + h^\alpha{}_\beta h^{\beta\gamma}\Big(h^\delta{}_{\delta;\alpha}h^\sigma{}_{\sigma;\gamma} - h_{\alpha\gamma;\delta}h_\sigma{}^{\sigma;\delta} + \frac{1}{2}h^{\delta\sigma}{}_{;\alpha}h_{\delta\sigma;\gamma} \\
& - h^\delta{}_{\alpha;\sigma}h^\sigma{}_{\gamma;\delta} - 2h^\delta{}_{\alpha;\sigma}h^\sigma{}_{\delta;\gamma} + h_{\alpha\gamma;\delta}h^{\delta\sigma}{}_{;\sigma} + h^\delta{}_{\delta;\alpha}h^\sigma{}_{\gamma;\sigma} - \frac{1}{2}h^\delta{}_{\delta;\alpha}h^\sigma{}_{\sigma;\gamma} \\
& + h^\delta{}_{\alpha;\sigma}h_{\gamma\delta}{}^{;\sigma}\Big) + h^{\alpha\gamma}h^{\beta\delta}\Big(h_{\alpha\gamma;\beta}h^\sigma{}_{\delta;\sigma} - h_{\alpha\gamma;\delta}h^\sigma{}_{\sigma;\beta} + \frac{1}{2}h_{\alpha\beta;\sigma}h_{\gamma\delta}{}^{;\sigma} \\
& - \frac{1}{2}h_{\alpha\gamma;\sigma}h_{\beta\delta}{}^{;\sigma} + h^\sigma{}_{\alpha;\beta}h_{\gamma\sigma;\delta} - h^\sigma{}_{\alpha;\beta}h_{\delta\sigma;\gamma} + h_{\alpha\beta;\delta}h^\sigma{}_{\sigma;\gamma} - 2h^\sigma{}_{\alpha;\beta}h_{\delta\gamma;\sigma} \\
& + h_{\alpha\gamma;\sigma}h^\sigma{}_{\beta;\delta}\Big) + R_{\alpha\beta}\Big(-2h^{\alpha\gamma}h_{\gamma\delta}h^{\delta\sigma}h_\sigma{}^\beta + h^\gamma{}_\gamma h^{\alpha\delta}h_{\delta\sigma}h^{\sigma\beta} + \frac{1}{2}h^{\alpha\gamma}h_\gamma{}^\beta h^{\delta\sigma}h_{\delta\sigma} \\
& - \frac{1}{4}h^{\alpha\gamma}h_\gamma{}^\beta h^\delta{}_\delta h^\sigma{}_\sigma + \frac{1}{3}h^{\alpha\beta}h^{\gamma\delta}h_{\delta\sigma}h^\sigma{}_\gamma - \frac{1}{4}h^{\alpha\beta}h^\gamma{}_\gamma h^{\delta\sigma}h_{\delta\sigma} + \frac{1}{24}h^{\alpha\beta}h^\gamma{}_\gamma h^\delta{}_\delta h^\sigma{}_\sigma\Big) \\
& + R\Big(-\frac{1}{192}h^\alpha{}_\alpha h^\beta{}_\beta h^\gamma{}_\gamma h^\delta{}_\delta + \frac{1}{16}h^\alpha{}_\alpha h^\beta{}_\beta h^{\gamma\delta}h_{\gamma\delta} + \frac{1}{4}h^{\alpha\beta}h_{\beta\gamma}h^{\gamma\delta}h_{\delta\alpha} \\
& - \frac{1}{16}h^{\alpha\beta}h_{\alpha\beta}h^{\gamma\delta}h_{\gamma\delta} - \frac{1}{6}h^\alpha{}_\alpha h^{\beta\gamma}h_{\gamma\delta}h^\delta{}_\beta\Big)\Big\}
\end{aligned}
$$

Imagine having to do calculations with these interaction vertices!

Still, they were done...

In 1963 I gave [Walter G. Wesley] a student of mine the problem of computing the cross section for a graviton-graviton scattering in tree approximation, for his Ph.D. thesis. The relevant diagrams are these:





Given the fact that the vertex function in diagram 1 contains over 175 terms and that the vertex functions in the remaining diagrams each contain 11 terms, leading to over 500 terms in all, you can see that this was not a trivial calculation, in the days before computers with algebraic manipulation capacities were available. And yet the final results were ridiculously simple. The cross section for scattering in the center-of-mass frame, of gravitons having opposite helicities, is

$$d\sigma/d\Omega = 4G^2E^2\cos^{12}\tfrac{1}{2}\theta/\sin^4\tfrac{1}{2}\theta$$

where G is the gravity constant and E is the energy.

From: Bryce DeWitt
arXiv:0805.2935
Quantum Gravity, Yesterday and Today

We now know that computing Feynman diagrams is not the simplest approach to the problem

Using the spinor helicity methods, the computation becomes doable

Using BCFW on-shell technology, the calculation becomes a homework exercise

Still, having a simpler off-shell description would be important

Linearized gauge-theoretic description (around de Sitter space)

a^i_μ infinitesimal SU(2) connection

Spinorial description: $\mu \to AA' \quad i \to (AB)$

$i = 1, 2, 3$

μ spacetime index

$$a^i_\mu \to a_{AA'}{}^{BC} \in S_+ \otimes S_- \otimes S_+^2 = S_+^3 \otimes S_- \oplus S_+ \otimes S_-$$

$$\boxed{\mathcal{L}^{(2)} \sim \left(\partial^{(A}_{A'} a^{BCD)A'}\right)^2}$$

$a^{(ABC)}_{A'}$

$a_{A'E}{}^{E}{}_{A}$

pure gauge (diffeomorphisms) part

explicitly non-negative (Euclidean signature) functional

only depends on the $S_+^3 \otimes S_-$ part of $a_{AA'}{}^{BC}$

$\dim(S_+^3 \otimes S_-) = 8$ (per point)

Count of propagating DOF:

$8 - 3 - 3 \to 2$ propagating DOF

SU(2) gauge rotations

after gauge-fixing only 8 connection components propagate (an irrep of Lorentz!)

