Functional Renormalization Group Equations from a Differential Operator Perspective

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D. Benedetti, K. Groh, P. Machado, and F.S., arXiv:1012.3081 [hep-th]

IHES, Bures-sur-Yvette, France

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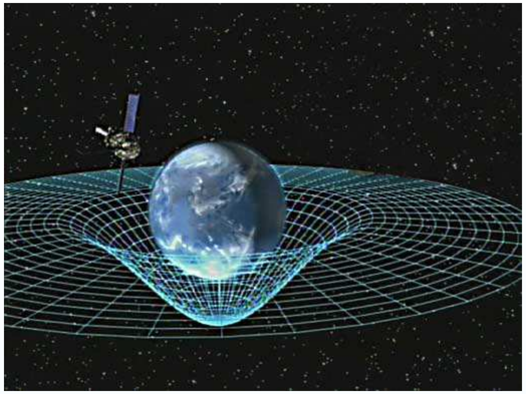
Outline

- Goal: Quantum Gravity
- QFT from the renormalization group perspective
- Functional renormalization group equations
- Results: pre-industrial computations
- Perspectives: industrial revolution
- Summary and Outlook

Classical General Relativity



Based on Einsteins equations

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\text{space-time curvature}} = \underbrace{-\Lambda g_{\mu\nu} + 8\pi G_N T_{\mu\nu}}_{\text{matter content}}$$

- Newton's constant: $G_N = 6.67 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}$
- cosmological constant: $\Lambda \approx 10^{-35}\ \mathrm{s}^{-2}$

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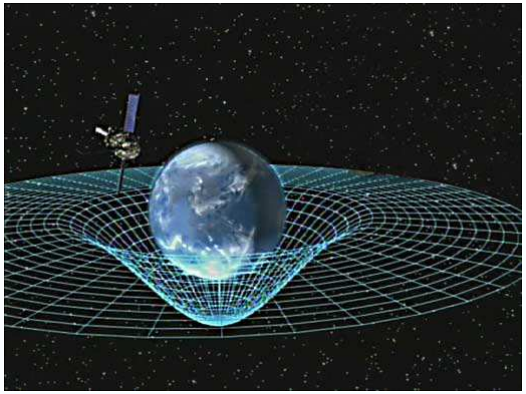

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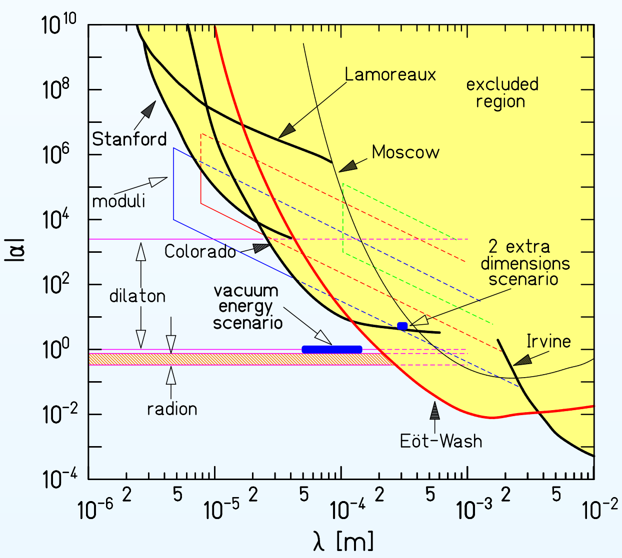

[Adelberger, et. al., 2004]

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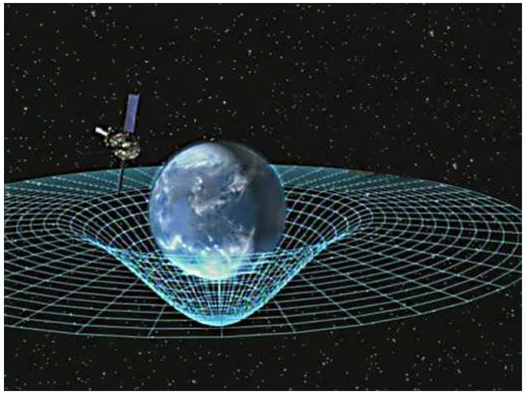

