BORCHERDS-KAC-MOODY LIE $SU_{\mathbf{s}}$PERALGEBRAS AND $SU_{\mathbf{r}}$PERGRAVITY

JAKOB PALMKVIST – IHES

Based on **1110.4892**, and work in progress …

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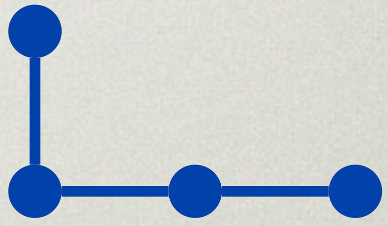
$D = 7$ $\qquad\qquad$ $\mathfrak{g} = \mathfrak{sl}(5, \mathbb{R})$



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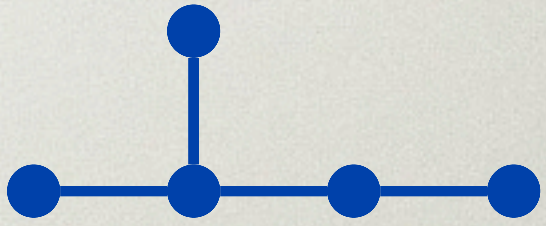
$D = 6$ $\qquad\qquad$ $\mathfrak{g} = \mathfrak{so}(5,5)$



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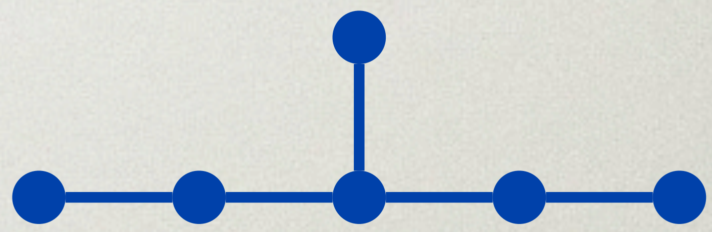

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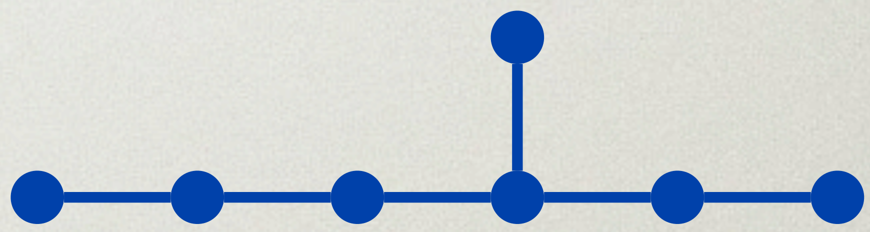
$D = 4$ $\qquad$ $\mathfrak{g} = E_7$



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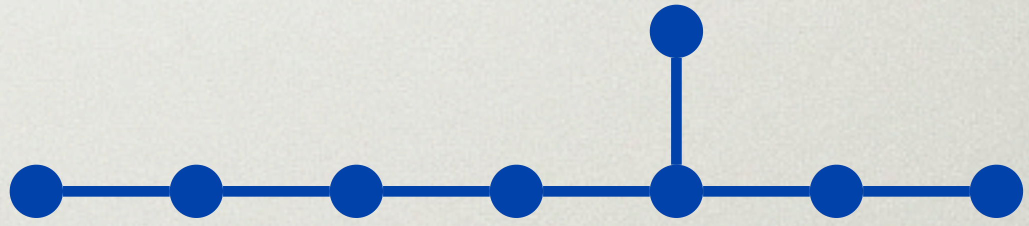
$D = 3$ $\qquad$ $\mathfrak{g} = E_8$



D	7
g	$\mathfrak{sl}(5,\mathbb{R})$
$\mathbf{r}_1$	$\overline{\mathbf{10}}$
$\mathbf{r}_2$	$\mathbf{5}$
$\mathbf{r}_3$	$\overline{\mathbf{5}}$
$\mathbf{r}_4$	$\mathbf{10}$

$$
\begin{aligned}
A_\mu{}^{\mathcal{M}} &= A_\mu{}^{[ab]} \\
A_{\mu\nu}{}^{\mathcal{M}\mathcal{N}} &= A_{\mu\nu}{}^{[abcd]} \\
A_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N}\mathcal{P}} &= A_{\mu\nu\rho}{}^{a} \\
A_{\mu\nu\rho\sigma}{}^{\mathcal{M}\mathcal{N}\mathcal{P}\mathcal{Q}} &= A_{\mu\nu\rho\sigma}{}^{[abc]}
\end{aligned}
$$