Interactions:

expansion around de Sitter $M^2 = \Lambda/3$

significantly more complicated expression in the metric case

<u>complete off-shell</u> cubic vertex

zero on-shell

$$\mathcal{L}^{(3)} = \frac{2}{MM_p}(\partial a)^{ABCD}(\partial a)^{M'N'}{}_{AB}(\partial a)_{M'N'CD} - \frac{1}{4MM_p}(\partial a)^{ABCD}(\partial a)_{AB}(\partial a)_{CD} + \frac{4M}{M_p}(\partial a)^{M'N'AB}(aa)_{M'N'AB}.$$

the only part that is relevant for MHV

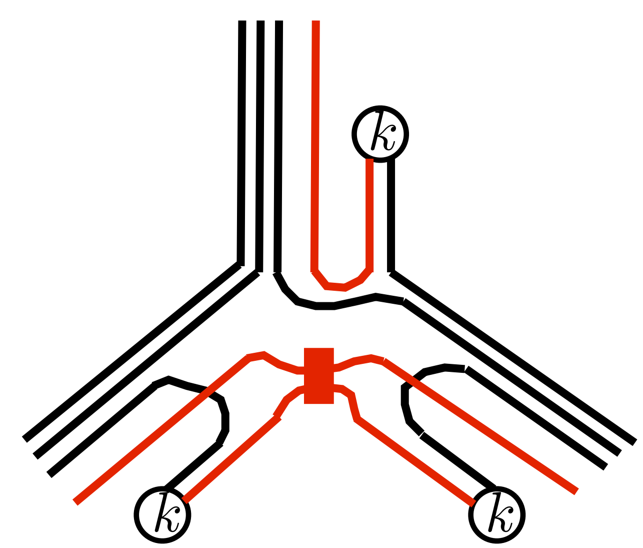
where the spinor contraction notations are

$$(\partial a)^{ABCD} = \partial^{(A}{}_{M'}a^{B)CDM'}$$

$$(\partial a)^{M'N'AB} = \partial^{C(M'}a_C{}^{ABN')}$$

$$(aa)^{M'N'CD} = a^{CD(AM'}a_{CD}{}^{B)N'}$$

graphical representation of the 3-derivative vertex



In terms of computational complexity, the above vertex is analogous to that of YM^2 type by Bern

Comparison with Yang-Mills:

can rewrite the YM Lagrangian as

gauge indices suppressed

$$\mathcal{L}_{\rm YM} = -\frac{1}{4g^2}(F^+_{\mu\nu})^2$$

F^+- self-dual part of the curvature

Spinorial description: $\mu \to AA'$

$$A_{AA'} \in S_+ \otimes S_-$$ spin 1

quadratic order (not gauge-fixed)

$$\mathcal{L}^2_{\rm YM} \sim (\partial^{(A}_{A'} A^{B)A'})^2$$

cubic order

$$\mathcal{L}^3_{\rm YM} \sim \left(\partial^{(A}_{A'} A^{B)A'}\right) A^{M'}{}_A A_{M'B}$$

our linearized graviton Lagrangian and the cubic vertex is just the generalization to the case

$$A_{ABCA'} \in S^3_+ \otimes S_-$$ spin 2

The gauge-theoretic formulation

- Simpler than the metric-based GR

convex action functional

vertices are much simpler in this formulation

conformal mode does not propagate even off-shell

- Suggests generalizations that are impossible to imagine in the metric formulation

GR is not the only theory of interacting massless spin 2 particles!

Suggests new (speculative at the moment) ideas as to what may be happening with gravity at very high energies

Self-duality

Riemannian signature for simplicity

$\Lambda^2 M$ two-forms on M

wedge product of forms gives a conformal metric

signature (3,3)

full metric if a volume form on M is chosen $\Lambda^2 M \otimes \Lambda^2 M \to \mathbb{R}$

If metric on M is chosen

Hodge dual

$* : \Lambda^2 M \to \Lambda^2 M$

$\Lambda^{\pm} M$ eigenspaces of $*$ of eigenvalues $\pm 1 (\pm i)$

$*^2 = \begin{cases} +1 & \text{Riemannian} \\ -1 & \text{Lorentzian} \end{cases}$

self (anti-self)-dual 2-forms

Thus, given a metric get $\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M$

restriction of the wedge-product metric on $\Lambda^+ M$ is positive-definite

Remark: Hodge dual is conformally-invariant

only need a conformal metric on M to get the above split

In the opposite direction the correspondence also holds

Split $\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M$ such that the first subspace is positive-definite $\Rightarrow$ conformal metric on M

Explicitly:

$$g_{\mu\nu} \sim \tilde{\epsilon}^{\alpha\beta\gamma\delta} \epsilon^{ijk} B^i_{\mu\alpha} B^j_{\nu\beta} B^k_{\gamma\delta}$$

