Affine symmetries in supergravity

work with Hermann Nicolai, Martin Weidner, Thomas Ortiz

IHES 05/2013

Henning Samtleben

motivation : 2D supergravity

symmetries

- classically integrable field theory
- affine symmetry group E_9 — solution generating (transitive)
- infinite-dimensional symmetries : $E_9 \longrightarrow E_{10} \longrightarrow E_{11}$

deformations

- affine symmetry also organizes the deformations of the theory
- infinite-dim. HW representations of non-propagating fields

supersymmetry

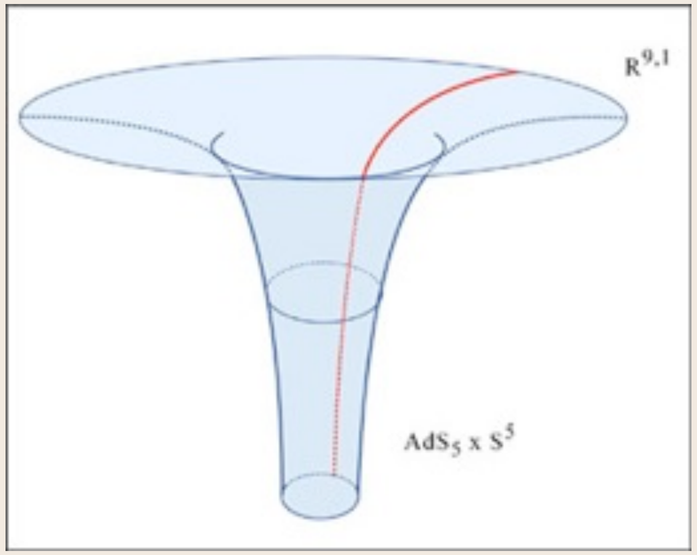
- SO(9) supergravity : first example of such a 2d deformation : IIA on S^8

  matrix model holography

motivation : SO(9) supergravity

Domain wall / QFT correspondence

[H.J. Boonstra, K. Skenderis, P. Townsend, 1999]



holography for Dp-branes : $AdS_{p+2} \times S^{8-p}$ warped

dual to SYM_{p+1} theory

gaugings of maximal supergravity

D6	IIA	$AdS_8 \times S^2$	d=8, SO(3)	[Salam, Sezgin, 1984]
D5	IIB	$AdS_7 \times S^3$	d=7, SO(4)	[Samtleben, Weidner, 2005]
D4	IIA	$AdS_6 \times S^4$	d=6, SO(5)	[Pernici, Pilch, van Nieuwenhuizen, 1984]
D3	IIB	$AdS_5 \times S^5$	d=5, SO(6)	[Günaydin, Romans, Warner, 1985]
D2	IIA	$AdS_4 \times S^6$	d=4, SO(7)	[Hull, 1984]
D1	IIB	$AdS_3 \times S^7$	d=3, SO(8)	[de Wit, Nicolai, 1982]
D0	IIA	$AdS_2 \times S^8$	d=2, SO(9)	??

Affine symmetries in supergravity

- motivation
- D=4 supergravity : symmetries and deformations
- D=2 supergravity : symmetries and deformations
- example : SO(9) supergravity
- conclusions

D=4 supergravity

symmetries and deformations

D=4 supergravity: some generic features

$$\mathcal{L} \;=\; R + G_{ij}(\phi)\,\partial_\mu\phi^i\,\partial^\mu\phi^j + \mathcal{I}_{\Lambda\Sigma}(\phi)\,F^\Lambda_{\mu\nu}\,F^{\mu\nu\,\Sigma} + \mathcal{R}_{\Lambda\Sigma}(\phi)\,F^\Lambda_{\mu\nu}\,{}^*F^{\mu\nu\,\Sigma} + \cdots$$

bosonic sector of maximal (N=8) D=4 supergravity

D=4 supergravity: symmetries

$$\mathcal{L} \;=\; R + G_{ij}(\phi)\,\partial_\mu\phi^i\,\partial^\mu\phi^j + \mathcal{I}_{\Lambda\Sigma}(\phi)\,F^{\Lambda}_{\mu\nu}\,F^{\mu\nu\,\Sigma} + \mathcal{R}_{\Lambda\Sigma}(\phi)\,F^{\Lambda}_{\mu\nu}\,{}^*F^{\mu\nu\,\Sigma} + \cdots$$

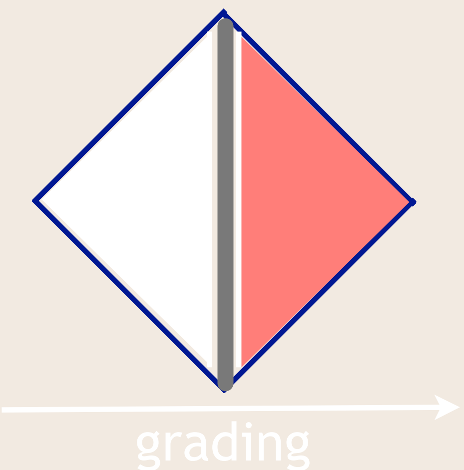
scalar sector: G/H coset space sigma model $\mathrm{E}_7 / \mathrm{SU}(8)$

$$\mathcal{V} \in \mathrm{E}_7 \qquad \mathcal{V} \approx \mathcal{V}\cdot H \qquad H \in \mathrm{SU}(8)$$

triangular gauge

$$\mathcal{V} \;=\; \exp\{\phi^m\,N_m\}\,\exp\{\phi^\lambda\,h_\lambda\}$$

nilpotent ↖ Cartan ↖



E_7 action $\quad \mathcal{V} \longrightarrow G\,\mathcal{V}\,H_{G,\mathcal{V}}$

- shift symmetries $\quad G = \exp\{\lambda^m N_m\}: \quad \phi^m \rightarrow \phi^m + \lambda^m$
- 'hidden' symmetries $\quad G = \exp\{\lambda^m N^\dagger_m\}$ **non-linear! (on ϕ^i) (linear on $\mathcal{V}$)**

D=4 supergravity: self-duality

$$\mathcal{L} \;=\; R + G_{ij}(\phi)\,\partial_\mu\phi^i\,\partial^\mu\phi^j + \mathcal{I}_{\Lambda\Sigma}(\phi)\,F^\Lambda_{\mu\nu}\,F^{\mu\nu\,\Sigma} + \mathcal{R}_{\Lambda\Sigma}(\phi)\,F^\Lambda_{\mu\nu}\,{}^*F^{\mu\nu\,\Sigma} + \cdots$$

self-duality (D=4: electric-magnetic duality for vectors)

field strength: $\mathcal{F}_{\mu\nu}{}^\Lambda = 2\,\partial_{[\mu}A_{\nu]}{}^\Lambda$

dual: $\mathcal{G}_{\mu\nu\,\Lambda} = -\varepsilon_{\mu\nu\rho\sigma}\,\dfrac{\partial\mathcal{L}}{\partial\mathcal{F}_{\rho\sigma}{}^\Lambda}$

Bianchi: $\partial_{[\mu}\mathcal{F}_{\nu\rho]}{}^\Lambda \;=\; 0$

eom: $\partial_{[\mu}\mathcal{G}_{\nu\rho]\,\Lambda} \;=\; 0$

dual vectors: $\mathcal{G}_{\mu\nu\,\Lambda} = 2\partial_{[\mu}A_{\nu]\,\Lambda}$

symplectic rotation

$$\begin{pmatrix}\mathcal{F}^\Lambda\\ \mathcal{G}_\Lambda\end{pmatrix} \longrightarrow \begin{pmatrix}U^\Lambda{}_\Sigma & Z^{\Lambda\Sigma}\\ W_{\Lambda\Sigma} & V_\Lambda{}^\Sigma\end{pmatrix}\begin{pmatrix}\mathcal{F}^\Sigma\\ \mathcal{G}_\Sigma\end{pmatrix}$$

non-local (on A^Λ_μ) !

