

Overview of Loop Quantum Gravity and Spin Foams

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- Preamble
- Loop Quantum Gravity (LQG)
 - Ashtekar-Barbero connection
 - Geometric Operators
 - Imposition of Constraints
 - Covariant Loop Quantum Gravity (CLQG)
- Spin Foam Models
 - BF and Plebanski Lagrangian
 - BC spin foam models and the new models.
- Conclusion

Loop Quantum Gravity (LQG) is a theory, still in construction, which aims at quantizing General Relativity (coupled to matter) while preserving the symmetries of General Relativity, which consist in diffeomorphisms of space time (Background independent quantization). The goal of LQG is to properly define a quantization of the Hamiltonian formulation of GR using canonical quantization of constrained system.

- States in this theory are described by spin networks which encodes the geometry of space.
- The transition amplitudes for states in LQG are given by certain type of generalized Feynman integrals which are described in terms of Spin Foam models.

I have been invited in IHES to give a *fair* overview of LQG and Spin Foam models and I have tried to give a precise introduction to this field which possess many interesting aspects but which still contains gaps that have to be stated clearly and hopefully positively understood.

I will only present here the main technical constructions in this field leaving aside two central subjects:

- the conceptual problems of space-time and quantum mechanics which are essential for a clear understanding of Quantum Gravity
- the physics of Quantum Gravity : black holes, initial singularity, eventual imprints on high energy physics.

Preamble 3

Here are good references that can be used to have precise ideas on Quantum gravity, on LQG and Spin Foam models:

S.Carlip: A progress report on Quantum Gravity gr-qc/ 0108040

C.Rovelli: Quantum Gravity Cambridge UK

T.Thiemann: Modern Canonical Quantum General Relativity UK

A.Perez: Spin Foam models for Quantum Gravity gr-qc/0301113

It is impossible to give credit in these slides to all the contributors of this field, they can be found in the review articles above.

LQG and Spin Foam models being scientific theories, unable for the present time to give clear predictions which could be tested with present state of technology, have to be thoroughly analyzed with a critical mind driven only by scientific issues:

H.Nicolai, K.Peeters and M.Zamaklar "LQG: an outside view"

S.Alexandrov and Ph.Roche "Critical overview of Loops and Foams" (to appear).

(\mathcal{M}, g) 3+1 Lorentzian manifold, G gravitational constant,

$$S_{EH}[g] = \frac{1}{G} \int_{\mathcal{M}} d^4x \sqrt{-g} R[g].$$

e_I orthonormal moving frame, e^I dual basis of one forms,
 $g = \eta_{IJ} e^I \otimes e^J$, $\eta_{IJ} = \text{diag}(-, +, +, +)$.

$$\nabla_{\mu} e_I = \omega_{\mu I}^J e_J$$

$$S_{EH}[g] = \frac{1}{2G} \int_{\mathcal{M}} d^4x \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}[\omega].$$

First order formalism of gravity (Palatini formulation)

e^I , $I = 0, 1, 2, 3$ one forms, ω $so(3, 1)$ connection,

$$S_{Palatini}[e, \omega] = \frac{1}{G} \int_{\mathcal{M}} d^4x \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}[\omega],$$

Variations w.r.t ω gives torsion of ω is zero.

Variations w.r.t e gives Einstein Equations $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$,
 with $g = \eta_{IJ} e^I \otimes e^J$.

Enhancement of this action, the Holst action:

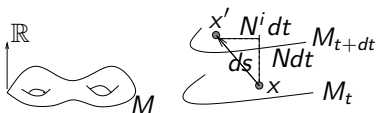
$$S_{Holst}[e, \omega] = \frac{1}{G} \int_{\mathcal{M}} d^4x \epsilon_{IJKL} (e^I \wedge e^J \wedge F^{KL}[\omega] + \frac{1}{\gamma} e^I \wedge e^J \wedge \star F^{KL}[\omega])$$

γ free parameter called *Immirzi parameter*. The second term does not modify the equation of motion.