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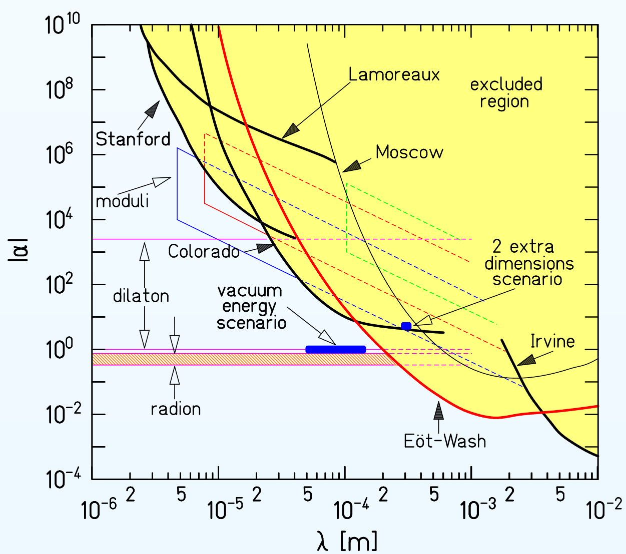

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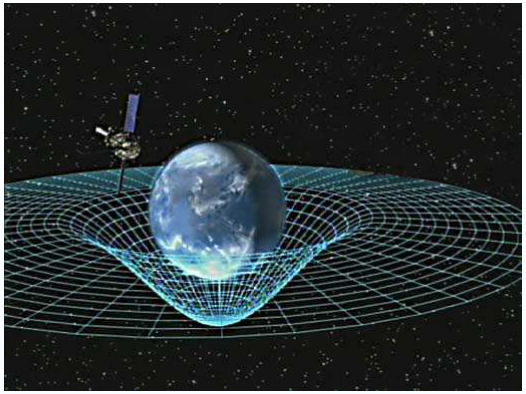

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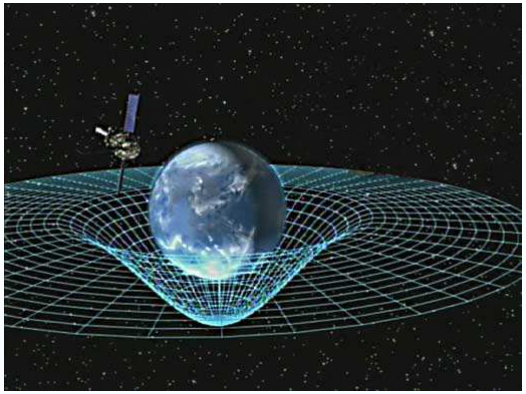

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Theoretical guidance: Quantum Theory for Gravity

Quantum Gravity 101: Quantizing General Relativity

perturbative quantization of the Einstein-Hilbert action:

- G_N has negative mass-dimension:
  - infinite number of counterterms
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Quantum Field Theory

Wilsonian Renormalization Group Perspective

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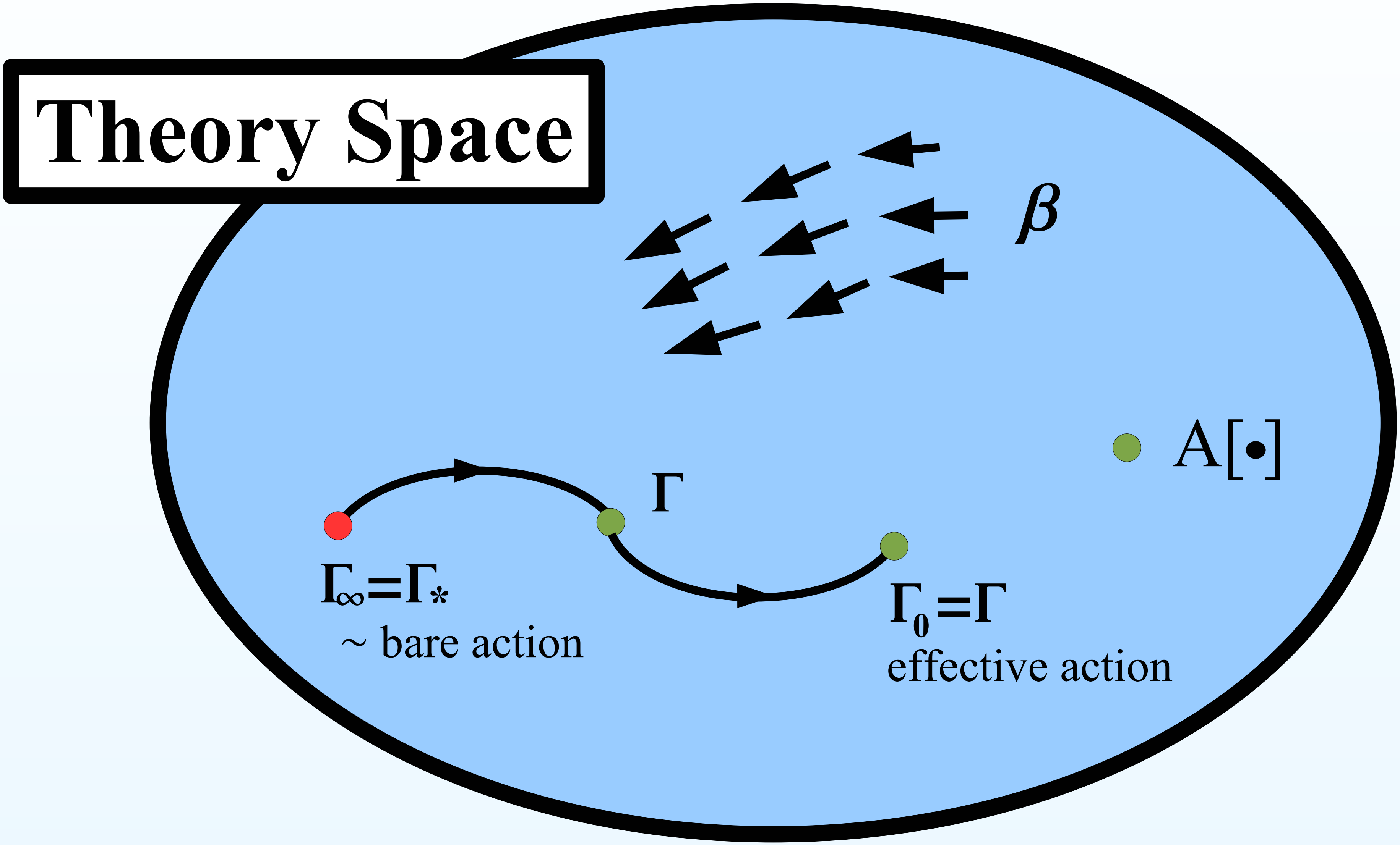
physics at different scales captured by family of effective descriptions