(local on $(A^\Lambda_\mu, A_{\mu\,\Lambda})$)

choice of an electric frame, analogous pattern for (n—1)-forms in D=2n

E_7 is realized (on-shell) on the combined set of 28 electric +28 magnetic vectors

D=4 supergravity: gauging

$$\mathcal{L} \;=\; R + G_{ij}(\phi)\,\partial_\mu\phi^i\,\partial^\mu\phi^j + \mathcal{I}_{\Lambda\Sigma}(\phi)\,F^{\Lambda}_{\mu\nu}\,F^{\mu\nu\,\Sigma} + \mathcal{R}_{\Lambda\Sigma}(\phi)\,F^{\Lambda}_{\mu\nu}\,{}^*F^{\mu\nu\,\Sigma} + \cdots$$

self-duality (D=4: electric-magnetic duality for vectors)

gauging (embedding tensor)

electric gauging (“standard”)

$$D_\mu = \partial_\mu - A_\mu{}^M\Theta_M{}^\alpha t_\alpha = \partial_\mu - A_\mu{}^\Lambda\Theta_\Lambda{}^\alpha t_\alpha - A_{\mu\,\Lambda}\Theta^{\Lambda\,\alpha}t_\alpha$$

magnetic gauging (“non-standard”)

consistency encoded in a set of algebraic constraints on the embedding tensor $\Theta_M{}^\alpha$

linear: (susy / consistent tensor hierarchy) $\Theta_{(M}{}^\alpha\, t_{\alpha,N}{}^P\,\Omega_{K)P} = 0$

56 x 133 = ~~56~~ + 912 + ~~6480~~

quadratic: (generalized Jacobi / locality)

$$f_{\alpha\beta}{}^\gamma\,\Theta_M{}^\alpha\,\Theta_N{}^\beta + (t_\alpha)_N{}^P\,\Theta_M{}^\alpha\Theta_P{}^\gamma \;=\; 0$$

$$\Longleftrightarrow \qquad \Omega^{MN}\,\Theta_M{}^\alpha\,\Theta_N{}^\beta \;=\; 0$$

D=4 supergravity: gauging

$$\mathcal{L} \;=\; R + G_{ij}(\phi)\,\partial_\mu\phi^i\,\partial^\mu\phi^j + \mathcal{I}_{\Lambda\Sigma}(\phi)\,F^\Lambda_{\mu\nu}\,F^{\mu\nu\,\Sigma} + \mathcal{R}_{\Lambda\Sigma}(\phi)\,F^\Lambda_{\mu\nu}\,{}^*F^{\mu\nu\,\Sigma} + \cdots$$

self-duality (D=4: electric-magnetic duality for vectors)

gauging

electric gauging ("standard")

$$D_\mu = \partial_\mu - A_\mu{}^M\Theta_M{}^\alpha t_\alpha = \partial_\mu - A_\mu{}^\Lambda\Theta_\Lambda{}^\alpha t_\alpha - A_{\mu\,\Lambda}\Theta^{\Lambda\,\alpha}t_\alpha$$

magnetic gauging ("non-standard")

off-shell formulation

$$\mathcal{L}_{\rm top} \;=\; -\tfrac{1}{8}\Theta^{\Lambda\alpha}\,B_\alpha\wedge\Big(2\,\partial A_\Lambda + X_{MN\Lambda}\,A^M\wedge A^N - \tfrac{1}{4}\Theta_\Lambda{}^\beta B_\beta\Big) \;+\; \cdots$$

upon introduction of additional two-forms (dual to scalars)
and BF couplings
gauging of on-shell symmetries

[de Wit, HS, Trigiante]

D=2 supergravity

affine symmetries

D=2 supergravity ungauged

Lagrangian

$$\mathcal{L} \;=\; -\tfrac{1}{4}\sqrt{-g}\,\rho\Big(-R+\mathrm{tr}[P^\mu P_\mu]\Big) \;+\; \mathcal{L}_{\rm ferm}(\psi^I,\psi_2^I,\chi^{\dot{A}})$$

coset space sigma model coupled to dilaton gravity $\quad \mathcal{V}^{-1}\partial_\mu\mathcal{V} = Q_\mu + P_\mu$

off-shell symmetry (target space isometries): E_8

field equations

dilaton

$$\Box\rho \;=\; 0$$

scalars

$$\partial_\mu J^\mu_M = 0 \qquad J_\mu \equiv \rho\mathcal{V}P_\mu\mathcal{V}^{-1}$$

conserved E_8 Noether current

has a remarkable structure :

(infinite tower of) dual scalar potentials

⟶ classical integrability, affine Lie-Poisson symmetry E_9

duality

$$\partial_\mu\tilde{\rho} = \epsilon_{\mu\nu}\,\partial^\nu\rho$$ dual dilaton

$$\partial_\mu Y_M \;\equiv\; \varepsilon_{\mu\nu}\,J^\nu_M$$ dual scalars

dual (D-2) forms

D=2 supergravity ungauged

duality $\partial_\mu \tilde{\rho} = \epsilon_{\mu\nu}\, \partial^\nu \rho$ dual dilaton

$\partial_\mu Y_M \;\equiv\; \varepsilon_{\mu\nu}\, J^\nu_M$ dual scalars

dual (D-2) forms

classical integrability, affine Lie-Poisson symmetry E_9

---

shift symmetries $\delta_1\, \tilde{\rho} = \lambda$ (1) $\Lambda^\alpha \delta_{\alpha,1}\, Y_1 = \Lambda$ (248)

$\Lambda^\alpha \delta_{\alpha,1}\, \mathcal{V} \;=\; 0$

'hidden' symmetries $\Lambda^\alpha \delta_{\alpha,-1}\, \mathcal{V} = [\Lambda, Y_1]\mathcal{V} - \tilde{\rho}\, \mathcal{V}[\mathcal{V}^{-1}\Lambda\mathcal{V}]_{\mathfrak{p}}$ (248)

---

extends to an infinite tower:

dual scalars

$$\partial_\pm Y_2 = \left(\pm\rho\tilde{\rho} + \frac{1}{2}\rho^2\right) \mathcal{V}P_\pm\mathcal{V}^{-1} + \frac{1}{2}[Y_1, \partial_\pm Y_1]\;,$$

$$\partial_\pm Y_3 = \left(\mp\frac{1}{2}\rho^3 \mp \rho\tilde{\rho}^2 - \rho^2\tilde{\rho}\right) \mathcal{V}P_\pm\mathcal{V}^{-1} + [Y_1, \partial_\pm Y_2] - \frac{1}{6}[Y_1, [Y_1, \partial_\pm Y_1]]]$$

'hidden' symmetries

$$\Lambda^\alpha \delta_{\alpha,-2}\, \mathcal{V} = \left\{[\Lambda, Y_2] + \frac{1}{2}[[\Lambda, Y_1], Y_1] - \tilde{\rho}[\Lambda, Y_1]\right\} \mathcal{V} + \left(\frac{1}{2}\rho^2 + \tilde{\rho}^2\right) \mathcal{V}[\mathcal{V}^{-1}\Lambda\mathcal{V}]_{\mathfrak{p}} \qquad (248)$$

etc...

close into (half of) the affine algebra !