$B^i_{\mu\nu}$ a basis in $\Lambda^+ M$

The above relation proves an isomorphism of group spaces

$\mathrm{SL}(4)/\mathrm{SO}(4)$		$\mathrm{SO}(3,3)/\mathrm{SO}(3)\times\mathrm{SO}(3)$
	$\Leftrightarrow$	
conformal metrics on M		Grassmanian of 3-planes in Λ^2

Conformal metrics can be encoded into the knowledge of which forms are self-dual

Curvature and self-duality

$$Riemann : \Lambda^2 M \to \Lambda^2 M$$

Or, in terms of the self-dual split

$$Riemann = \begin{array}{cc} & \begin{array}{cc} + & - \end{array} \\ \begin{array}{c} + \\ - \end{array} & \left(\begin{array}{c|c} Q & N \\ \hline N^T & P \end{array} \right) \end{array}$$

Q,P - symmetric 3x3 matrices

Einstein condition

$$Riemann = \Lambda \ metric \quad \Leftrightarrow \quad \left. \begin{array}{l} N = 0 \\ \mathrm{Tr} Q + \mathrm{Tr} P = 2\Lambda \end{array} \right\}$$

10 equations

Bianchi identity $\Leftrightarrow$ $\mathrm{Tr} Q = \mathrm{Tr} P$

Differential Bianchi identity $\Leftrightarrow$ $\Lambda = const$

don't need the 10th equation as independent

Connection on $\Lambda^+ M$

Levi-Civita connection on $T^* M$ $\Rightarrow$ connection on $\Lambda^+ M$

(self-dual part of Levi-Civita)

Its curvature F^+ is the self-dual part of the Riemann Q, N parts

Einstein condition:

Curvature of the metric connection on $\Lambda^+ M$
is self-dual as a two-form N=0

Differential Bianchi identity $\Rightarrow$

$$\mathrm{tr} Q = const$$

Remarks on $\Lambda^+ M$ bundle

wedge product metric + volume form on M

$\Rightarrow$ metric (positive-definite) in fibers of $\Lambda^+ M$

connection in $\Lambda^+ M$ preserves this metric

$\Rightarrow$ all $\Lambda^+ M$ are SO(3) bundles

Which bundle?

Hitchin

$(F^+)^2 + (F^-)^2$

$$\int_M (Riemann)^2 = 2\chi(M)$$

Euler characteristic of M

$(F^+)^2 - (F^-)^2$

$$\int_M (Riemann)(Riemann)^* = 3\tau(M)$$

Signature of M

$(F^+)^2 = \mathrm{Tr} Q^2 - \mathrm{Tr} N^2$

first Pontrjagin form

$$p_1 = \int_M (F^+)^2 = 2\chi(M) + 3\tau(M) \quad \Rightarrow$$

for Einstein manifolds

$2\chi(M) + 3\tau(M) \geq 0$

$\Downarrow$

bundle is fixed by the topology of M

Plebanski formulation of GR

Idea: encode metric in the split $\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M$

Since all $\Lambda^+ M$ bundles are isomorphic, fix the principal SO(3) bundle over M with $p_1 = 2\chi(M) + 3\tau(M)$

Let $E \to M$ be the associated bundle

fibers $\mathfrak{su}(2)$
the usual SO(3) invariant metric in the fibers

Let $\boxed{B : E \to \Lambda^2 M}$ such that the pullback of the wedge product metric to E coincides with the SO(3) invariant metric in E

defined modulo SO(3) rotations

$B \wedge B \sim \delta$

Declare the image of E in $\Lambda^2 M$ to be $\Lambda^+ M$ $\Rightarrow$ conformal metric

Declare the image of the (inverse) metric in E in $\Lambda^2 M \otimes \Lambda^2 M$ (composed with the wedge product) to be the volume form

$\Rightarrow$ full metric

Connections

The metric connection in $\Lambda^+ M$ is also encoded in B

Lemma: $\exists$ unique (modulo gauge) SO(3) connection A in E such that $d_A B = 0$

Lemma: It coincides with the pull-back (under B) of the metric connection in $\Lambda^+ M$

Thus B can be taken as the basic object

Einstein equations

Theorem: The metric encoded in B is Einstein iff

$$\exists Q \in \mathrm{End}(E) : F(A(B)) = QB$$

curvature of the self-dual part of the Levi-Civita connection is self-dual as a two-form

Action principle:

$$S[B, A, Q] = \int_M \mathrm{Tr}\left(B \wedge F(A) - \frac{1}{2} QB \wedge B\right)$$

$$\mathrm{Tr}Q = \Lambda$$

trace of Q is fixed and is not varied

varying wrt Q gives the condition that B is an isometry, other equations also follow

Euler-Lagrange equations

Q variation: $B \wedge B \sim \delta$

A variation: $d_A B = 0$

B variation: $F(A) = QB$

To get a better feel for this, consider linearization

$$B \wedge b \sim \delta \quad \Rightarrow \quad b = \tilde{b} + \phi B$$

where $\tilde{b} \in E \otimes \Lambda^- M$, $\phi \in C(M)$

$$b \in E \otimes \Lambda^2 M$$

the tracefree part of the metric variation

the trace part

But B provides an isomorphism $E \sim \Lambda^+ M \quad \Rightarrow \quad B(\tilde{b}) \in \Lambda^+ M \otimes \Lambda^- M$