One performs an ADM decomposition $\mathcal{M} = \mathbb{R} \times M$, and imposes the *time gauge* ($e^0 = Ndt$) which amounts to choose a coordinate system where the cotetrad expresses as:

$$e^0 = Ndt, e^a = E_i^a N^i dt + E_i^a dx^i$$

$$ds^2 = -N^2 dt^2 + q_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$



Densitized triad: $\tilde{E}_a^i = \sqrt{q}E_a^i$

Ashtekar-Barbero $su(2)$ connection on M_t : $A_i^a = \Gamma_i^a(E) - \gamma K_i^a$ with $\Gamma_i^a(E)$ Levi Civita connexion of (M_t, q) expressed in the basis E^a and $K_i^a = \omega_i^{0a}$. One finally obtains that A_i^a, \tilde{E}_a^i are canonical variables i.e:

$$\{A_i^a(x), A_j^b(y)\} = \{\tilde{E}_a^i(x), \tilde{E}_b^j(y)\} = 0, \{A_i^a(x), \tilde{E}_b^j(y)\} = \gamma \delta_i^j \delta_b^a \delta(x, y).$$

Canonical analysis gives 3 sets of first class constraints:

$$G_a = \partial_i \tilde{E}_a^i - \epsilon_{ab}{}^c A_i^b \tilde{E}_c^i \approx 0, \text{ (Gauss constraint)}$$

$$H_i = \tilde{E}_a^k F_{ik}^a[A] \approx 0, \text{ (Diffeomorphism constraint)}$$

$$H = \tilde{E}_a^i \tilde{E}_b^j \left(\epsilon^{ab}{}^c F_{ij}^c[A] - (1 + \gamma^2) K_{[i}^a K_{j]}^b \right) \approx 0. \text{ (Hamiltonian constraint)}$$

Classical observables are functions of the canonical variables which Poisson commute with the above constraints. They are complicated (discussion:problem of time) Originally Ashtekar connection was defined for $\gamma = \pm i$ (discussion:problem of reality conditions)

Ultimate goal of LQG is to quantize this constrained system using appropriate generalization of Dirac program i.e:

- Construct unitary representation of a quantized algebra $\hat{A}_a^i(x), \hat{E}_a^i(x)$ satisfying commutations relations of canonical variables acting on a Hilbert space \mathcal{H} of certain functions of the connection A .
- Extract in \mathcal{H} the subspace of physical states which means that

$$\mathcal{H}_{phys} = \{\psi \in \mathcal{H}, \hat{G}_a \psi = \hat{H}_i \psi = \hat{H} \psi = 0\}.$$

A precise definition of the quantization of the constraints $\hat{G}_a, \hat{H}_i, \hat{H}$ has to be given.

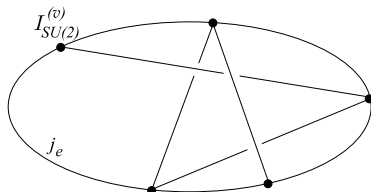
- Endow \mathcal{H}_{phys} with a structure of Hilbert space on which quantum observables will act unitarily
- Find inside \mathcal{H}_{phys} appropriate vectors which describes in the classical limit general relativity.

What is the present status of this program?

One proceeds in stages by imposing constraints one after the other (I am cheating here: Dirac algebra of constraint gives $\{G, G\} = G, \{G, H_i\} = \{G, H\} = 0, \{H_i, H_j\} = H_k, \{H_i, H\} = H, \{H, H\} = (*H)_i$).

One first defines the Hilbert space \mathcal{H} of functions of the connection A which are cylindrical i.e. which depends only the holonomies of the connection along finitely many curves e_1, \dots, e_n immersed in M .

In this space one can define the subspace invariant under Gauss constraint, the so called Kinematical Hilbert space \mathcal{H}_{kin} which basis is given by *spin networks* embedded in M and hermitian form \langle, \rangle defined on cylindrical functions by integrating with the Haar measure and turning the basis of spin networks in an orthonormal basis. Such a state Ψ_Γ is labelled by a colored graph Γ embedded in M . The embedded graph is just a finite number of points $\{v\}$ connected by a finite number of smooth embedded curves $\{e\}$ in M , whereas the coloring associates irreducible representations of $SU(2)$ (half-integer spins j_e) to the edges e and $SU(2)$ invariant intertwiners I^v to the vertices v . The corresponding state Ψ_Γ is constructed by contracting holonomies of A along edges in representations j_e with invariant intertwiners I^v at vertices.



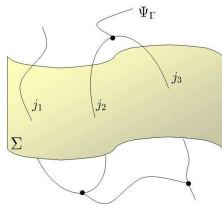
It is possible to give regularization of the measure of area and volume (Rovelli and Smolin) and to find the action of these geometric operators on \mathcal{H}_{kin} .