D=2 supergravity ungauged

linear system

the equations of motion can be encoded as integrability conditions
of a linear system [Belinskii, Zakharov / Maison / Julia / Nicolai, Warner]

$$\hat{\mathcal{V}}^{-1}\partial_\pm\hat{\mathcal{V}} \;=\; Q_\pm + \frac{1\mp\gamma}{1\pm\gamma}\,P_\pm \qquad \text{(light-cone-coord. } x^\pm\text{)}$$

for a group-valued function $\hat{\mathcal{V}}(\gamma)$ and the spectral parameter

$$\gamma = \frac{1}{\rho}\left(w+\tilde{\rho}-\sqrt{(w+\tilde{\rho})^2-\rho^2}\right)$$

expansion in w gives rise to the infinite series of dual scalars

$$\hat{\mathcal{V}} \;=\; \ldots e^{Y_3 w^{-3}} e^{Y_2 w^{-2}} e^{Y_1 w^{-1}}\mathcal{V}$$

$$\begin{aligned}
\partial_\pm Y_1 &= \pm\rho\mathcal{V}P_\pm\mathcal{V}^{-1}\\
\partial_\pm Y_2 &= -(\pm\rho\tilde{\rho}+\tfrac{1}{2}\rho^2)\mathcal{V}P_\pm\mathcal{V}^{-1}+\tfrac{1}{2}[Y_1,\partial_\pm Y_1]\\
\partial_\pm Y_3 &= \ldots
\end{aligned}$$

D=2 supergravity ungauged

affine symmetry group E₉

action parametrized by a meromorphic function $\Lambda(w)$

$$\delta\sigma = \kappa - \mathrm{tr}\left\langle \Lambda(w)\,\partial_w\hat{\mathcal{V}}(w)\hat{\mathcal{V}}^{-1}(w)\right\rangle_w$$

$$\mathcal{V}^{-1}\,\delta\mathcal{V} = \left\langle \frac{2\gamma(w)}{\rho\,(1-\gamma(w)^2)}\,\tilde{\Lambda}_{\mathfrak{k}}(w)\right\rangle_w$$

$$\hat{\mathcal{V}}^{-1}\Lambda(w)\hat{\mathcal{V}} \;=\; \tilde{\Lambda}_{\mathfrak{h}} + \tilde{\Lambda}_{\mathfrak{k}}$$

$$\langle f(w)\rangle_w \;\equiv\; \oint \frac{dw}{2\pi i}\,f(w)$$

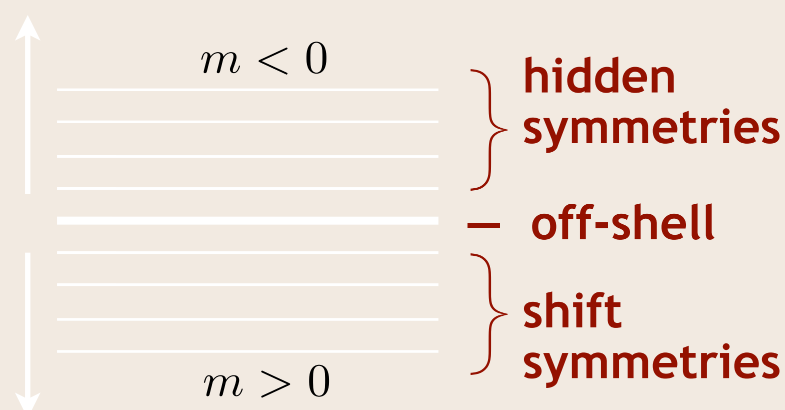
extends to the set of dual scalars

$$\delta\tilde{\rho} = \lambda$$

$$\hat{\mathcal{V}}^{-1}\delta\hat{\mathcal{V}}(w) = \lambda\,\hat{\mathcal{V}}^{-1}\partial_w\hat{\mathcal{V}}(w) + \tilde{\Lambda}(w) - \left\langle \frac{1}{v-w}\left(\tilde{\Lambda}_{\mathfrak{h}}(v) + \frac{\gamma(v)\,(1-\gamma^2(w))}{\gamma(w)\,(1-\gamma^2(v))}\tilde{\Lambda}_{\mathfrak{k}}(v)\right)\right\rangle_v$$

coset action E₉ / K(E₉)

$$\{t^\alpha_m,\, L_1,\, k\}$$



Virasoro $L_1\,\tilde{\rho} = 1$

central extension $k\,\sigma = 1$ [Julia]

⟶ deformations : gauge part of this nonlinear, nonlocal, on-shell symmetry

D=2 supergravity

deformations

[HS, Martin Weidner]

gauging D=2 supergravity

gauged Lagrangian

$$\mathcal{L} \;=\; \partial^\mu \rho\, D_\mu \sigma - \tfrac{1}{2}\rho\, \mathrm{tr}(\mathcal{P}_\mu \mathcal{P}^\mu) \;+\; \mathcal{L}_{\rm top}$$

gauged sigma-model, with covariantized derivatives (embedding tensor)

$$D_\mu \;=\; \partial_\mu - A_\mu^{\mathcal{M}} \Theta_{\mathcal{M}}{}^{\mathcal{A}}\, t_{\mathcal{A}} \;=\; \partial_\mu - A_\mu^\alpha\, t_\alpha - B_\mu\, L_1 - C_\mu\, k$$

and a "topological" term

$$\begin{aligned}\mathcal{L}_{\rm top} \;=\; & +\epsilon^{\mu\nu}\mathrm{tr}\Big\langle A_\mu(w)\,(\partial_\nu \hat{\mathcal{V}} - \hat{\mathcal{V}} Q_\nu)\,\hat{\mathcal{V}}^{-1} - \frac{1+\gamma^2}{1-\gamma^2}\, A_\mu(w) \hat{\mathcal{V}} P_\nu \hat{\mathcal{V}}^{-1}\Big\rangle_w + \epsilon^{\mu\nu}\Big(C_\mu - \mathrm{tr}\Big\langle A_\mu(w)\,\partial_w \hat{\mathcal{V}}(w)\hat{\mathcal{V}}^{-1}(w)\Big\rangle_w\Big)\partial_\nu\tilde{\rho} \\ & - \tfrac{1}{2}\epsilon^{\mu\nu} C_\mu B_\nu + \tfrac{1}{2}\,\epsilon^{\mu\nu}\mathrm{tr}\Big\langle\Big\langle \frac{1}{v-w}[\tilde{A}_\mu(w)]_{\mathfrak{h}}[\tilde{A}_\nu(v)]_{\mathfrak{h}} + \frac{(\gamma(v)-\gamma(w))^2 + (1-\gamma(v)\gamma(w))^2}{(v-w)(1-\gamma(v))^2(1-\gamma(w))^2}[\tilde{A}_\mu(w)]_{\mathfrak{k}}[\tilde{A}_\nu(v)]_{\mathfrak{k}}\Big\rangle_v\Big\rangle_w\end{aligned}$$

the Lagrangian carries dual scalars and vector fields (topological) such that variation w.r.t. the vector fields yields the linear system!

$$\delta\mathcal{L} \;=\; \delta C^{\pm}\,(\partial_\pm \rho \mp \partial_\pm \tilde{\rho}) + \mathrm{tr}\,\Big\langle \delta A^{\pm}\,(\partial_\pm \hat{\mathcal{V}}\hat{\mathcal{V}}^{-1} - \hat{\mathcal{V}}(Q_\pm + \frac{1\mp\gamma}{1\pm\gamma}\,P_\pm)\hat{\mathcal{V}}^{-1})\Big\rangle_w$$

and part of the former on-shell symmetry is gauged!

simplest case : gauging of target-space isometries E8
(theories of D=3 origin...)

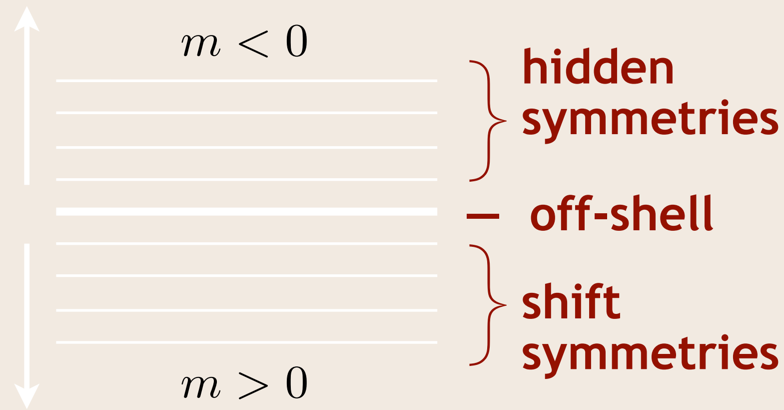
$$\mathcal{L}_{\rm top} \;=\; \epsilon^{\mu\nu} F_{\mu\nu}{}^M\, \Theta_{MN}\, Y^N + \ldots$$

[Hull, Spence]

gauging D=2 supergravity

group theory (for consistent deformations)

affine global symmetry $\{t^{\alpha}_{m},\, L_1,\, k\}$



- **vector fields** (nonpropagating in D=2)
  restore by embedding known examples: basic representation of **E₉**

$$\chi_{\omega 0} \;=\; 1 + 248q + 4124q^2 + 34752q^3 + 213126q^4 + 1057504q^5 + 4530744q^6 + \ldots$$