Theory space underlying the Functional Renormalization Group



Fixed points of the RG flow

Central ingredient in Wilsons picture of renormalization

Definition:

- fixed point $\{g_i^*\} \Longleftrightarrow \beta$-functions vanish $\left(\beta_{g_i}(\{g_i\})\big|_{g_i=g_i^*} \stackrel{!}{=} 0 \right)$

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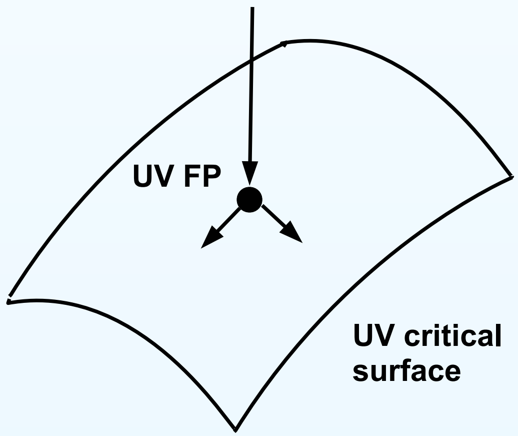
Properties:

- well-defined continuum limit
  - trajectory captured by FP in UV has no unphysical UV divergences
- 2 classes of RG trajectories:
  - relevant $=$ attracted to FP in UV
  - irrelevant $=$ repelled from FP in UV
- predictivity:
  - number of relevant directions $=$ free parameters (determine experimentally)



Renormalization: asymptotic freedom and asymptotic safety

Renormalization via UV fixed points $\Longrightarrow$ two classes of renormalizable QFTs

- Gaussian Fixed Point (GFP)
  - perturbatively renormalizable field theories
  - fundamental theory: free
  - asymptotic freedom (example: QCD)

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Wilsonian picture: generalization of perturbative renormalization

asymptotic safety as predictive as asymptotic freedom

Examples: Asymptotically Safe Theories

Theories with non-Gaussian UV fixed point

- $O(N)$-sigma model ($d = 2 + \epsilon$)

  [Brézin, Zinn-Justin '76]

  - critical exponents of Heisenberg ferromagnets

- Gross-Neveu model ($d = 2 + \epsilon$)

  [Gawedzki, Kupiainen '85]

- Grosse-Wulkenhaar model (non-commutative ϕ^4-theory)

  [Grosse, Wulkenhaar '05; Disertori, Gurau, Magnen, Rivasseau '07]

- Gravity in $2 + \epsilon$ dimensions

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Weinberg's asymptotic safety conjecture (1979):

gravity in $d = 4$ has non-Gaussian UV fixed point

Renormalizing gravity

Wilsonian formulation:

- UV fixed points allow two classes of renormalizable Quantum Field Theories

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Gravity

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Flows on Theory Space

Functional Renormalization Group Equations

RG flows beyond perturbation theory

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For the purpose of studying gravity:

- flow equation for effective average action Γ_k

[C. Wetterich, Phys. Lett. **B301** (1993) 90]

- adapted to gravity

[M. Reuter, Phys. Rev. D **57** (1998) 971, hep-th/9605030]

Effective action Γ in scalar field theory

- start: generic action $S_{\hat{k}}[\chi]$

$$S_{\hat{k}}[\chi] = \int d^d x \left\{ \tfrac{1}{2}(\partial_\mu \chi)^2 + \tfrac{1}{2} m^2 \chi^2 + \text{interactions} \right\}$$

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- generating functional for connected Green functions

$$W[J] = \ln \int \mathcal{D}\chi \exp \left\{ -S_{\hat{k}}[\chi] + \int d^d x \, J \, \chi \right\}$$

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- effective action $\Gamma[\phi]$ gives 1PI correlation functions

$$\Gamma[\phi] = \int d^d x \, J\phi - W[J]$$

  - classical (expectation value) field

$$\phi = \langle \chi \rangle = \frac{\delta W[J]}{\delta J}$$

Effective average action Γ_k in scalar field theory

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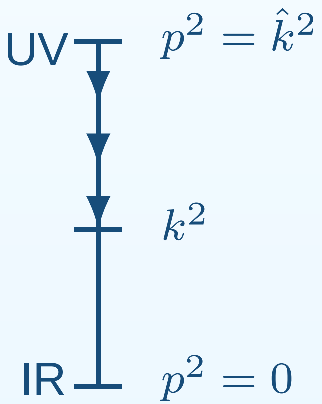
- introduce scale-dependent mass term $\Delta_k S[\chi]$ in $W[J]$

$$W_k[J] = \ln \int \mathcal{D}\chi \exp \left\{ -S_{\hat{k}}[\chi] - \Delta_k S[\chi] + \int d^d x \, J \, \chi \right\}$$

$$\Delta_k S[\chi] = \tfrac{1}{2} \int \tfrac{d^d p}{(2\pi)^d} \mathcal{R}_k(p^2) \, |\widehat{\chi}|^2$$

  - discriminate between low/high-momentum modes

$$\mathcal{R}_k(p^2) = \begin{cases} k^2 & p^2 \ll k^2 \\ 0 & p^2 \gg k^2 \end{cases}$$



    - high momentum modes: integrated out
    - low momentum modes: suppressed by mass term

Effective average action Γ_k for scalars

- scale-dependent generating functional for connected Green functions

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- Effective average action = modified Legendre-transform of $W_k[J]$

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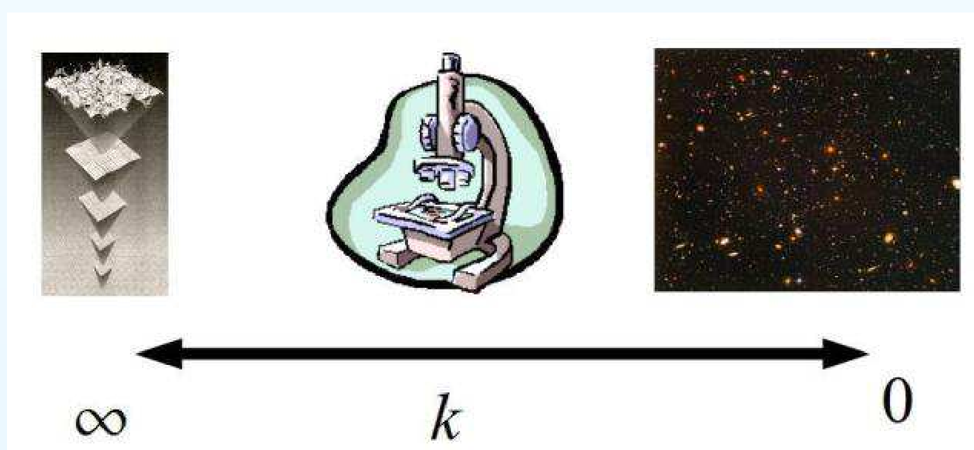
- k-dependence governed by Functional RG Equation (FRGE)

$$k\partial_k \Gamma_k[\phi] = \tfrac{1}{2} \mathrm{Tr}\left[\left(\delta^2 \Gamma_k + \mathcal{R}_k \right)^{-1} k\partial_k \mathcal{R}_k \right]$$

  - upon specifying $\mathcal{R}_k$ = “vector field” on theory space



Effective average action Γ_k for gravity

- start: generic diffeomorphism invariant action for metric $S_{\hat{k}}^{\text{grav}}[g_{\mu\nu}]$
- complication: diffeomorphisms