D	7	6
g	$\mathfrak{sl}(5,\mathbb{R})$	$\mathfrak{so}(5,5)$
$\mathbf{r}_1$	$\overline{\mathbf{10}}$	$\mathbf{16}_c$
$\mathbf{r}_2$	$\mathbf{5}$	$\mathbf{10}$
$\mathbf{r}_3$	$\overline{\mathbf{5}}$	$\mathbf{16}_s$
$\mathbf{r}_4$	$\mathbf{10}$	

$$A_\mu{}^{\mathcal{M}} = A_\mu{}^{\alpha}$$
$$A_{\mu\nu}{}^{\mathcal{M}\mathcal{N}} = A_{\mu\nu}{}^{a}$$
$$A_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N}\mathcal{P}} = A_{\mu\nu\rho}{}^{\dot{\alpha}}$$

D	7	6	5
g	$\mathfrak{sl}(5,\mathbb{R})$	$\mathfrak{so}(5,5)$	E_6
$\mathbf{r}_1$	$\overline{\mathbf{10}}$	$\mathbf{16}_c$	$\overline{\mathbf{27}}$
$\mathbf{r}_2$	$\mathbf{5}$	$\mathbf{10}$	$\mathbf{27}$
$\mathbf{r}_3$	$\overline{\mathbf{5}}$	$\mathbf{16}_s$	
$\mathbf{r}_4$	$\mathbf{10}$		

D	7	6	5	4
g	$\mathfrak{sl}(5,\mathbb{R})$	$\mathfrak{so}(5,5)$	E_6	E_7
$\mathbf{r}_1$	$\overline{\mathbf{10}}$	$\mathbf{16}_c$	$\overline{\mathbf{27}}$	$\mathbf{56}$
$\mathbf{r}_2$	$\mathbf{5}$	$\mathbf{10}$	$\mathbf{27}$	
$\mathbf{r}_3$	$\overline{\mathbf{5}}$	$\mathbf{16}_s$		
$\mathbf{r}_4$	$\mathbf{10}$			

D	7	6	5	4
g	$\mathfrak{sl}(5,\mathbb{R})$	$\mathfrak{so}(5,5)$	E_6	E_7
$\mathbf{r}_1$	$\overline{\mathbf{10}}$	$\mathbf{16}_c$	$\overline{\mathbf{27}}$	$\mathbf{56}$
$\mathbf{r}_2$	$\mathbf{5}$	$\mathbf{10}$	$\mathbf{27}$	
$\mathbf{r}_3$	$\overline{\mathbf{5}}$	$\mathbf{16}_s$		
$\mathbf{r}_4$	$\mathbf{10}$			

Hodge duality: p-forms $\longleftrightarrow$ $(D-2-p)$-forms

$$\mathbf{r}_p = \bar{\mathbf{r}}_{D-2-p}$$

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$\mathbf{r}_2$	$\mathbf{5}$	$\mathbf{10}$	$\mathbf{27}$	$\mathbf{133}$	
$\mathbf{r}_3$	$\overline{\mathbf{5}}$	$\mathbf{16}_s$	$\mathbf{78}$		
$\mathbf{r}_4$	$\mathbf{10}$	$\mathbf{45}$			
$\mathbf{r}_5$	$\mathbf{24}$				

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Two ways of deriving these representations, and predicting which $(D-1)$- and D-forms we can add

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In this talk I will relate them to each other.

Gauged supergravity

$\mathbf{r}_1$ index

- Gauge field: the vector field $A_\mu{}^{\mathcal{M}}$ transforming in a representation $\mathbf{r}_1$ of G
- Gauge group: a subgroup G_0 of G $(\dim \mathrm{G}_0 \leq \dim \mathbf{r}_1)$
- Covariant derivative: $D_\mu = \partial_\mu - g A_\mu{}^{\mathcal{M}} X_{\mathcal{M}}$
- Generators of G_0: $X_{\mathcal{M}} = \Theta_{\mathcal{M}}{}^{\alpha} t_\alpha$
- Generators of G: t_α
- Embedding tensor: $\Theta_{\mathcal{M}}{}^{\alpha}$

adj index

[de Wit, Nicolai, Samtleben, ... 0801.1294, ...]

The embedding tensor $\Theta_{\mathcal{M}}{}^{\alpha}$ transforms in a subrepresentation of $\bar{\mathbf{r}}_1 \times \mathbf{adj}$.