McKay-Thompson series of class 3C for the monster

- **embedding tensor** linear constraint:
  transforms in the **dual** representation

$$\Lambda_{\rm adj} \otimes \Lambda_1 \;=\; \boxed{\Lambda_1} \oplus \cdots$$

$$D_\mu \;=\; \partial_\mu - A_\mu^{\mathcal{M}} \Theta_{\mathcal{M}}{}^{\mathcal{A}}\, t_{\mathcal{A}} \;=\; A_\mu^{\mathcal{M}} \Theta_{\mathcal{N}}\, \eta^{\mathcal{A}\mathcal{B}}\, t_{\mathcal{A}\,\mathcal{M}}{}^{\mathcal{N}}\, t_{\mathcal{B}}$$

➜ infinite-dimensional parameter space of deformations!

gauging D=2 supergravity

quadratic constraint

$$f_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}}\,\Theta_{\mathcal{M}}{}^{\mathcal{A}}\,\Theta_{\mathcal{N}}{}^{\mathcal{B}} + t_{\mathcal{A}\mathcal{N}}{}^{\mathcal{P}}\,\Theta_{\mathcal{M}}{}^{\mathcal{A}}\Theta_{\mathcal{P}}{}^{\mathcal{C}} \;=\; 0$$

lives in the tensor product $\Theta\,\Theta$

$$\chi_{\omega 0}\,\chi_{\omega 0} \;=\; (1+q^2+q^3+q^4+2q^5+\ldots)\,\chi_{2\omega 0} + (1+q+q^2+\ldots)\,\chi_{\omega 7}$$

structure of multiplicities organized by

$$\frac{\mathbf{E_9}_{\,1} \oplus \mathbf{E_9}_{\,1}}{\mathbf{E_9}_{\,2}}$$

coset CFT:
Ising model

$$\chi_{\omega 0}\,\chi_{\omega 0} \;=\; \chi^{\rm vir}_{(1,1)}\,\chi_{2\omega 0} + \chi^{\rm vir}_{(2,1)}\,\chi_{\omega 7}$$

quadratic constraint translates into

$$\eta^{\mathcal{A}\mathcal{B}}\,t_{\mathcal{A}\,\mathcal{M}}{}^{\mathcal{P}}t_{\mathcal{B}\,\mathcal{N}}{}^{\mathcal{Q}}\,\Theta_{\mathcal{P}}\Theta_{\mathcal{Q}} \;=\; (L_1^G - L_1^H)\,\Theta\,\Theta \;=\; L_1^{G/H}\,\Theta\,\Theta \;\equiv\; 0$$

quasiprimary states in the tensor product

- every **Θ** satisfying this constraint defines a consistent deformation

gauging D=2 supergravity

embedding tensor — basic representation

branching under E8 identifies gaugings of 3d origin

$$D_\mu \equiv \partial_\mu - g A_\mu{}^{\mathcal{M}} \Theta_{\mathcal{M}}{}^{\mathcal{A}} T_{\mathcal{A}} \qquad \Theta_{\mathcal{M}}{}^{\mathcal{A}} = (T_{\mathcal{B}})_{\mathcal{M}}{}^{\mathcal{N}} \eta^{\mathcal{AB}} \theta_{\mathcal{N}}$$

$\theta_{\mathcal{M}}$

- 1 ▹ **flux of 3d Kaluza-Klein vector field**
- 248 ▹ **Scherk-Schwarz reductions from 3d**
- $1 + 248 + 3875$ ▹ **torus reduction of 3d gaugings**
- $1 + 2 \cdot 248 + 3875 + 30380$ ▹ ...?
- $2 \cdot 1 + 3 \cdot 248 + 2 \cdot 3875 + 30380 + 27000 + 147250$ ▹ ...??
- $\vdots$ ▹ ...???

quadratic constraint ...

gauging D=2 supergravity

embedding tensor — basic representation

$$D_\mu \equiv \partial_\mu - g A_\mu{}^{\mathcal{M}} \Theta_{\mathcal{M}}{}^{\mathcal{A}} T_{\mathcal{A}} \qquad \Theta_{\mathcal{M}}{}^{\mathcal{A}} = (T_{\mathcal{B}})_{\mathcal{M}}{}^{\mathcal{N}} \eta^{\mathcal{A}\mathcal{B}} \theta_{\mathcal{N}}$$

$\theta_{\mathcal{M}}$

1

248

$1 + 248 + 3875$

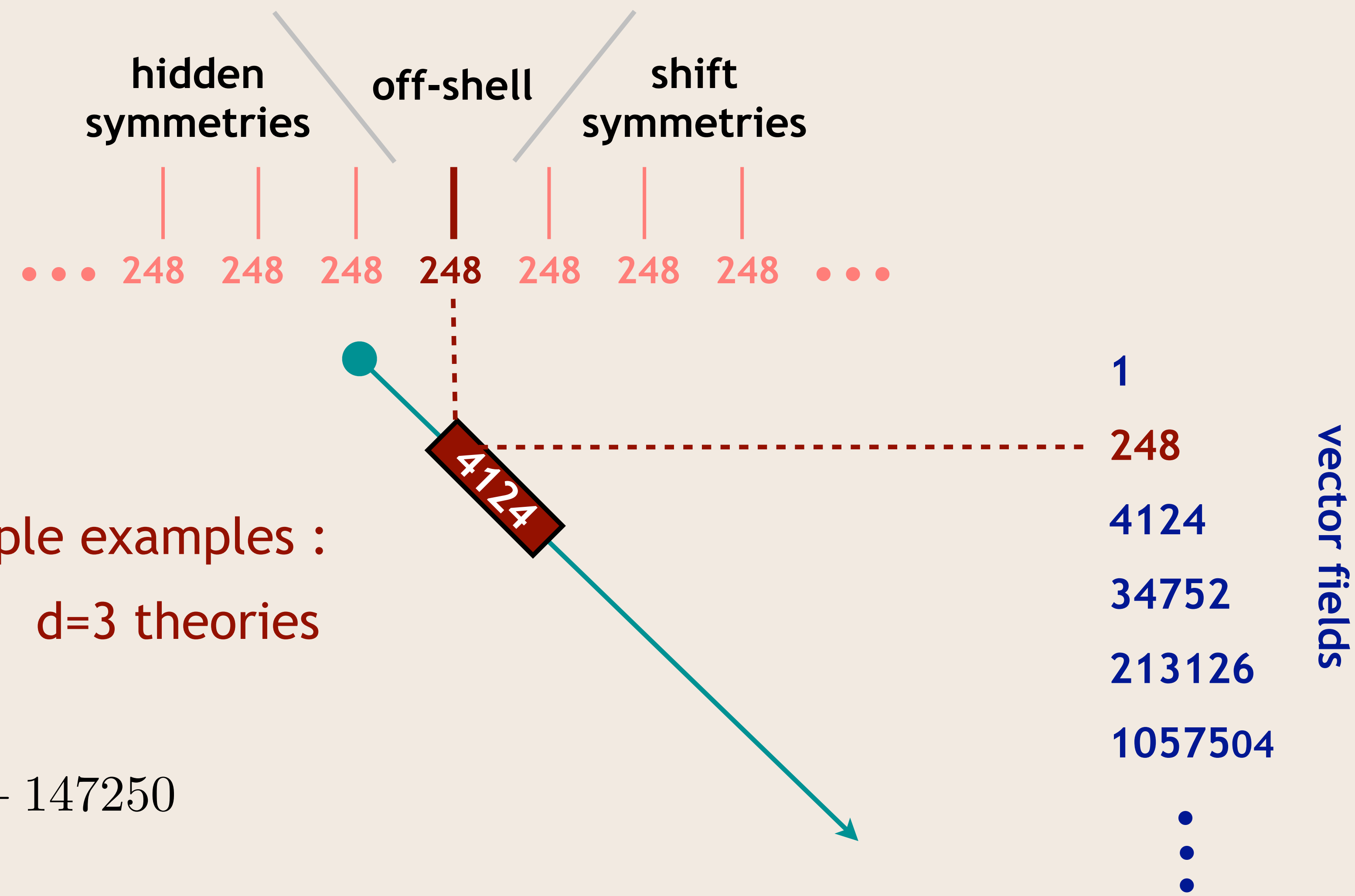
$1 + 2 \cdot 248 + 3875 + 30380$

$2 \cdot 1 + 3 \cdot 248 + 2 \cdot 3875 + 30380 + 27000 + 147250$

$\vdots$

simple examples :