Let a surface $\Sigma \subset M$,

$$A_\Sigma = \int_\Sigma d^2\sigma \sqrt{n_i n_j g^{ij}}, \quad g^{ij} = \delta^{ab} \tilde{E}_a^i \tilde{E}_b^j$$

where n_i is the normal to the surface.

$$\hat{A}_\Sigma \Psi_\Gamma = a_{\Sigma, \Gamma} \Psi_\Gamma, \quad a_{\Sigma, \Gamma} = \gamma \ell_p^2 \sum_{e \in \Sigma \neq \emptyset} \sqrt{j_e(j_e + 1)},$$



Properties of the area spectrum

The expression in the square root is nothing else but the Casimir operator of $SU(2)$. Thus the LQG spectrum of the area operator is discrete and has a minimal non zero eigenvalue. Nice discrete structure (Spin networks are often said to be the "atoms of space") but there are puzzling questions:

- The spectrum is proportional to a parameter which has no classical meaning and corresponds to a choice of canonical coordinates.
- This spectrum is sensitive to regularization (there are other regularization which gives the equally spaced spectrum $j_e + \frac{1}{2}$ instead of $\sqrt{j_e(j_e + 1)}$)
- The spectrum is discrete because Ashtekar-Barbero is a $su(2)$ connection (because of the choice of time gauge)
- In 2+1 dimension quantum gravity is a topological theory and in this case the spectrum of length of space like curves is *continuous*
- The area operator *is not an observable* in the sense of Dirac: it does not commute with H_i (one can imagine how to treat this problem using matter), it does not commute with H .

Discussion. More on this question with CLQG.

Volume operator

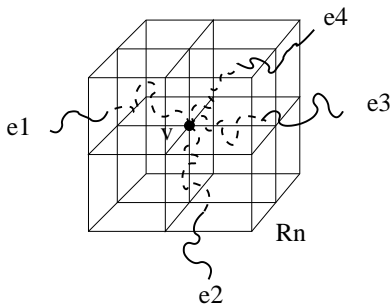
The volume of region $R \subset M$ is given by the following integral

$$V_R = \int_R d^3x \sqrt{h} = \int_R d^3x \left| \frac{1}{3!} \epsilon_{ijk} \epsilon^{abc} \tilde{E}_a^i \tilde{E}_b^j \tilde{E}_c^k \right|^{1/2}.$$

it admits the following quantization

$$\hat{V}_R \Psi_\Gamma = \gamma^{3/2} \ell_p^3 \sum_{v \in R \cap \Gamma} \left| \frac{i C_{\text{reg}}}{8} \sum_{I, J, K} \epsilon_v(e_I, e_J, e_K) \epsilon_{abc} X_{v, e_I}^a X_{v, e_J}^b X_{v, e_K}^c \right|^{1/2} \Psi_\Gamma,$$

$\epsilon_v(e_I, e_J, e_K) \in \{-1, 1, 0\}$ is the sign of the orientation of the three tangent vectors at v of the curves e_I, e_J, e_K .



The computation of the spectrum of the volume operator in the version of Ashtekar-Lewandowski is complicated, spin networks are no more eigenvectors and numerical simulations on space of spinnetworks of valence 5 and 6 indicate that there is no non zero lower bound.

Diffeomorphism constraints

In LQG one implements the constraint of diffeomorphism as follows: One defines \mathcal{H}_{diff} as being the subspace of linear forms on \mathcal{H}_{kin} by "averaging" with the group of diffeomorphisms of M , and defining

$$\langle \Psi_{[\Gamma]} | := \sum_{\phi(\Gamma), \phi \in Diff(M)} \langle \Psi_{\phi(\Gamma)} |$$

and endow \mathcal{H}_{diff} with a structure of Hilbert space

$$\langle \Psi_{[\Gamma]} | \Psi_{[\Gamma']} \rangle := \langle \Psi_{[\Gamma]} | \Psi_{[\Gamma']} \rangle.$$

It remains to implement the Hamiltonian constraint. There are two approaches to this problem (called the problem of dynamics in LQG): Quantum Spin Dynamics and Spin Foam models.

Hamiltonian constraint: QSD 1

Quantum Spin Dynamics of Thiemann

A proper regularization of $H[N] = \int_M d^3x N[x] H(x)$ is proposed which action on spin networks is finite.