  $\Longrightarrow$ fix via background field method ($g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$)

  - adds gauge fixing term $S_{\text{gf}}[h;\bar{g}]$
  - adds ghost action $S_{\text{gh}}[h,\bar{\xi},\xi;\bar{g}]$

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- flow equation: analogous to scalar case

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goal: study RG flow described by this equation

Non-perturbative approximation: derivative expansion of Γ_k

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- expand Γ_k in derivatives and truncate series:

$$\Gamma_k[\Phi] = \sum_{i=1}^{N} \bar{u}_i(k)\, \mathcal{O}_i[\Phi]$$

  - $\Longrightarrow$ substitute into FRGE
  - $\Longrightarrow$ projection of flow gives β-functions for running couplings

$$k\partial_k \bar{u}_i(k) = \beta_i(\bar{u}_i; k)$$

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- testing the reliability:
  - within a given truncation:
    - cutoff-scheme dependence of physical quantities (= vary $\mathcal{R}_k$)
  - stability of results wrt extended truncation

Pre-industrial computations

probing theory space by hand

Gravitational theory space: FKWC-basis for $\Gamma_k^{\text{grav}}[g]$

Einstein-Hilbert truncation

$\vdots$			
R^8	$\dots$		
R^7	$\dots$		
R^6	$\dots$		
R^5	$\dots$		
R^4	$\dots$		
R^3	$C_{\mu\nu}{}^{\rho\sigma} C_{\rho\sigma}{}^{\kappa\lambda} C_{\kappa\lambda}{}^{\mu\nu}$	$R \Box R$	+ 7 more
R^2	$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$	$R_{\mu\nu} R^{\mu\nu}$	
R			
$\mathbb{1}$			

The Einstein-Hilbert truncation: setup

Einstein-Hilbert truncation: two running couplings: $G(k), \Lambda(k)$

$$\Gamma_k = \frac{1}{16\pi G(k)} \int d^4x \sqrt{g} \left[-R + 2\Lambda(k) \right] + S_{\rm gf} + S_{\rm gh}$$

- project flow onto G-$\Lambda-$plane

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explicit β-functions for dimensionless couplings $g_k := k^2 G(k)\,,\ \lambda_k := \Lambda(k) k^{-2}$

- Particular choice of $\mathcal{R}_k$ (optimized cutoff)

$$k\partial_k g_k = (\eta_N + 2) g_k \,,$$

$$k\partial_k \lambda_k = -(2 - \eta_N)\,\lambda_k - \tfrac{g_k}{2\pi}\left[5 \tfrac{1}{1-2\,\lambda_k} - 4 - \tfrac{5}{6}\tfrac{1}{1-2\lambda_k}\eta_N\right]$$

- anomalous dimension of Newton's constant:

$$\eta_N = \frac{g B_1}{1 - g B_2}$$

$$B_1 = \tfrac{1}{3\pi}\left[5\,\tfrac{1}{1-2\lambda} - 9\,\tfrac{1}{(1-2\lambda)^2} - 7\right]\,,\ B_2 = -\tfrac{1}{12\pi}\left[5\,\tfrac{1}{1-2\lambda} + 6\,\tfrac{1}{(1-2\lambda)^2}\right]$$

Einstein-Hilbert truncation: Fixed Point structure

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microscopic theory $\Longleftrightarrow$ fixed points of the β-functions

$$\beta_g(g^*,\lambda^*) = 0\,, \qquad \beta_\lambda(g^*,\lambda^*) = 0$$

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microscopic theory $\Longleftrightarrow$ fixed points of the β-functions