⟶ d=3 theories



gauging D=2 supergravity

embedding tensor — basic representation

$$D_\mu \equiv \partial_\mu - g A_\mu{}^{\mathcal{M}} \Theta_{\mathcal{M}}{}^{\mathcal{A}} T_{\mathcal{A}} \qquad \Theta_{\mathcal{M}}{}^{\mathcal{A}} = (T_{\mathcal{B}})_{\mathcal{M}}{}^{\mathcal{N}} \eta^{\mathcal{AB}} \theta_{\mathcal{N}}$$

$\theta_{\mathcal{M}}$

1

248

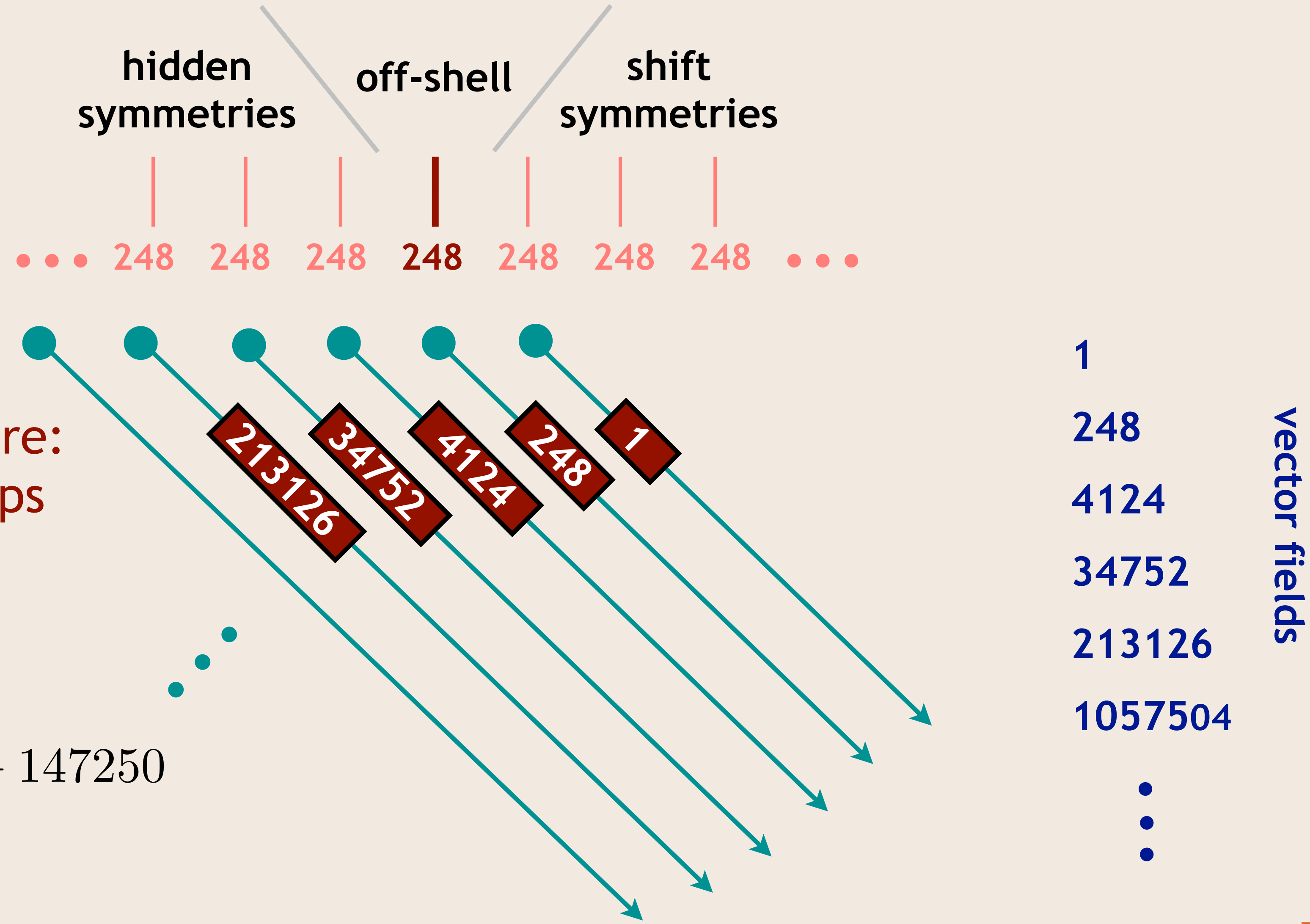
$1 + 248 + 3875$

$1 + 2 \cdot 248 + 3875 + 30380$

$2 \cdot 1 + 3 \cdot 248 + 2 \cdot 3875 + 30380 + 27000 + 147250$

⋮

the full structure:
inf-dim HW reps



gauging D=2 supergravity

embedding tensor — basic representation

$$D_\mu \equiv \partial_\mu - g A_\mu{}^{\mathcal{M}} \Theta_{\mathcal{M}}{}^{\mathcal{A}} T_{\mathcal{A}} \qquad \Theta_{\mathcal{M}}{}^{\mathcal{A}} = (T_{\mathcal{B}})_{\mathcal{M}}{}^{\mathcal{N}} \eta^{\mathcal{A}\mathcal{B}} \theta_{\mathcal{N}}$$