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Closure constraint: $[X_{\mathcal{M}}, X_{\mathcal{N}}] = -(X_{\mathcal{M}})_{\mathcal{N}}{}^{\mathcal{P}} X_{\mathcal{P}}$

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Closure constraint: $[X_{\mathcal{M}}, X_{\mathcal{N}}] = -(X_{\mathcal{M}})_{\mathcal{N}}{}^{\mathcal{P}} X_{\mathcal{P}}$

But in general $(X_{\mathcal{M}})_{\mathcal{N}}{}^{\mathcal{P}} \neq -(X_{\mathcal{N}})_{\mathcal{M}}{}^{\mathcal{P}}$

[de Wit, Nicolai, Samtleben, ... 0801.1294, ...]

Problem: The field strength of ${A_\mu}^{\mathcal{M}}$

$$F_{\mu\nu}{}^{\mathcal{P}} = 2\,\partial_{[\mu} A_{\nu]}{}^{\mathcal{P}} + g(X_{\mathcal{M}})_{\mathcal{N}}{}^{\mathcal{P}} A_{[\mu}{}^{\mathcal{M}} A_{\nu]}{}^{\mathcal{N}}$$

is not covariant under gauge transformations:

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$$\delta A_\mu{}^{\mathcal{M}} = D_\mu \Lambda^{\mathcal{M}} = \partial_\mu \Lambda^{\mathcal{M}} + g A_\mu{}^{\mathcal{P}} (X_{\mathcal{P}})_{\mathcal{Q}}{}^{\mathcal{M}} \Lambda^{\mathcal{Q}} \Rightarrow$$

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$$\begin{aligned} \delta F_{\mu\nu}{}^{\mathcal{M}} &= g\Lambda^{\mathcal{P}} (X_{\mathcal{N}})_{\mathcal{P}}{}^{\mathcal{M}} F_{\mu\nu}{}^{\mathcal{N}} - 2g(X_{(\mathcal{P}})_{\mathcal{Q})}{}^{\mathcal{M}} A_{[\mu}{}^{\mathcal{P}} \delta A_{\nu]}{}^{\mathcal{Q}} \\ &\neq -g\Lambda^{\mathcal{P}} (X_{\mathcal{P}})_{\mathcal{N}}{}^{\mathcal{M}} F_{\mu\nu}{}^{\mathcal{N}} \end{aligned}$$

[de Wit, Nicolai, Samtleben, ... 0801.1294, ...]

Recipe to regain covariance:

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Recipe to regain covariance:

- Add a term to $\delta A_\mu{}^{\mathcal{M}}$ involving a new parameter$\Lambda\ {}_\mu{}^{\mathcal{MN}}$

$$\delta A_\mu{}^{\mathcal{M}} \to \delta A_\mu{}^{\mathcal{M}} + 2g(X_{(\mathcal{M}})_{\mathcal{N})}{}^{\mathcal{P}}\Lambda_\mu{}^{\mathcal{MN}}$$

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- Add a term to $F_{\mu\nu}{}^{\mathcal{P}}$ involving a new tensor field $A_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}$

$$F_{\mu\nu}{}^{\mathcal{P}} \to F_{\mu\nu}{}^{\mathcal{P}} - 2g(X_{(\mathcal{M}})_{\mathcal{N})}{}^{\mathcal{P}} A_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}$$

[de Wit, Nicolai, Samtleben, ... 0801.1294, ...]

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$$F_{\mu\nu}{}^{\mathcal{P}} \to F_{\mu\nu}{}^{\mathcal{P}} - 2g(X_{(\mathcal{M}})_{\mathcal{N})}{}^{\mathcal{P}} A_{\mu\nu}{}^{\mathcal{MN}}$$

- Define the gauge transformations of $A_{\mu\nu}{}^{\mathcal{M}}$ appropriately

[de Wit, Nicolai, Samtleben, ... 0801.1294, ...]

This solves the problem with the field strength of ${A_\mu}^{\mathcal{M}}$, but leads to the same problem for the field strength of ${A_{\mu\nu}}^{\mathcal{M}\mathcal{N}}$.

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However, we can apply the same solution again: introduce a new parameter${\Lambda\ _{\mu\nu}}^{\mathcal{M}\mathcal{N}\mathcal{P}}$ and a new field ${A_{\mu\nu\rho}}^{\mathcal{M}\mathcal{N}\mathcal{P}}$.