It uses as central tool the fact that one can recover the extrinsic curvature K_i^a from Poisson brackets with the volume function V_M , more precisely we have

$$H_E = \tilde{E}_a^i \tilde{E}_b^j \epsilon^{ab}{}_c F_{ij}^c[A] = \epsilon^{ijk} \delta_{ab} F_{ij}^a \{A_k^b, V_M\},$$

where V_M is the volume of space. The central identities are

$$K_i^a = \{A_i^a, K\} \quad \text{and} \quad K = \{V_M, \int_M H_E(x) d^3x\}.$$

After having quantized V_M using the the volume operator of Ashtekar-Lewandowski \hat{V}_M one can define $\hat{H}[N]$ by replacing every where Poisson Bracket by commutators.

The Dirac algebra of constraints can be shown to be satisfied in a weak sense.

However drawbacks:

- No construction of physical scalar product on states annihilated by $\hat{H}[N]$ ("physical states")
- No good control of semiclassical physical states (problem: how to recover gravity in this scheme)
- No control of the eventual higher corrections in \hbar
- No result on the spectra of Dirac observables.

These difficulties are hoped to be cured using the "master constraint" program of T.Thiemann.

Covariant Loop Quantum Gravity 1

The choice of time gauge implies that the Ashtekar-Barbero connection is a $su(2)$ connection which implies that the spectrum of the area operator is discrete with a non zero lower bound and depends on a new constant γ . Are the results of LQG sensible to this choice of gauge?

Covariant Loop Quantum Gravity, developed by S.Alexandrov, aims at quantizing GR in first order formalism without making this choice and introduces new field χ^a :

$$e^0 = Ndt + \chi_a E_i^a dx^i, \quad e^a = E_i^a N^i dt + E_i^a dx^i.$$

$$\tilde{P}_{IJ}^i = \begin{cases} \tilde{E}_a^i & I = 0, J = a \\ \tilde{E}_a^i \chi_b - \tilde{E}_b^i \chi_a & I = a, J = b \end{cases}$$

These covariant generalization of \tilde{E}_a^i satisfy *second class* constraints

$$\phi^{ij} = \epsilon^{ijkl} \tilde{P}_{IJ}^i \tilde{P}_{KL}^j \approx 0.$$

The \tilde{P}_{IJ}^i would be conjugated to ω_i^{IJ} if not the second class constraints which have to be implemented by the use of Dirac Bracket. As a result, in order to find the "almost" canonical conjugate variable to \tilde{P}_{IJ}^i (out of which one constructs geometric operators) one has to add to ω_i^{IJ} constraints and one finally obtains a two parameters family of $so(3, 1)$ connection ${}^{(a,b)}\Omega$.

- There is a notion of spin network associated to these connections (projected spin networks) where edges are labelled by principal unitary representations (k, ρ) of $so(3, 1)$ as well as $su(2)$ spins j .
- An edge e of this projected spin network intersecting Σ gives a unit of area given by:

$$\ell_p^2 \left((a^2 + (1-b)^2) C_{SU(2)} - (1-b)^2 C_{SO(3,1)}^{(1)} + a(1-b) C_{SO(3,1)}^{(2)} \right)^{1/2}$$

$$\text{with } C_{SO(3,1)}^{(1)} = k^2 - \rho^2 - 1, C_{SO(3,1)}^{(2)} = 2k\rho, C_{SU(2)} = j(j+1).$$

There are two interesting choices

- $a = -\gamma, b = 0$ in this case the connection $(-\gamma, 0)\Omega_i$ is the Lorentz extension of the Ashtekar-Barbero connection ($\chi = 0$) and satisfies $\{\Omega, \Omega\} = 0$. This is however not a space time connection (problem with evolution of H), the spectrum is discrete and contains γ .
- $a = b = 0$ in this case the connection $(0, 0)\Omega_i$ is a space-time connection but it is now non commutative $\{(0, 0)\Omega_i, (0, 0)\Omega_j\}_D \neq 0$ and quite complicated. The spectrum is continuous, does not contains γ , the connection behaves well under space time diffeomorphism but there is a real difficulty to give a sense to this theory if there is not even a kinematical representation up to now.

Hamiltonian constraint: Spin foams

Spin Foam methods.