$\theta_{\mathcal{M}}$

1

248

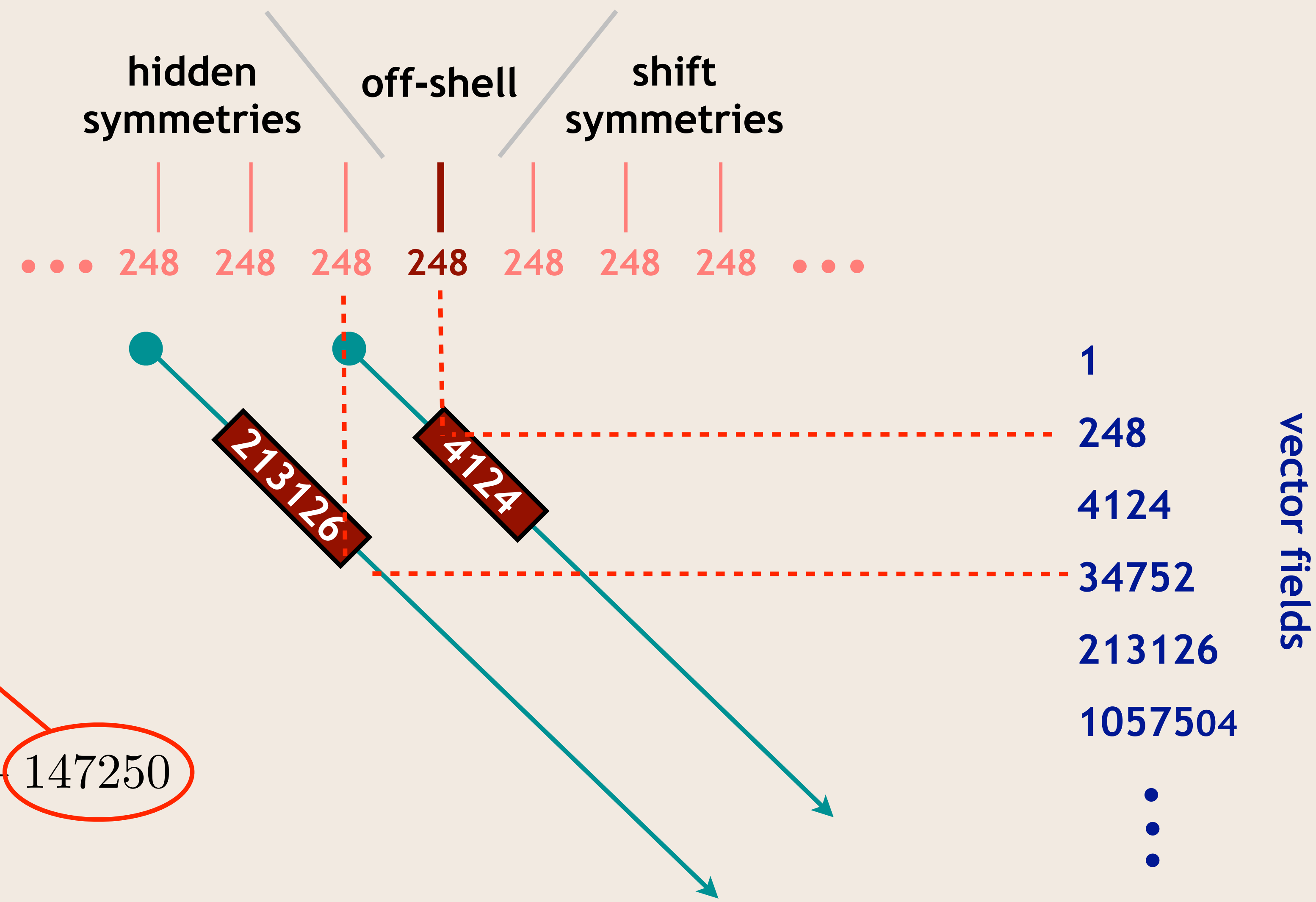
$1 + 248 + 3875$ → 1

$1 + 2 \cdot 248 + 3875 + 30380$

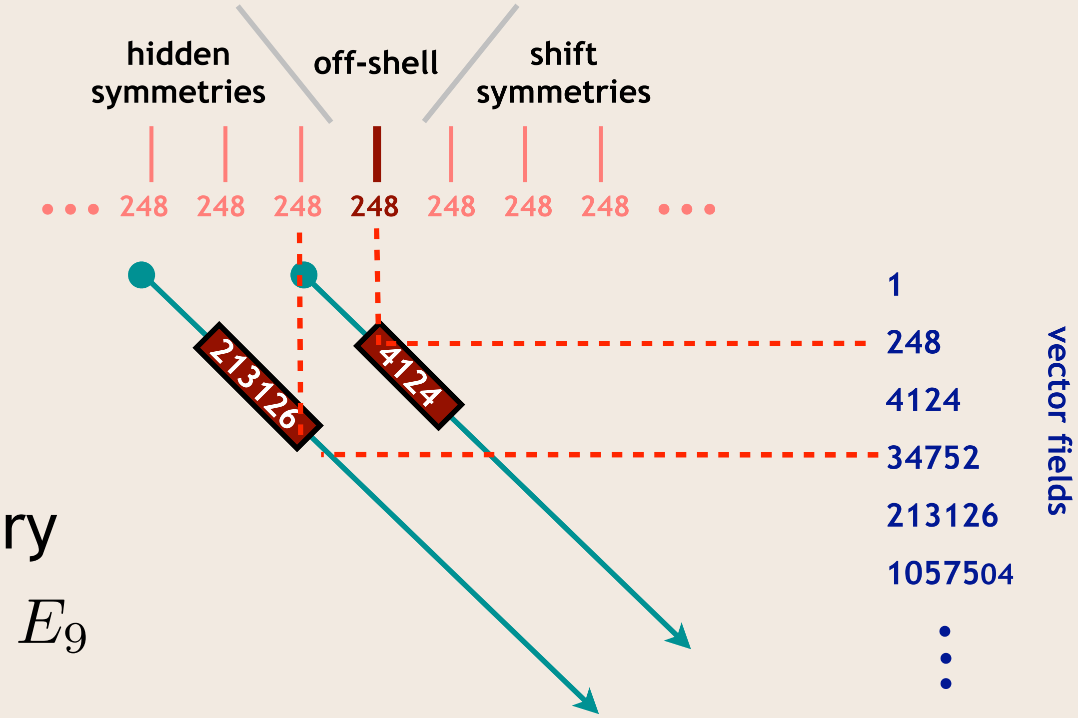
$2 \cdot 1 + 3 \cdot 248 + 2 \cdot 3875 + 30380 + 27000 + 147250$ → 1

⋮

new example:
the **SO(9)** theory



gauging D=2 supergravity



- the SO(9) theory is a genuine d=2 theory

  in particular $SO(9) \not\subset E_8$ but $SO(9) \subset E_9$

- the full gauge group is infinite-dimensional (shift symmetries)

- the theory in the “E_8 frame” looks rather miserable

  in particular the gauge group is $G = \mathrm{SO}(8) \ltimes \Big(\underbrace{(\mathbb{R}_+^{28} \times \mathbb{R}_+^{8})_0}_{\text{off-shell}} \times \underbrace{(\mathbb{R}_+^{8})_{-1}}_{\text{hidden (on-shell)}} \Big)$

- still the Yukawa couplings and the scalar potential are missing

example : SO(9) supergravity [H.S., Thomas Ortiz]

- go to a “T-dual frame” in which SO(9) is among the off-shell symmetries
- the proper embedding of the gauge group :

$$SO(9) \not\subset E_8$$

but $$SO(9) \subset SL(9) \subset \widehat{SL(9)} \subset E_9$$

SO(9) supergravity

affine E_8 with L_0 grading

shift	248_{+3}
shift	248_{+2}
shift	248_{+1}
off-shell	248_{0}
hidden	248_{-1}
hidden	248_{-2}

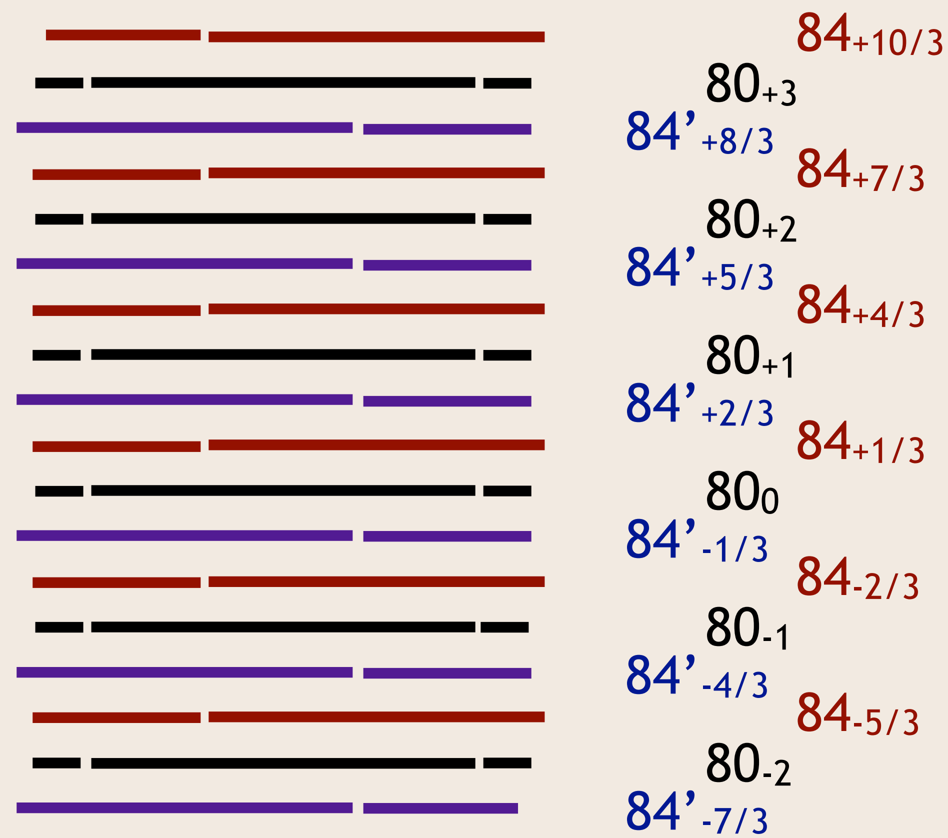
$$SO(9) \not\subset E_8$$

but $$SO(9) \subset SL(9) \subset \widehat{SL(9)} \subset E_9$$

SO(9) supergravity

affine E_8 with $\widehat{L_0}$ grading : decomposition under $\widehat{SL}(9)$

off-shell symmetry $SL(9) \ltimes \mathbb{T}^{84}$ {



"T-dual frame" : change some of the target space coordinates for their duals

coset sigma model $E_8/SO(16)$ $\longrightarrow$ coset sigma model $\left(SL(9) \ltimes \mathbb{T}^{84}\right)/SO(9)$ with WZ term