This is the known identification

$$\boxed{\Lambda^+ M \otimes \Lambda^- M \sim S_0^2 T^* M}$$

symmetric tracefree 2-tensors

The second linearized equation $db + [a, B] = 0 \quad \Rightarrow \quad a(b)$

$$a \in E \otimes \Lambda^1 M$$ linearized connection

To disentangle the content, consider tracefree perturbations only $b = \tilde{b}$

Introduce a new exterior derivative

$$d_- : E \otimes \Lambda^1 M \to E \otimes \Lambda^- M$$

Lemma: $a(\tilde{b}) = d_-^* \tilde{b}$

The last equation linearized $da(b) = qB + Qb$

Let us take its $\Lambda^- M$ part (for tracefree perturbations) $\boxed{d_- d_-^* \tilde{b} = Q\tilde{b}}$

This is the tracefree part of the (linearized) Einstein condition!

(on an arbitrary background!)

Towards the "pure connection" formulation

Idea: Take A in $E \to M$ as the main variable

Consider Einstein metrics such that $F(A^+)$ spans a definite 3-dimensional subspace in $\Lambda^2 M$ $\Rightarrow$ conformal metric

Fine, Panov

Definition: An SO(3) connection is called *definite* if $F(A) \wedge F(A)$ is a definite matrix

For a definite A, declare the subspace spanned by F(A) to be $\Lambda^+ M$
$\Rightarrow$ conformal metric

To get the full metric and Einstein equations consider
$F(A) \wedge F(A) \in \Lambda^4 M \otimes \mathrm{End}(E)$

In Plebanski this is
$Q^2 \mathrm{Tr}(B \wedge B)$

volume form

Define the volume form via

$$\Lambda^2(\mathrm{vol}) := \left(\mathrm{Tr}\sqrt{F \wedge F}\right)^2$$

$\Lambda = s/4$

Theorem: Let A be a definite connection in the principal SO(3) bundle over M with $p_1 = 2\chi(M) + 3\tau(M)$

Let F be its curvature 2-form. Define

$$\Lambda B := \mathrm{Tr}(\sqrt{F \wedge F})(F \wedge F)^{-1/2} F$$

(so that $B : E \to \Lambda^2 M$ is an isometry).

Then the metric defined by F (or B) is Einstein if

$$d_A B = 0$$

Einstein metrics with $s \neq 0$ and definite $\frac{s}{12} + W_+$

$\longleftrightarrow$

SO(3) connections (on a specific bundle over M) satisfying

$$d_A B = 0$$

Examples not covered:

$S^2 \times S^2$

Kahler-Einstein

alternatively, metrics for which $F(A^+)$ spans $\Lambda^+ M$

Variational Principle

KK arXiv:1103.4498

partial results on zero scalar curvature in early 90's

$$S_{\rm GR}[A] = \frac{1}{16\pi G\Lambda}\int_M \left(\mathrm{Tr}(\sqrt{F\wedge F})\right)^2$$

well-defined on the space of definite connections

Euler-Lagrange equations $d_A B = 0$

should take a given p_1 bundle to get GR

where $\Lambda B := \mathrm{Tr}(\sqrt{F\wedge F})(F\wedge F)^{-1/2} F$

thus precisely Einstein metrics

Remark: Recalling the metric as defined by F

action is just the volume

$$\boxed{S_{\rm GR}[A] = \Lambda M_p^2\, \mathrm{Vol}(M)}$$

The new functional from Plebanski

can solve the B equation $B = Q^{-1}F$

$$S[A,Q] = \int Q^{-1}F \wedge F + \mu(\mathrm{Tr}(Q) - \Lambda)$$

now minimize wrt Q, keeping the trace fixed

$$\Rightarrow \quad \mu Q^2 = F \wedge F$$

$$\Rightarrow \quad \Lambda\sqrt{\mu} = \mathrm{Tr}\sqrt{F \wedge F}$$

$$\Rightarrow \quad S[A] = \frac{1}{\Lambda}\int \left(\mathrm{Tr}\sqrt{F \wedge F}\right)^2$$

precisely the same procedure as one that leads to the so-called <u>Eddington's</u> formulation of GR

(also <u>Schrodinger</u>)

$$S[\Gamma] = \frac{1}{\Lambda}\int d^4x \sqrt{\det(R_{\mu\nu})}$$

as in our formulation,
it is just the volume

Half-flat metrics

The gauge-theoretic reformulation of GR gives a simple characterization of half-flat metrics $Q|_{\rm tf} = 0,\ P|_{\rm tf} \neq 0$

Theorem: A connection whose curvature viewed as a map $F : E \to \Lambda^2 M$ is an isometry $F \wedge F \sim \delta$ gives a half-flat metric (instanton) of non-zero scalar curvature

Capovilla, Dell, Jacobson '90

Fine '10

Proof: Define $B = F$

Satisfies $d_A B = 0$ as well as $B \wedge B \sim \delta$

Thus gives an Einstein metric with $Q \sim \delta$

$\Rightarrow$ Weyl curvature is purely anti-self-dual

To get a better feel for the new functional, let us consider linearization (around an ASD connection)

Lemma: $$\tilde{b} = d_- a$$ $$\phi = \mathrm{Tr}\, B^{-1}(d_+ a)$$

where $B^{-1}(d_+ a) \in E \otimes E$

variation of the conformal factor is not independent!