$$\beta_g(g^*, \lambda^*) = 0\,, \qquad \beta_\lambda(g^*, \lambda^*) = 0$$

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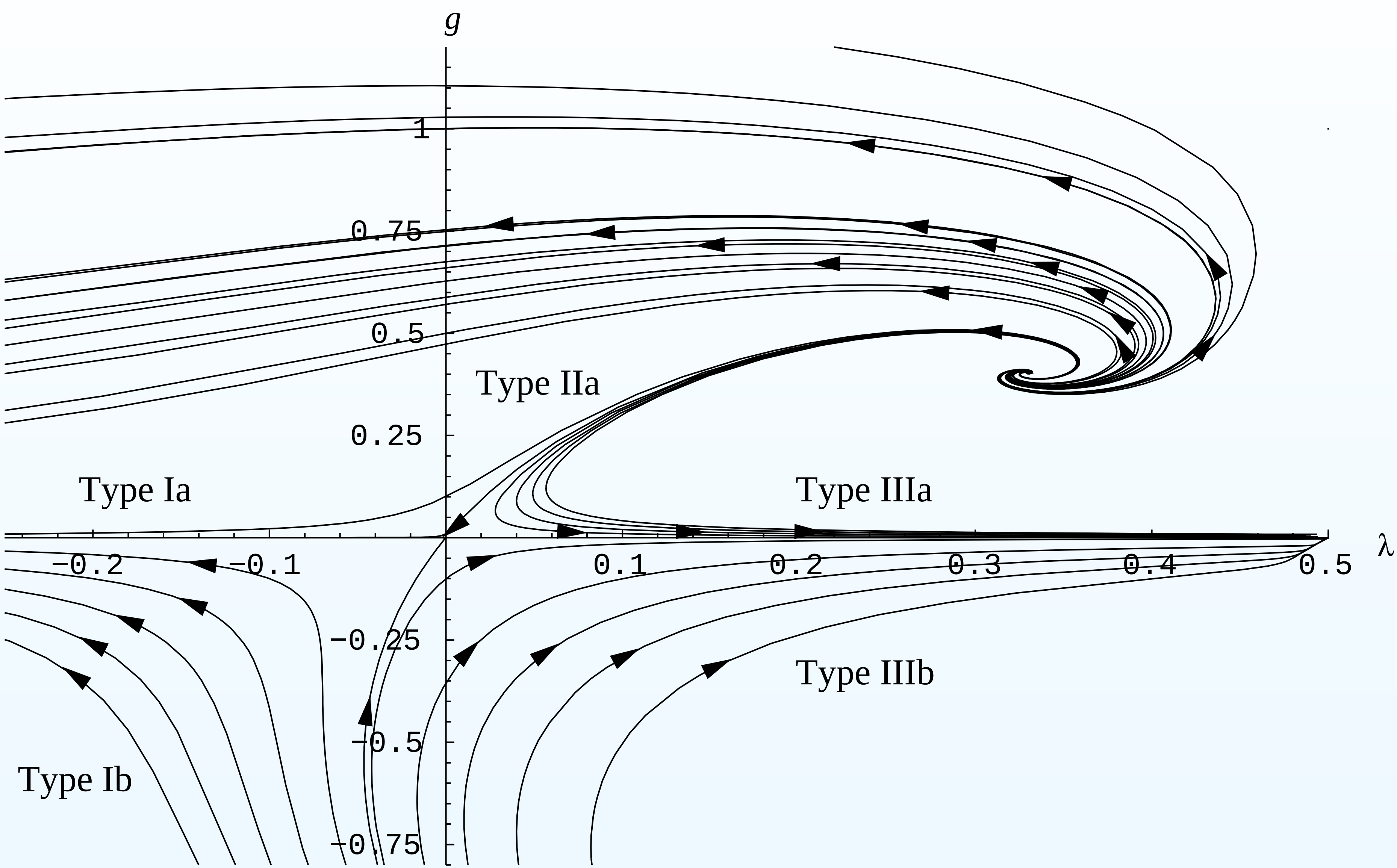
Asymptotic safety: non-Gaussian Fixed Point is UV completion for gravity

Einstein-Hilbert-truncation: the phase diagram



Gravitational theory space: FKWC-basis for $\Gamma_k^{\mathrm{grav}}[g]$

Einstein-Hilbert truncation

polynomial $f(R)$-truncation

$R^2 + C^2$-truncation

$\vdots$

R^8	$\ldots$		
R^7	$\ldots$		
R^6	$\ldots$		
R^5	$\ldots$		
R^4	$\ldots$		
R^3	$C_{\mu\nu}{}^{\rho\sigma} C_{\rho\sigma}{}^{\kappa\lambda} C_{\kappa\lambda}{}^{\mu\nu}$	$R \Box R$	+ 7 more
R^2	$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$	$R_{\mu\nu} R^{\mu\nu}$	
R			
$\mathbb{1}$			

Exploring the gravitational theory space

Some key results . . .

- all computations confirm existence of NGFP
  - $\Rightarrow$ strong evidence for asymptotic safety
- predictivity
  - $\Rightarrow$ UV-criticial surface is finite dimensional (possibly 3 relevant parameters)

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. . . and open questions:

- existence of NGFP in extended truncations?
  (convergence of position, critical exponents, . . .)