SO(9) supergravity

"T-dual frame" :

coset sigma model $\left(SL(9) \ltimes \mathbb{T}^{84}\right)/SO(9)$ with WZ term

$$
\begin{aligned}
\mathcal{L}_0 \;=\;& -\frac{1}{4}\rho R + \frac{1}{4}\rho\, P^{\mu\,ab} P_\mu^{ab} + \frac{1}{12}\,\rho^{1/3}\, M_{il} M_{jm} M_{kn}\, \partial^\mu \phi^{ijk} \partial_\mu \phi^{lmn} \\
& + \frac{1}{648}\,\varepsilon^{\mu\nu} \varepsilon_{klmnpqrst}\, \phi^{klm}\, \partial_\mu \phi^{npq}\, \partial_\nu \phi^{rst}
\end{aligned}
$$

target space :

SL(9)/SO(9) coset currents $P_\mu^{ab} = (\mathcal{V}^{-1}\partial_\mu \mathcal{V})^{(ab)}$

84 extra scalars ϕ^{abc} with kinetic matrix $M = \mathcal{V}\mathcal{V}^T$

and WZ term $84 \wedge 84 \wedge 84 \longrightarrow 1$

in fact this is the d=11 theory reduced on a torus T^9 ...

SO(9) supergravity

"T-dual frame" :

coset sigma model $\left(SL(9) \ltimes \mathbb{T}^{84}\right) / SO(9)$ with WZ term

$$
\begin{aligned}
\mathcal{L}_0 \;=\;& -\frac{1}{4}\rho R + \frac{1}{4}\rho\, P^{\mu\,ab} P_\mu^{ab} + \frac{1}{12}\,\rho^{1/3}\, M_{il} M_{jm} M_{kn}\, \partial^\mu \phi^{ijk} \partial_\mu \phi^{lmn} \\
& + \frac{1}{648}\, \varepsilon^{\mu\nu} \varepsilon_{klmnpqrst}\, \phi^{klm}\, \partial_\mu \phi^{npq}\, \partial_\nu \phi^{rst}
\end{aligned}
$$

fermionic part :

$$
\begin{aligned}
& -\rho e^{-1} \varepsilon^{\mu\nu} \bar{\psi}_2^I D_\mu \psi_\nu^I - \frac{i}{2}\, \bar{\psi}_\nu^I \gamma^\nu \psi_\mu^I\, \partial^\mu \rho - \frac{i}{2}\, \rho\, \bar{\chi}^{aI} \gamma^\mu D_\mu \chi^{aI} + \frac{i}{2}\, \rho^{2/3}\, \bar{\chi}^{aI} \gamma^3 \gamma^\mu \chi^{bJ} \Gamma^c_{IJ}\, \varphi_\mu^{abc} - \frac{i}{24}\, \rho^{2/3}\, \bar{\chi}^{aI} \gamma^3 \gamma^\mu \chi^{aJ} \Gamma^{bcd}_{IJ}\, \varphi_\mu^{bcd} \\
& -\frac{1}{4}\, \rho^{2/3}\, \bar{\chi}^{aI} \gamma^3 \gamma^\nu \gamma^\mu \psi_\nu^J \Gamma^{bc}_{IJ}\, \varphi_\mu^{abc} - \frac{i}{12}\, \rho^{2/3}\, \bar{\chi}^{aI} \gamma^\mu \psi_2^J \Gamma^{bc}_{IJ}\, \varphi_\mu^{abc} - \frac{1}{2}\, \rho\, \bar{\chi}^{aI} \gamma^\nu \gamma^\mu \psi_\nu^J \Gamma^b_{IJ} P_\mu^{ab} - \frac{i}{2}\, \rho\, \bar{\chi}^{aI} \gamma^3 \gamma^\mu \psi_2^J \Gamma^b_{IJ} P_\mu^{ab} \\
& + \frac{i}{54}\, \rho^{2/3}\, \bar{\psi}_2^I \gamma^3 \gamma^\mu \psi_2^J \Gamma^{abc}_{IJ}\, \varphi_\mu^{abc} + \frac{1}{24}\, \rho^{2/3}\, \bar{\psi}_2^I \left(\gamma^\mu \gamma^\nu - \frac{1}{3} \gamma^\nu \gamma^\mu\right) \psi_\nu^J\, \Gamma^{abc}_{IJ}\, \varphi_\mu^{abc}
\end{aligned}
$$

off-shell symmetry $SL(9) \ltimes \mathbb{T}^{84}$ $\supset\ SO(9)$ gauging

SO(9) supergravity

"T-dual frame" :

gauged coset sigma model $\left(SL(9) \ltimes \mathbb{T}^{84}\right)/SO(9)$ with WZ term

$$
\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{4}\rho R + \frac{1}{4}\rho\, P^{\mu\,ab} P_\mu^{ab} + \frac{1}{12}\,\rho^{1/3}\, M_{il} M_{jm} M_{kn}\, D^\mu \phi^{ijk} D_\mu \phi^{lmn} \\
& + \frac{1}{648}\,\varepsilon^{\mu\nu} \varepsilon_{klmnpqrst}\, \phi^{klm}\, D_\mu \phi^{npq}\, D_\nu \phi^{rst}
\end{aligned}
$$

fermion couplings and Yukawa terms

$$
\begin{aligned}
{}^{-1}\mathcal{L}_{\rm Yuk} \;=\; & -\frac{1}{2}\, e^{-1} \rho\, \varepsilon^{\mu\nu} \left(\bar\psi^I_\nu \psi^J_\mu B_{IJ} + \bar\psi^I_\nu \gamma^3 \psi^J_\mu \tilde B_{IJ} - 2i \bar\psi^I_2 \gamma_\nu \psi^J_\mu A_{IJ}\right) + i\rho\, \bar\psi^I_2 \gamma^\mu \psi^J_\mu \tilde A_{IJ} \\
& + i\rho\, \bar\chi^{aI} \gamma^\mu \psi^J_\mu C^a_{IJ} \; - i\rho\, \bar\chi^{aI} \gamma^3 \gamma^\mu \psi^J_\mu \tilde C^a_{IJ} \; + \rho\, \bar\psi^I_2 \psi^J_2 D_{IJ} \; + \rho\, \bar\psi^I_2 \gamma^3 \psi^J_2 \tilde D_{IJ} \\
& + \rho\, \bar\chi^{aI} \psi^J_2 E^a_{IJ} \; + \rho\, \bar\chi^{aI} \gamma^3 \psi^J_2 \tilde E^a_{IJ} \; + \rho\, \bar\chi^{aI} \chi^{bJ} F^{ab}_{IJ} \; + \rho\, \bar\chi^{aI} \gamma^3 \chi^{bJ} \tilde F^{ab}_{IJ}
\end{aligned}
$$