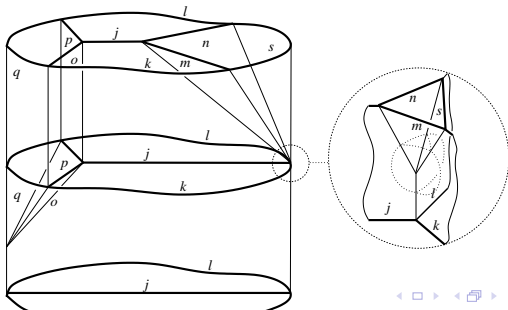
Formal use of "group averaging" method applied to the operator $\hat{H}[N]$ and which amounts to define a projector operator $P : \mathcal{H}_{kin} \rightarrow \mathcal{H}_{phys}$ by

$$P = \int [DN] \exp(i\hat{H}[N]) = \prod_x \delta(H(x))$$

the physical scalar product

$$\langle \Psi_{\Gamma}, \Psi_{\Gamma'} \rangle_{phys} = \langle \Psi_{\Gamma} | P | \Psi_{\Gamma'} \rangle_{kin}$$

can be expanded in powers of N as a spin foam model as follows:



Spin foam models gives transition amplitudes for spin network states, i.e

$$\langle \Psi_{\Gamma}, \Psi_{\Gamma'} \rangle_{phys} := \sum_{C, \partial C = \Gamma \cup \Gamma'} w(C) \sum_{J, l_v} \prod_f A_f \prod_e A_e \prod_v A_v,$$

the sum goes over all 2-complexes C fitting the given graph of the spin networks at the boundaries and over all colorings (J, l_v) of each C fitting the coloring of the spin networks, possibly with some additional restrictions on allowed representations and intertwiners. The weight $w(C)$ is usually some symmetry coefficient and A_f , A_e , A_v are *face*, *edge* and *vertex amplitudes*.

Numerous problems to be solved:

- find spin foam models describing gravity in 3+1 dimensions, i.e find the explicit form of A_f , A_e , A_v .
- find appropriate scheme to give meaning to these infinite sums (and integrals !) which moreover should corresponds to some physical interpretation.

Strategy for building spin foam models of gravity:

- find field theories which have an exact description in term of spin foam models (usually topological field theory)
- impose constraints on these field theories in order to obtain gravity
- implement these constraints in the spin foam models in order to obtain a spin foam description of gravity.
- consistency check: analyze the vertex amplitude in different regimes (for example a semiclassical one) and compare it to the would be properties of the amplitude of quantum gravity.

In $2 + 1$ dimensions the action of gravity in first order formalism has the form

$$S = \int \epsilon_{IJK} e^I \wedge F^{JK}[\omega]$$

this is a topological theory which (in the Riemannian case) has an exact description as a spin foam model known as Ponzano-Regge, A_V is given by $6j$ symbols of $SU(2)$ group (In this case there is no need for summing over all triangulation, still IR divergent but can be regularized by adding cosmological constant and corresponding spin foam model is Turaev-Viro with $6j$ symbols of $U_q(su(2))$ q root of unit).

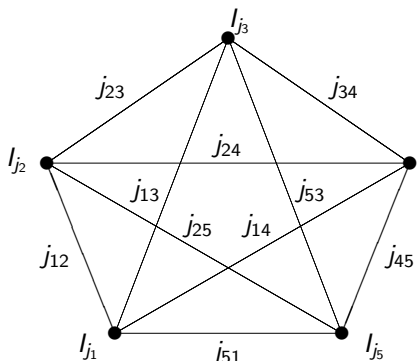
The $3+1$ dimensional analog of this is the BF theory given by the action

$$S_{BF} = \int B_{IJ} \wedge F[\omega]^{IJ}$$

with ω $so(3,1)$ connection and B a 2-form with values in $so(3,1)$ equivalent to B^{IJ} 2-forms. The equation of motion are just $F[\omega] = 0$ and $D_\omega B = 0$, (Topological theory). If one constraint B^{IJ} to be of the form $B^{IJ} = \star(e^I \wedge e^J)$ one recover the action of gravity.

Spin foam formulation of BF theory

BF theory has an exact formulation in term of spin foam model, one can discretize exactly this theory and one finally obtain that in the case of $su(2)$ the summation is over spins J taking half integer values and the summation is over a basis of interwiner I_j , the vertex amplitude being given by:


$$I_{j_4} = A_V^{BF, SU(2)}(\{j_{ab}\}; \{I_{j_a}\})$$

Simplicity Constraints

How to implement the condition $B^{IJ} = \star(e^I \wedge e^J)$ on the BF spin foam model?

This was originally done using the so called simplicity constraints, which can be reformulated as

$$\epsilon^{\mu\nu\rho\sigma} B_{\mu\nu}^{IJ} B_{\rho\sigma}^{KL} = \frac{\mathcal{V}}{4!} \epsilon^{IJKL} .$$

because simplicity constraint imply $B^{IJ} = \pm \star(e^I \wedge e^J)$ (gravitational sector) or $B^{IJ} = \pm(e^I \wedge e^J)$ (topological sector, which was originally discarded but caused much problem later).