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Continuing in this way, we obtain a tensor hierarchy of parameters, p-form fields and field strengths.

$$\Lambda^{\mathcal{M}},\ {\Lambda_\mu}^{\mathcal{M}\mathcal{N}},\ {\Lambda_{\mu\nu}}^{\mathcal{M}\mathcal{N}\mathcal{P}},\ \ldots$$

$${A_\mu}^{\mathcal{M}},\ {A_{\mu\nu}}^{\mathcal{M}\mathcal{N}},\ {A_{\mu\nu\rho}}^{\mathcal{M}\mathcal{N}\mathcal{P}},\ \ldots$$

$${F_{\mu\nu}}^{\mathcal{M}},\ {F_{\mu\nu\rho}}^{\mathcal{M}\mathcal{N}},\ {F_{\mu\nu\rho\sigma}}^{\mathcal{M}\mathcal{N}\mathcal{P}},\ \ldots$$

[de Wit, Nicolai, Samtleben, ... 0801.1294, ...]

The term that we add to $\delta A_{\mu_1 \cdots \mu_p}{}^{\mathcal{M}_1 \cdots \mathcal{M}_p}$ has the form

$$gY^{\mathcal{M}_1 \mathcal{M}_2 \cdots \mathcal{M}_p}{}_{\mathcal{N}_1 \mathcal{N}_2 \cdots \mathcal{N}_{p+1}} \Lambda_{\mu_1 \cdots \mu_p}{}^{\mathcal{N}_1 \mathcal{N}_2 \cdots \mathcal{N}_{p+1}}$$

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The intertwiners are defined recursively

$$\begin{aligned} Y^{\mathcal{M}_1 \mathcal{M}_2 \cdots \mathcal{M}_p}{}_{\mathcal{N}_1 \mathcal{N}_2 \cdots \mathcal{N}_{p+1}} &= -\delta_{\mathcal{N}_1}{}^{\langle \mathcal{M}_1} Y^{\mathcal{M}_2 \cdots \mathcal{M}_p \rangle}{}_{\mathcal{N}_2 \cdots \mathcal{N}_{p+1}} \\ &\quad - (X_{\mathcal{N}_1})_{\mathcal{N}_2 \cdots \mathcal{N}_{p+1}}{}^{\langle \mathcal{M}_1 \mathcal{M}_2 \cdots \mathcal{M}_p \rangle} \end{aligned}$$

$$Y^{\mathcal{P}}{}_{\mathcal{M}\mathcal{N}} = -(X_{\mathcal{M}})_{\mathcal{N}}{}^{\mathcal{P}} - (X_{\mathcal{N}})_{\mathcal{M}}{}^{\mathcal{P}}$$

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$$Y^{\mathcal{P}}{}_{\mathcal{M}\mathcal{N}} = -(X_{\mathcal{M}})_{\mathcal{N}}{}^{\mathcal{P}} - (X_{\mathcal{N}})_{\mathcal{M}}{}^{\mathcal{P}}$$

The lower indices of $Y^{\mathcal{M}_1 \cdots \mathcal{M}_p}{}_{\mathcal{N}_1 \cdots \mathcal{N}_{p+1}}$ define $\mathbf{r}_{p+1}$.

[de Wit, Nicolai, Samtleben, ... 0801.1294, ...]

D	7	6	5	4	3
g	$\mathfrak{sl}(5,\mathbb{R})$	$\mathfrak{so}(5,5)$	E_6	E_7	E_8
$\mathbf{r}_1$	$\overline{\mathbf{10}}$	$\mathbf{16}_c$	$\overline{\mathbf{27}}$	$\mathbf{56}$	$\mathbf{248}$
$\mathbf{r}_2$	$\mathbf{5}$	$\mathbf{10}$	$\mathbf{27}$	$\mathbf{133}$	$\mathbf{3875}$
$\mathbf{r}_3$	$\overline{\mathbf{5}}$	$\mathbf{16}_s$	$\mathbf{78}$	$\mathbf{912}$	$\mathbf{3875}+\mathbf{147250}$
$\mathbf{r}_4$	$\mathbf{10}$	$\mathbf{45}$	$\overline{\mathbf{351}}$	$\mathbf{133}+\mathbf{8645}$	
$\mathbf{r}_5$	$\mathbf{24}$	$\mathbf{144}_s$	$\mathbf{27}+\mathbf{1728}$		
$\mathbf{r}_6$	$\overline{\mathbf{15}}+\mathbf{40}$	$\mathbf{10}+\mathbf{126}_s+\mathbf{320}$			
$\mathbf{r}_7$	$\mathbf{5}+\overline{\mathbf{45}}+\mathbf{70}$				