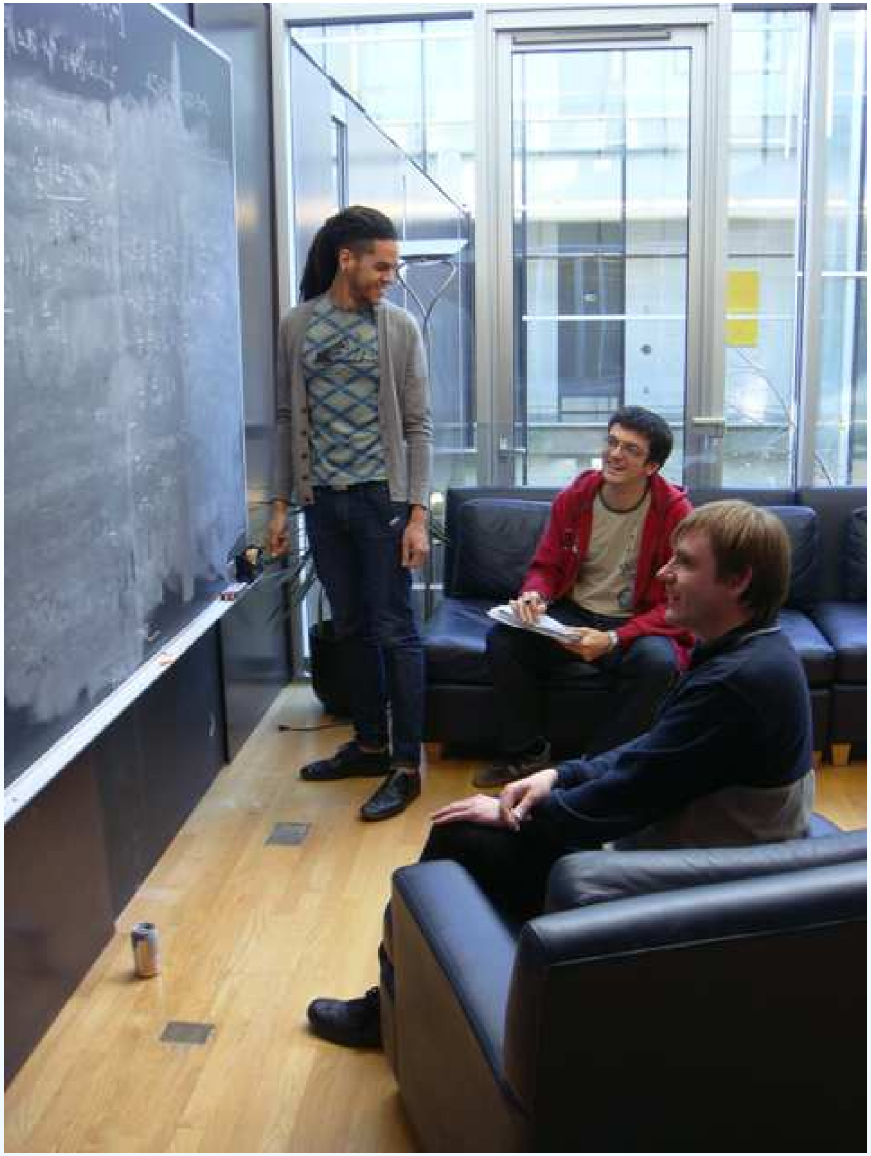
- dimension of its UV-critical surface?
  (= number of parameters to be determined experimentally)

- universality class of the fundamental theory?

Industrial Revolution

The Universal RG Machine

Perimeter Institute, Nov. '09:



What if one could track the flow of 20, 100, . . . couplings???

Obstructions

All previous constructions rely on special background $\bar{g}$:

- Interaction monomials become indistinguishable

  example: curvature^2-terms on background sphere

$$\int d^4x \sqrt{\bar{g}} \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} = \frac{1}{4} \int d^4x \sqrt{\bar{g}} \bar{R}^2 \, , \quad \int d^4x \sqrt{\bar{g}} \bar{R}_{\mu\nu\alpha\beta} \bar{R}^{\mu\nu\alpha\beta} = \frac{1}{6} \int d^4x \sqrt{\bar{g}} \bar{R}^2$$

Insufficient to probe theory space in full generality

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Insufficient to probe theory space in full generality

Working with general background $\bar{g}$:

- Tr contains very complicated operator structures

  $\Rightarrow$ not computable by “standard” methods

The universal RG machine: blueprint

goal: systematic derivative expansion of

$$k\partial_k \Gamma_k[h,\xi,\bar{\xi};\bar{g}] = \frac{1}{2}\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\partial_k \mathcal{R}_k\right]$$

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4-step implementation:

1. expand $\Gamma_k[h;\bar{g}]$ to second order in fluctuations
2. reduce operator structure of $\Gamma_k^{(2)}$ to “standard” form
3. perturbative inversion of $[\Gamma_k^{(2)} + \mathcal{R}_k]$
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virtues:

- lifts technical requirement of having “simple” backgrounds $\bar{g}$
- each step can be handled by a computer algebra software
- no numerical integrations