the space of connections mod gauge transforms is only 9 functions per point

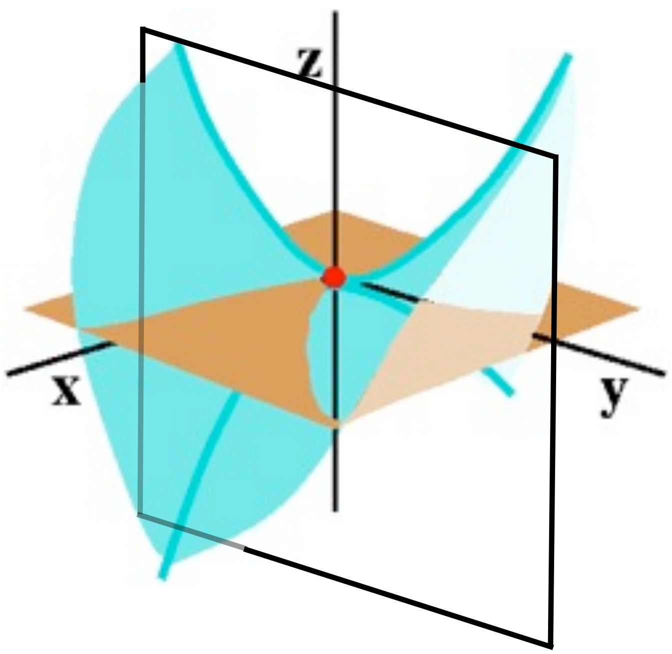
Lemma: $$\delta^2 S_{\rm GR} = \int_M \left(B^{-1}(d_+ a)\Big|_{sym, tracefree} \right)^2$$

At a critical point corresponding to ASD Einstein metric the functional is non-negative

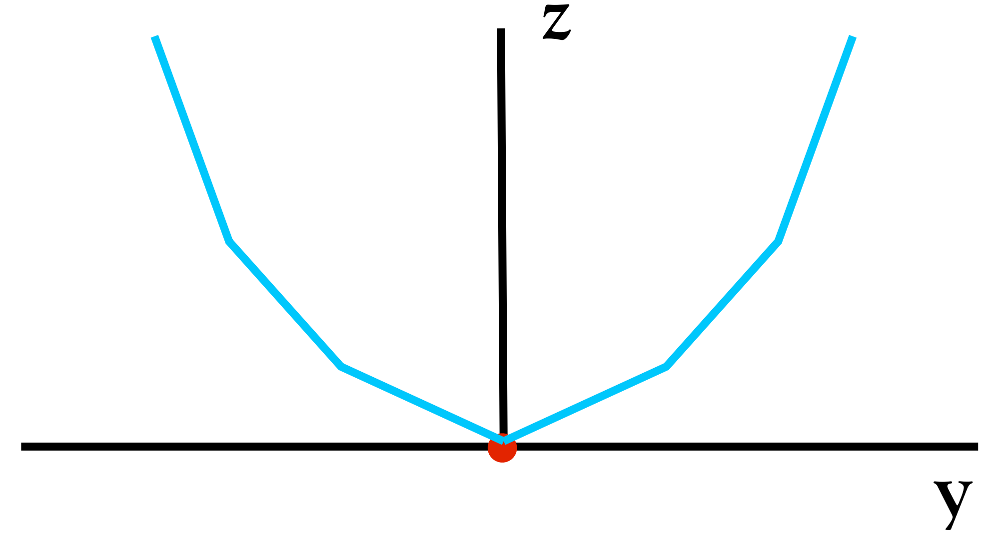
convex after gauge-fixing $\Rightarrow$ local rigidity of ASD Einstein metrics (of positive scalar curvature)

Description of GR without the conformal mode problem!

On-shell equivalent description of gravitons



space of metrics



space of conformal metrics = SU(2) connections/gauge

(Euclidean) EH functional is not convex (conformal mode problem)

The new action (its Euclidean version) is a convex functional

Restriction of the EH action to a smaller space gives a convex functional

Same critical points!

The conformal mode has been "integrated out" and is now absent even off-shell

How to do calculations: Scattering Amplitudes

Gauge-fixing:

gauge-fixing condition invariant under shifts $\partial^\mu \left(P^{(3/2,1/2)} a^i_\mu \right) = 0$

in spinor terms $(\partial a)^{BC} \equiv \partial^A_{A'} a^{BCA'}_A = 0$ where $a^{ABC}_{A'} \in S^3_+ \otimes S_-$

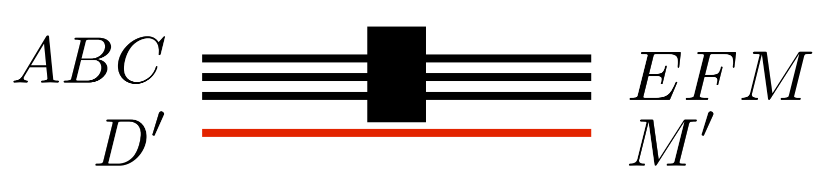
gauge-fixed Lagrangian - functional on $\mathcal{C}^\infty(S^3_+ \otimes S_-)$

Analog of Feynman gauge in YM

$$\mathcal{L}^{(2)} + \mathcal{L}^{(2)}_{gf} = \left(\partial^{(A}_{A'} a^{BCD)A'} \right)^2 + \frac{3}{4} \left((\partial a)^{AB} \right)^2 = -\frac{1}{2} a_{ABC}{}^{A'} \partial^2 a^{ABC}{}_{A'}$$