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$\boxed{(D-2)\text{-forms}}$ ⟷ adjoint representation

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$(D-2)$-forms ⟷ adjoint representation

$(D-1)$-forms ⟷ supersymmetry constraint

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$(D-2)$-forms ⟷ adjoint representation

$(D-1)$-forms ⟷ supersymmetry constraint

D-forms ⟷ closure constraint

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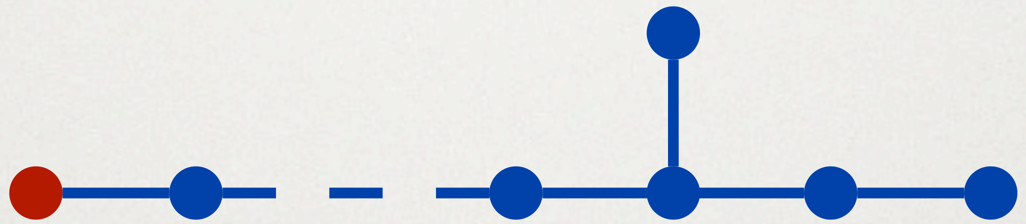
Extend the Lie algebra E_n
to a BKM superalgebra U_{n+1}



[Henry-Labordère, Julia, Paulot: hep-th/0203070, 0303178]
[Henneaux, Julia, Levie: 1007.5241]

Another way to obtain (almost) the same representations:

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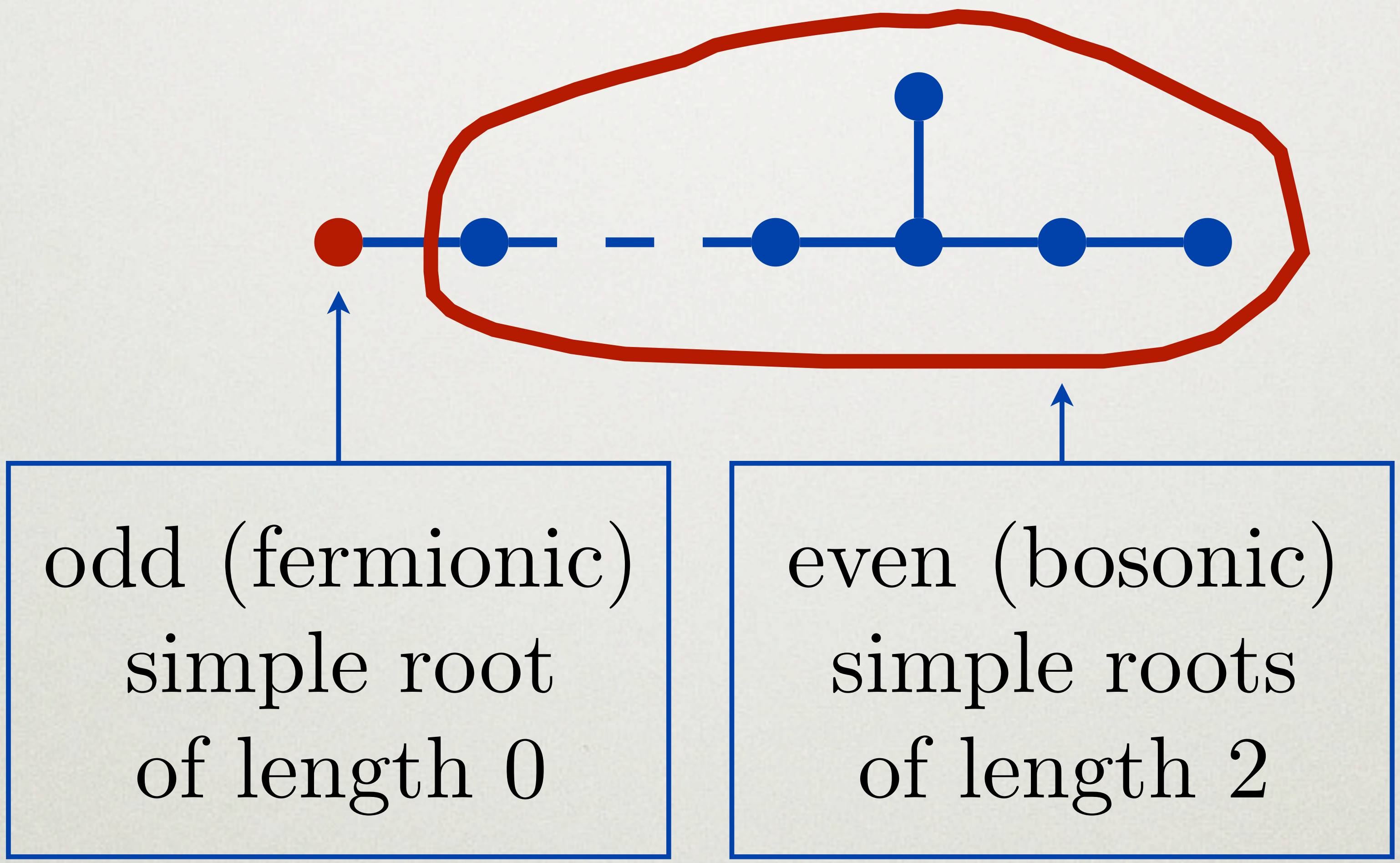