A field theory implementing this constraint is Plebanski theory

$$S^{PI}[\omega, B, \varphi] = S_{BF}[\omega, B] + \frac{1}{4} \int_{\mathcal{M}} d^4x \varphi^{\mu\nu\rho\sigma} B_{\mu\nu}^{IJ} B_{\rho\sigma}^{KL} \epsilon_{IJKL} .$$

satisfying the tracelessness condition $\epsilon_{\mu\nu\rho\sigma} \varphi^{\mu\nu\rho\sigma} = 0$.

The simplicity constraints integrated on a triangle f gives the following relation on $B_f^{IJ} = \int_{\Delta_f} B^{IJ}$

$$\epsilon_{IJKL} B_f^{IJ} B_{f'}^{KL} = 0, \pm \mathcal{V}_v$$

depending on the position of f, f' .

Barrett-Crane model

Barrett-Crane model (1998) imposes the simplicity constraint at the level of the BF model for the group $so(3, 1)$. The set of representations of $so(3, 1)$ are the irreducible unitary ones which enter in Plancherel formula i.e these are the principal unitary representations $\Pi_{(k, \rho)}$ with k half-integer and ρ real parameter.

The constraint coming from simplicity constraint is imposed as:

$$\epsilon_{IJKL} \Pi_{(k, \rho)}(T^{IJ}) \Pi_{(k, \rho)}(T^{KL}) = 0$$

which select the spherical ones $k = 0$. One can show that the simplicity constraint amounts to select a specific interwiner, the BC intertwiner I_{BC} .

$$A_V^{BC}(\rho_i, i = 1, \dots, 10) = A_V^{BF, SO(3,1)}((0, \rho_i); I_{BC}, \dots, I_{BC}).$$

This model has different problems:

- No correct asymptotics when ρ_i are large (one should recover $\exp(\frac{i}{\hbar} S_{Regge})$)
- the problem of ultralocality: no propagation from one simplex to the other
- not capable of reproducing the graviton propagator
- the labels on the faces are continuous ρ_i where as the labels on the spin networks are half integers spins.

New spin foam models

If one wants to pursue in the direction of spin foam models one needs other Spin Foam models which solve these difficulties.

The new models of Engle-Livine-Pereira-Rovelli (2008) and Freidel-Krasnov (2008) are attempts to correct the BC model by removing the degenerate sector and by imposing new relations implying the simplicity constraint, namely one imposes that there exist for each tetrahedron t a non-zero vector x_t such that $*B_f^{IJ}(x_t)_J = 0$, x_t now becomes an additional variable.

There is a huge activity on these new spin foam models ranging from their construction to the analysis of the behaviour of the vertex amplitude (which is defined as a $15j$ of $so(3,1)$).

However there are puzzling questions on these models about the handling of the second class constraints which may have been overlooked.

Summing over coloring and on complexes

A very pressing and not very much studied question in spin foam models is the control of the sum over the coloring of 2-complexes and of the sum over the 2 complexes.

The integration over coloring on a fixed complex has been studied in numerous work. (Discussion)

The question of the behaviour of the sum over the 2 complexes can be investigated using the notion of *group field theory* and are under active study. There is at present no conclusive answers to this question.

Conclusion 1

Is LQG a viable theory of Quantum Gravity?

The advocates of LQG are defending enthusiastically their theories which addresses very good questions, poorly addressed in String theory such as: background independence, the problem of observables in Quantum Gravity, the problem of time in Quantum gravity, the quantization of geometric operators.

One of the interest of LQG for describing the world is that this approach is minimalistic in the sense that it does not incorporate supersymmetry, higher dimensional spaces and inclusion of other fields.

In 2+1 dimension quantum gravity can be well studied with the structures which appear in LQG: spin networks, representations of holonomy algebras and which triggered my interest in this field.

There are three major questions which are not solved (but are definitely addressed in LQG):

- The problem of construction of Dirac Observables and the computation of their spectra.
- The problem of dynamics: how to implement the Hamiltonian constraint and to have a good understanding of physical states which approximate a classical solution of GR.
- A precise scheme to make computations (even perturbatively in some parameter) which would allow to make prediction when LQG is coupled to matter.

The future of LQG (and SpinFoam models) as a viable theory of quantum gravity depends on the ability of providing answers to these major problems.