$$
\begin{aligned}
A_{IJ} &= \frac{7}{9}\,\delta_{IJ}\, b - \frac{5}{9}\,\Gamma^a_{IJ}\, b^a + \frac{1}{9}\,\Gamma^{abcd}_{IJ}\, b^{abcd}\,, \\
\tilde A_{IJ} &= \frac{2}{9}\,\Gamma^{ab}_{IJ}\, b^{ab} - \frac{4}{9}\,\Gamma^{abc}_{IJ}\, b^{abc}\,, \\
B_{IJ} &= \Gamma^{ab}_{IJ}\, b^{ab} + \Gamma^{abc}_{IJ}\, b^{abc}\,, \\
\tilde B_{IJ} &= \delta_{IJ}\, b + \Gamma^a_{IJ}\, b^a + \Gamma^{abcd}_{IJ}\, b^{abcd}\,, \\
C^a_{IJ} &= \frac{8}{9}\,\delta_{IJ}\, b^a - \frac{1}{9}\Gamma^{ab}_{IJ}\, b^b + \frac{20}{9}\,\Gamma^{bcd}_{IJ}\, b^{abcd} - \frac{4}{9}\,\Gamma^{abcde}_{IJ}\, b^{bcde} + c^{ab}\,\Gamma^b_{IJ}\,, \\
\tilde C^a_{IJ} &= -\frac{14}{9}\,\Gamma^b_{IJ}\, b^{ab} + \frac{2}{9}\,\Gamma^{abc}_{IJ}\, b^{bc} + \frac{2}{3}\,\Gamma^{bc}_{IJ}\, b^{abc} - \frac{1}{9}\,\Gamma^{abcd}_{IJ}\, b^{bcd} + c^{a,bc}\,\Gamma^{bc}_{IJ}\,,
\end{aligned}
$$

$$
\begin{aligned}
b &= \frac{1}{4}\,\rho^{-2/9}\, T\,, \\
b^a &= -\rho^{-14/9}\, \mathcal{V}^{-1km}{}_{bc}\, \theta_{ml}\, \varphi^{abc} Y_k{}^l + \frac{1}{144}\,\rho^{-14/9}\, \varepsilon^{bcdefghij} T^{kl} \varphi^{kef} \varphi^{lgh} \varphi^{aij} \varphi^{bcd} \\
b^{ab} &= -\frac{1}{2}\,\rho^{-11/9}\, \mathcal{V}^{-1[km]}{}_{ab}\, \theta_{ml} Y_k{}^l + \frac{1}{144}\,\rho^{-11/9}\, \varepsilon^{abcdefghi} T^{jk} \varphi^{jcd} \varphi^{kef} \varphi^{ghi}\,, \\
b^{abc} &= \frac{1}{4}\,\rho^{-5/9}\, T^{d[a} \varphi^{bc]d}\,, \\
b^{abcd} &= -\frac{1}{8}\,\rho^{-8/9}\, T^{ef} \varphi^{e[ab} \varphi^{cd]f}\,, \\
c^{ab} &= -\frac{1}{2}\,\rho^{-2/9} \left(T^{ab} - \frac{1}{9}\delta^{ab} T\right), \\
c^{a,bc} &= \frac{1}{3}\,\rho^{-5/9} \left(T^{da} \varphi^{bcd} + T^{d[b} \varphi^{c]da}\right)\,,
\end{aligned}
$$

(

SO(9) supergravity

"T-dual frame" :

gauged coset sigma model $\left(SL(9) \ltimes \mathbb{T}^{84}\right)/SO(9)$ with WZ term

$$\mathcal{L} = -\frac{1}{4}\rho R + \frac{1}{4}\rho\, P^{\mu\, ab} P_\mu^{ab} + \frac{1}{12}\,\rho^{1/3}\, M_{il} M_{jm} M_{kn}\, D^\mu \phi^{ijk} D_\mu \phi^{lmn}$$
$$+\frac{1}{648}\,\varepsilon^{\mu\nu}\varepsilon_{klmnpqrst}\,\phi^{klm}\, D_\mu \phi^{npq}\, D_\nu \phi^{rst}$$

vector fields couple via

$$\mathcal{L}_F = \varepsilon^{\mu\nu}\, F_{\mu\nu}{}^{mn}\, \mathcal{Y}_{mn}$$

with auxiliary (dual scalar) fields $\mathcal{Y}_{mn}$

which also enter Yukawa couplings and scalar potential

scalar potential

$$V_{\rm pot} = \frac{1}{8}\,\rho^{5/9}\left((\mathrm{tr}\,T)^2 - 2\,\mathrm{tr}(T^2) + 18\,\rho^{-2/3}\,T^{d[a}\varphi^{bc]d}T^{ea}\varphi^{bce} - 16\,\rho^{-2/3}\,T^{d[b}\varphi^{c]ad}T^{eb}\varphi^{cae}\right)$$
$$-\rho^{-13/9}\,T^{ac}T^{bc}\,Y_{ad}Y_{bd} + \mathcal{O}(\phi^3)$$

$T \equiv \left(\mathcal{V}^{\rm T}\mathcal{V}\right)^{-1}$

$\varphi \equiv \phi\cdot\mathcal{V}$

$Y \equiv \mathcal{V}^{\rm T}\mathcal{Y}\,\mathcal{V}$

eighth order polynomial in ϕ

the dilaton powers precisely support the correct DW solution (near horizon of AdS_2 x S^8)

SO(9) supergravity

different presentations

gauged coset sigma model $\left(SL(9) \ltimes \mathbb{T}^{84}\right)/SO(9)$ with WZ term

$$\mathcal{L} = -\frac{1}{4}\rho R + \frac{1}{4}\rho\, P^{\mu\, ab} P_\mu^{ab} + \frac{1}{12}\,\rho^{1/3}\, M_{il} M_{jm} M_{kn}\, D^\mu \phi^{ijk} D_\mu \phi^{lmn}$$
$$+\frac{1}{648}\,\varepsilon^{\mu\nu}\varepsilon_{klmnpqrst}\,\phi^{klm}\, D_\mu \phi^{npq}\, D_\nu \phi^{rst} + \varepsilon^{\mu\nu}\, F_{\mu\nu}{}^{mn}\, \mathcal{Y}_{mn} + V_{\rm pot}(\rho, \mathcal{V}, \phi, \mathcal{Y})$$

integrate out the auxiliary scalars $\mathcal{Y}_{mn}$

upon using their field equations $F_{\mu\nu}{}^{mn} = \frac{\partial \mathcal{L}}{\partial \mathcal{Y}_{mn}} + \text{fermions}$

leads to

$$\mathcal{L}_2 = -\frac{1}{4}\rho R + F_{\mu\nu}{}^{mn} F^{\mu\nu\, kl}\, \mathcal{R}_{mn,kl}(\rho, \mathcal{V}, \phi) + \ \ldots\ + \tilde{V}_{\rm pot}(\rho, \mathcal{V}, \phi)$$

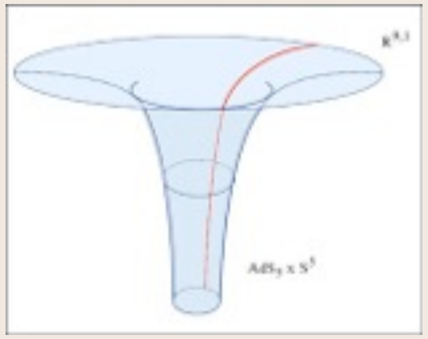
gauged sigma model coupled to d=2 SYM

the $U(1)^4$ truncation can be shown to arise as consistent truncation from IIA

concluding

affine symmetries in supergravity

- general structure of deformations of the two-dimensional theory
- truly affine structures at work (basic representation of the embedding tensor)
- maximally supersymmetric d=2 supergravity with gauge group SO(9)
- last missing gauged supergravity around Dp near-horizon geometries



warped

D6	IIA	$AdS_8 \times S^2$	d=8, SO(3)	[Salam, Sezgin, 1984]
D5	IIB	$AdS_7 \times S^3$	d=7, SO(4)	[Samtleben, Weidner, 2005]
D4	IIA	$AdS_6 \times S^4$	d=6, SO(5)	[Pernici, Pilch, van Nieuwenhuizen, 1984]
D3	IIB	$AdS_5 \times S^5$	d=5, SO(6)	[Günaydin, Romans, Warner, 1985]
D2	IIA	$AdS_4 \times S^6$	d=4, SO(7)	[Hull, 1984]
F1/D1	IIA/B	$AdS_3 \times S^7$	d=3, SO(8)	[de Wit, Nicolai, 1982]
D0	IIA	$AdS_2 \times S^8$	d=2, SO(9)	

outlook

- holography : d=1 supersymmetric matrix quantum mechanics ...!
- first supersymmetric example of a d=2 gauging, general structure of susy (?)
- general structure of gauge groups, gradings of E_9
- descend further in dimension : gauging E_{10} structures