[Henry-Labordère, Julia, Paulot: hep-th/0203070, 0303178]
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The Cartan matrix of the BKM superalgebra U_{n+1} :

$$A_{IJ} = \begin{pmatrix} 0 & -1 & 0 & \cdots \\ -1 & 2 & -1 & \cdots \\ 0 & -1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The Cartan matrix A_{ij} of $\mathfrak{g} = E_n$

Each row and column corresponds to two generators e_I and f_I, which are odd elements for the first row and column ($I = 0$), and even elements for the others ($I = i$).

The BKM superalgebra U_{n+1} is the Lie superalgebra generated by $e_I,\ f_I$ and $h_I = [\![e_I,\ f_I]\!]$ modulo the Chevalley-Serre relations

$$h_0 = \{e_0,\ f_0\}$$
$$h_i = [e_i,\ f_i]$$

$$[\![h_I, e_J]\!] = A_{IJ} e_J \qquad \{e_0, e_0\} = \{f_0, f_0\} = 0$$

$$[\![h_I, f_J]\!] = -A_{IJ} f_J \qquad (\text{ad } e_I)^{1-A_{IJ}}(e_J) = 0$$

$$[\![e_I, f_J]\!] = \delta_{IJ} h_J \qquad (\text{ad } f_I)^{1-A_{IJ}}(f_J) = 0$$

[Ray 1994] [Kac 1977]

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$\mathbb{Z}_2$-grading: $U = U_{(0)} \oplus U_{(1)}$ $\qquad e_0,\ f_0 \in U_{(1)}$

$[\![U_{(M)}, U_{(N)}]\!] \subset U_{(M+N)} \bmod \mathbb{Z}_2$ $\qquad e_i,\ f_i \in U_{(0)}$

[Ray 1994] [Kac 1977]

The odd generators e_0 and f_0 give rise to a $\mathbb{Z}$-grading of U

$$[\![U_p, U_q]\!] \subseteq U_{p+q}$$

$$U = \cdots + U_{-2} + U_{-1} + \left(\mathfrak{g} + \mathbb{R}\right) + U_1 + U_2 + \cdots$$

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and a level decomposition of the adjoint of U

$$\mathbf{adj}\; U = \cdots + \bar{\mathbf{s}}_2 + \bar{\mathbf{s}}_1 + \Big(\mathbf{adj}\; \mathfrak{g} + \mathbf{1}\Big) + \mathbf{s}_1 + \mathbf{s}_2 + \cdots$$

It turns out that the representations $\mathbf{s}_p$ coming from the level decomposition coincide with $\mathbf{r}_p$ coming from the tensor hierarchy of gauged supergravity (except for an additional part of $\mathbf{s}_2$ and $\mathbf{s}_3$ for $D = 3$)

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How can we understand this?

For any $k \geq 0$, define $\tilde{U}_{-k}$ recursively as the vector space of all linear maps $\tilde{U}_1 \to \tilde{U}_{-k+1}$, let $\tilde{U}_+ = \tilde{U}_1 + \tilde{U}_2 + \cdots$ be the free Lie superalgebra generated by $\tilde{U}_1 = U_1$, and extend the Lie superbracket on $\tilde{U}_+$ suitably to the whole of

$$\tilde{U} = \tilde{U}_- + \tilde{U}_0 + \tilde{U}_+ = \cdots + \tilde{U}_{-1} + \tilde{U}_0 + \tilde{U}_1 + \cdots$$

[Kantor 1970, Palmkvist 0905.2468]

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From $\tilde{U}$ we can construct a graded Lie superalgebra U by specifying a subspace $U_{-1} \subset \tilde{U}_{-1}$ (Not necessarily the same U and U_{-1} as above!)