Thus, the propagator

$$\Delta(k)_{EFM}{}^{M'ABC}{}_{D'} = \frac{\epsilon_E{}^{(A} \epsilon_F{}^B \epsilon_M{}^{C)} \epsilon_{D'}{}^{M'}}{k^2}$$



$$\frac{1}{k^2}$$

only the (3/2,1/2) component propagates

Spinor helicity states

$$\varepsilon^+(k)^{ABC}{}_{D'} = \frac{1}{M}\frac{k^A k^B k^C p_{D'}}{[kp]}, \qquad \varepsilon^-(k)^{ABC}{}_{D'} = M\frac{q^A q^B q^C k_{D'}}{(kq)^3}$$

here, as usual p^A, q^A are arbitrary spinors not aligned with k^A

and $[kp] := k_{A'}p^{A'}$, $(kp) := k^A p_A$ are spinor products

To take the $M \to 0$ limit

need to make the (positive helicity) <u>external</u> momenta slightly <u>massive</u>

$$k^{AA'} = k^A k^{A'} + \frac{M^2 q^A q^{A'}}{(kq)[kq]} \quad \text{so that} \quad k^{AA'}k_{AA'} = -2M^2$$

Usual spinor helicity calculations! Same amplitudes (e.g. graviton-graviton, MHV)

the only headache is taking the $M \to 0$ limit

arXiv:1210.6215

Relation to the metric description

both valid on-shell only

$$h_{ABA'B'} \sim \frac{1}{M} (\partial a)_{ABA'B'} \qquad a^{ABCA'} \sim \frac{1}{M} \partial^{(A}_{B'} h^{BC)A'B'}$$

both are true on $k^2 = 2M^2$

then our helicity states are just images of the usual metric states

3-vertex in the metric language

Bern's 3-vertex for GR

square of the YM vertex

$$\mathcal{L}^{(3)} \sim \frac{1}{M_p} \left(\partial^{(A}_{A'} \partial^{B}_{B'} h^{CD)A'B'} \right) h^{M'N'}{}_{AB} h_{M'N'CD}$$

our calculations are exactly the same as ones done with the usual metric helicity states and the above vertex

Summary so far:

- Using $S_+^3 \otimes S_-$ instead of $S_+^2 \otimes S_-^2$ to describe gravitons

parity invariance non-manifest!

- "Restriction" of the EH action to a smaller space of conformal metrics gives a convex functional

- Much simpler linearized action, much simpler interaction vertices!

e.g. off-shell 4-vertex contains only 7 terms, as compared to a page in the metric-based case

- Formulation in which the off-shell 3-vertex is (basically) (YM vertex)^2

became possible because the conformal mode does not propagate even off-shell

in this respect similar to Bern's reformulation

Generalization: Diffeomorphism invariant gauge theories

$\mathfrak{g}$ - Lie algebra of G

Let f be a function on $\mathfrak{g} \otimes_S \mathfrak{g}$ satisfying

$$f : X \to \mathbb{R}(\mathbb{C})$$

defining function

$$X \in \mathfrak{g} \otimes_S \mathfrak{g}$$

1) $$f(\alpha X) = \alpha f(X)$$ homogeneous degree 1

2) $$f(gXg^T) = f(X), \quad \forall g \in G$$ gauge-invariant

Then $f(F \wedge F)$ is a well-defined 4-form (gauge-invariant)

Can define a gauge and diffeomorphism invariant action

$$F = dA + (1/2)[A, A]$$

$$\boxed{S[A] = \mathrm{i} \int_M f(F \wedge F)}$$

no dimensionful coupling constants!

Lorentzian signature functional

Field equations: $d_A B = 0$

where $B = \frac{\partial f}{\partial X} F$ and $X = F \wedge F$

Second-order (non-linear) PDE's

compare Yang-Mills equations: $d_A B = 0$

where $B = {}^* F$

$*$ - encodes the metric

Dynamically non-trivial theory with 2n-4 propagating DOF

apart from the single point $f_{\rm top} = {\rm Tr}(F \wedge F)$

Gauge symmetries:

$\delta_\phi A = d_A \phi$ gauge rotations

$\delta_\xi A = \iota_\xi F$ diffeomorphisms

The simplest non-trivial theory:

G=SU(2) - gravity

(interacting massless spin 2 particles)

Define the metric by:

$$\mathrm{Span}\{F(A)\} = \Lambda^+ M \qquad (\mathrm{vol}) \sim f(F \wedge F)$$

The functional is just the volume: $\boxed{S[A] \sim \mathrm{Vol}(M)}$

The linearization (around de Sitter) is the same for any f()

For any choice of f() - a theory of interacting massless spin 2 particles

specific f() - GR

Deformations of GR

All other choices of f() lead to different (from GR) interacting theories of massless spin 2 particles

can be shown to correspond to the EH Lagrangian with an infinite set of counterterms added

seemingly impossible due to the GR uniqueness, but specific (sometimes innocuous) assumptions that go into each version of the uniqueness theorems are explicitly violated here