[Kantor 1970, Palmkvist 0905.2468]

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$$U_- + U_0\,(= \{U_{-1}, U_1\}) + U_+ = U$$

If U_{-1} is spanned by all operators $\mathcal{F}^{\mathcal{N}} : U_1 \to \operatorname{End} U_1$ given by $\mathcal{F}^{\mathcal{N}}(E_{\mathcal{M}}) = \operatorname{ad}\{E_{\mathcal{M}},\, F^{\mathcal{N}}\}$, then we get back the original BKM superalgebra, which we now call $U_{\mathbf{s}}$.

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If U_{-1} is spanned by all operators $Y : U_1 \to \operatorname{End} U_1$ given by $Y(E_{\mathcal{M}}) = -\operatorname{ad} X_{\mathcal{M}} = -\operatorname{ad} \Theta_{\mathcal{M}}{}^{\alpha} t_{\alpha}$, then we get a different Lie superalgebra, which we call $U_{\mathbf{r}}$.

The components $Y^{\mathcal{N}_1\cdots\mathcal{N}_{p-1}}{}_{\mathcal{M}_1\cdots\mathcal{M}_p}$ defining the tensor hierarchy representations now reappear in the relation

$$[\![Y,\, E_{\mathcal{M}_1\cdots\mathcal{M}_p}]\!] = Y^{\mathcal{N}_1\cdots\mathcal{N}_{p-1}}{}_{\mathcal{M}_1\cdots\mathcal{M}_p} E_{\mathcal{N}_1\cdots\mathcal{N}_{p-1}}$$

where $E_{\mathcal{M}_1\cdots\mathcal{M}_p} \equiv [\![E_{\mathcal{M}_1}, [\![E_{\mathcal{M}_2}, \ldots, [\![E_{\mathcal{M}_{p-1}}, E_{\mathcal{M}_p}]\!] \cdots]\!]]\!]$

It follows that $\mathbf{r}_p$ is the E_n representation on $(U_{\mathbf{r}})_p$.

The supersymmetry constraint on the embedding tensor $\Theta_{\mathcal{M}}{}^{\alpha}$ can now be written $(U_{\mathbf{r}})_2 \subseteq (U_{\mathbf{s}})_2$ or $(U_{\mathbf{r}})_+ \subseteq (U_{\mathbf{s}})_+$, which implies that $\mathbf{r}_p \subseteq \mathbf{s}_p$ for all p.

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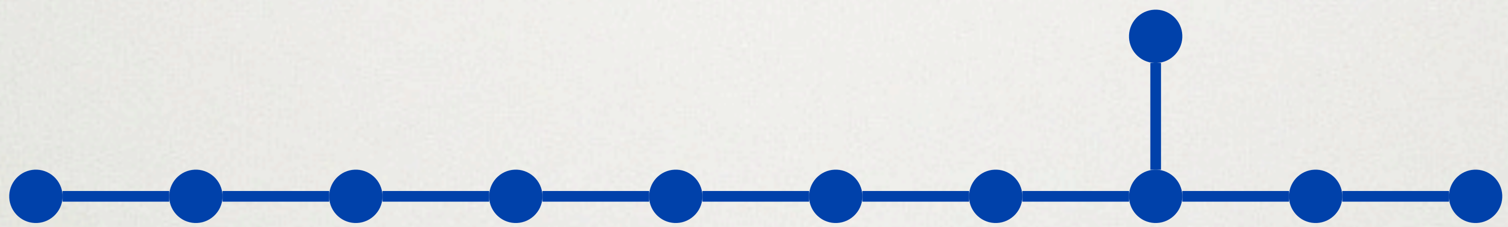
In fact, $\mathbf{r}_p = \mathbf{s}_p$ at least up to $p = D$, for $D \neq 3$.

The closure constraint can be written $\{Y, Y\} = 0$.

The representations $\mathbf{s}_p$, up to $p = D$ (or $p = D - 1$), also appear in a corresponding level decomposition of the indefinite Kac-Moody algebra E_{11} (or E_{10}).

[Bergshoeff, Nutma, De Baetselier: 0705.1304] [Riccioni, West: 0705.0752] [Henneaux, Julia, Levie: 1007.5241] [Palmkvist: 1110.4892, 1203.5107]



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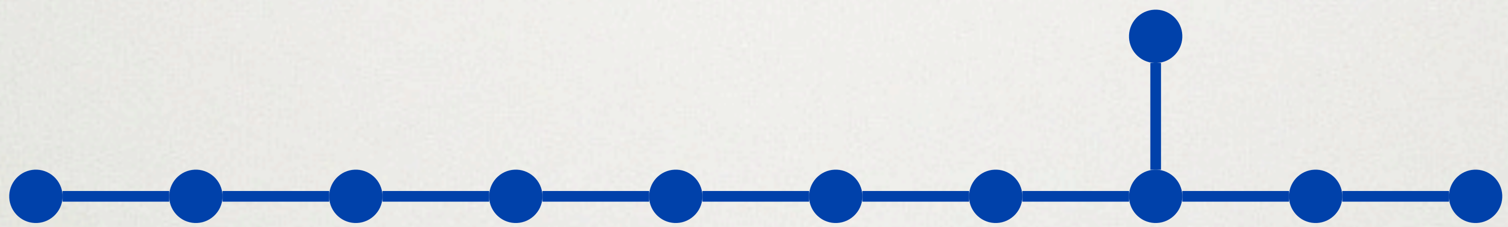


In the same way as

$$E_{11} \supset E_{10} \supset \cdots \supset E_{n+1} \supset E_n \supset \cdots$$

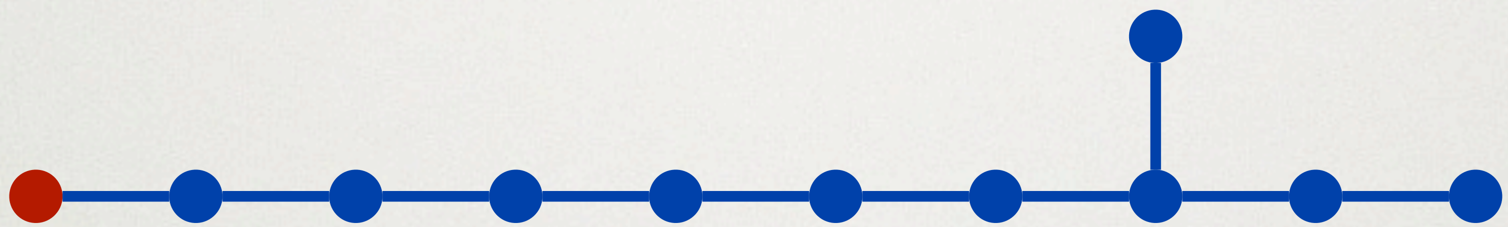
it can be shown that

$$U_{11} \supset U_{10} \supset \cdots \supset U_{n+1} \supset U_n \supset \cdots$$

[Work in progress with A. Kleinschmidt]

Maybe U_{11} or U_{10} can play a role in supergravity or M-theory similar to the ones proposed for E_{11} and E_{10}

[West, Riccioni, ...] [Damour, Henneaux, Nicolai, Kleinschmidt, ...]



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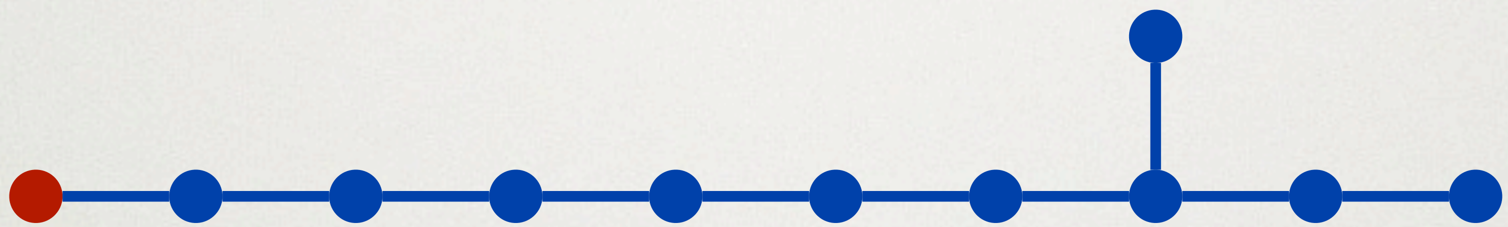

To be continued ...