Not a dynamical theory of $g_{\mu\nu}$

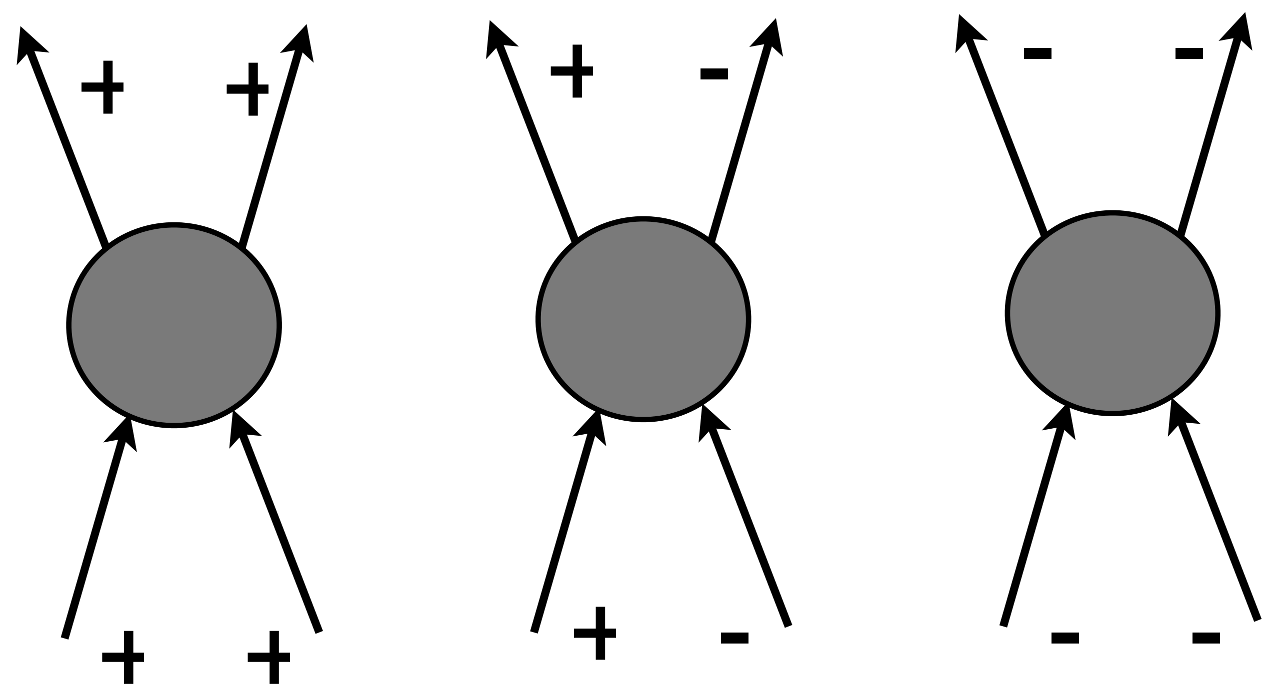
(in its second-order formulation)

A generic theory is not parity invariant!

Modified gravity theories with 2 propagating DOF - a very interesting object of study

Parity violation is quantified in scattering amplitudes

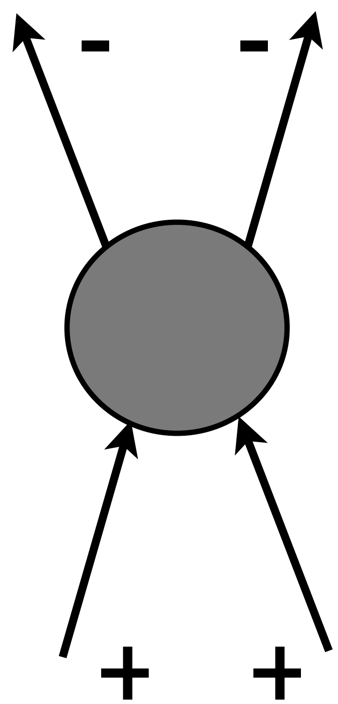
In GR only parity-preserving processes:



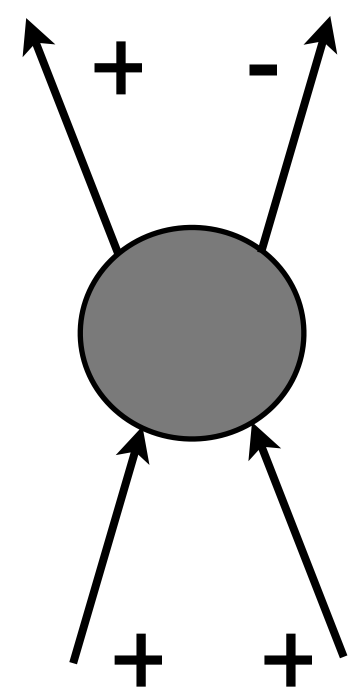
amplitude $$\mathcal{A} \sim \frac{1}{M_p^2}\frac{s^3}{tu} \sim \left(\frac{E}{M_p}\right)^2$$

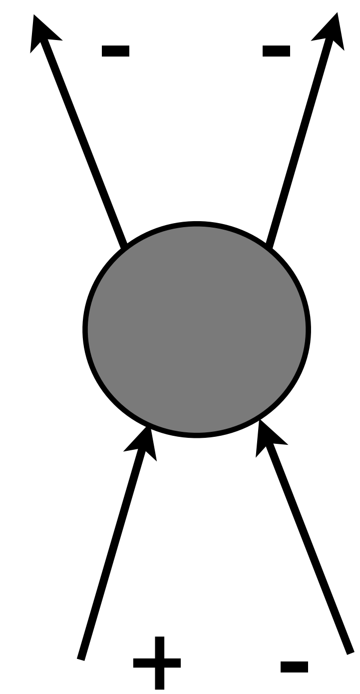
becomes larger than unity at Planck energies, cannot trust perturbation theory

In a general theory from our family parity-violating processes become allowed:



$$\mathcal{A} \sim \frac{s^4 + t^4 + u^4}{M_p^8} \sim \left(\frac{E}{M_p}\right)^8$$





$$\mathcal{A} \sim \frac{stu}{M_p^6} \sim \left(\frac{E}{M_p}\right)^6$$

A general theory likes negative helicity gravitons!

Can speculate that at high energies these processes will dominate and all gravitons will get converted into negative helicity ones (strongly coupled by the parity-preserving processes)

Quantum Theory Hopes

Remark: no dimensionful coupling constants in any of these gravitational theories

(negative) dimension coupling constant comes when expanded around a background

Non-renormalizable in the usual sense

Hope: the class of theories - all possible f() - is large enough to be closed under renormalization

$$\frac{\partial f(F \wedge F)}{\partial \log \mu} = \beta_f(F \wedge F)$$

I.e. physics at higher energies continues to be described by theories from the same family

= no new DOF appear at Planck scale, just the dynamics changes

The speculative RG flow

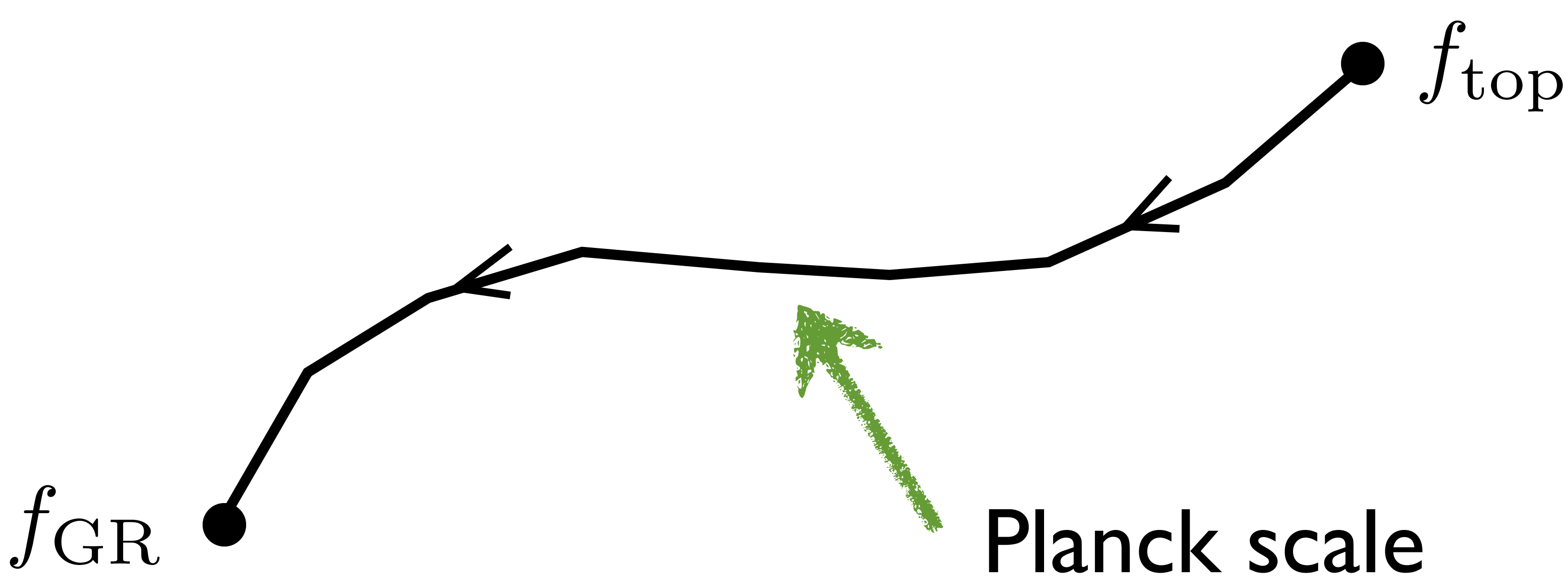
strongly coupled negative helicity gravitons at high energies

$\Rightarrow$ no propagating DOF ? $\Rightarrow$ topological theory ?

$$f_{\rm top}(F\wedge F) = {\rm Tr}(F\wedge F)$$

necessarily a fixed point of the RG flow

corresponds to a topological theory (no propagating DOF)



flow from very steep in IR towards very flat in UV potential

Summary:

- Dynamically non-trivial diffeomorphism invariant gauge theories
- The simplest non-trivial such theory G=SU(2) - gravity
- GR can be described in this language (on-shell equivalent only)

  $\Rightarrow$ possibly different quantum theory

- Computationally efficient alternative to the usual description
  (no propagating conformal mode even off-shell)
- Different from GR (parity-violating)
  theories of interacting massless spin 2 particles
- If this class of theories is closed under renormalization

  $\Rightarrow$ understanding of the gravitational RG flow
  description of the Planck scale physics

Open problems

- Chiral, thus complex description. Unitarity?
- Coupling to matter?

  Enlarging the gauge group - rather general types of matter coupled to gravity can be obtained. Fermions?
- Closedness under renormalization?

  Are these just some effective field theory models, or they are UV complete as Yang-Mills?