Step 1: expand $\Gamma_k[h;\bar{\phi}]$ to second order

Notation:

- $\bar{\phi}$: multiplet of background fields (e.g. $\{\bar{g}_{\mu\nu}, \bar{A}_\mu, \bar{h}\}$)
- h: multiplet of fluctuations around $\bar{\phi}$ (e.g. $\{h_{\mu\nu}, h_\mu, h\}$)

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$\Gamma_k^{(2)}$: results from Taylor expansion:

$$\Gamma_k[h,\bar\phi] = \ldots + \frac{1}{2}\int d^dx\sqrt{\bar g}\; h_i\, \left[\Gamma_k^{(2)}[\bar\phi]\right]^{ij}\, h_j + \ldots$$

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Example: free massive scalar field h in curved space-time $\bar g_{\mu\nu}$:

$$\Gamma_k[h,\bar g] = \tfrac{1}{2}\int d^dx\sqrt{\bar g}\, h\, \left[-\bar D^2 + m_k^2\right]\, h$$

- $\Gamma_k^{(2)}[\bar g]$: Laplace-type differential operator

$$\Gamma_k^{(2)}[\bar g] = -\bar D^2 + m_k^2$$

Step 2: simplify operator structure of $\Gamma_k^{(2)}$

generically, $\Gamma_k^{(2)}$ has non-Laplacian part:

$$\left[\Gamma_k^{(2)}\right]^{ij} = \underbrace{\mathbb{K}(\Delta)\,\delta^{ij}\,\mathbb{1}_i}_{\text{kin. terms}} + \underbrace{\mathbb{D}(D_\mu)}_{\text{uncontracted derivatives}} + \underbrace{\mathbb{M}(R, D_\mu)}_{\text{background curvature}}$$

Examples:

$$\mathbb{D} = (1-\alpha) D_\mu D^\nu \quad , \quad \mathbb{D} = D^\mu D^\nu D_\alpha D_\beta$$

$$\mathbb{M} = R^{\mu\nu} D_\mu D_\nu$$

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$\mathbb{D} \neq 0 \Leftrightarrow \Gamma_k^{(2)}$ is non-minimal differential operator:

- obstructs evaluation of traces via “standard” heat-kernel formulae

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general solution: Transverse decomposition of fluctuation fields

[York, ’75; Reuter, Lauscher ’02]

- vector: $h_\mu = h_\mu^{\mathrm{T}} + D_\mu h \;, \qquad D^\mu h_\mu^{\mathrm{T}} = 0$
- graviton: $h_{\mu\nu} = h_{\mu\nu}^{\mathrm{T}} + D_\mu \xi_\nu + D_\nu \xi_\mu - \frac{1}{2} g_{\mu\nu} D^\alpha \xi_\alpha + \frac{1}{4} g_{\mu\nu} h$

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Removes $\mathbb{D}$-part from $[\Gamma_k^{(2)}]^{ij}$

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choose $\mathcal{R}_k$ to regulate kinetic terms:

$$\mathbb{K}(\Delta) \mapsto \mathbb{P}(\Delta)\,, \quad \text{following} \qquad \Delta \mapsto P_k = \Delta + R_k$$

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Perturbative expansion of inverse matrix elements in $\mathbb{M}$:

$$\left[\Gamma^{(2)} + \mathcal{R}_k\right]^{-1}_{11} = \frac{1}{\mathbb{P}_1} - \frac{1}{\mathbb{P}_1}\,\mathbb{M}_1\,\frac{1}{\mathbb{P}_1} + \ldots$$

- Two classes of traces: pure Laplacian and Laplacian + vertex insertions
- truncation $\Longleftrightarrow$ expansion terminates at finite order:

Step 4a: evaluate operator traces without insertions $\mathbb{M}$

Uses: standard heat-kernel expansion of Laplace-operators

- Laplace transform $W(\Delta) \to \tilde{W}(s)$

$$\mathrm{Tr}_i \left[W(\Delta) \right] = \int_0^\infty ds \, \tilde{W}(s) \, \mathrm{Tr}_i \left[\mathrm{e}^{-s\Delta} \right]$$

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- Coefficients for transverse fields:
  - construct via projection operators

$$\mathrm{Tr}_{1T} \left[\mathrm{e}^{-s\Delta} \right] \equiv \mathrm{Tr}_1 \left[\mathrm{e}^{-s\Delta} \Pi_T \right]$$
$$= \mathrm{Tr}_1 \left[\mathrm{e}^{-s\Delta} \right] + \mathrm{Tr}_0' \left[D_\mu \Delta^{-1} D^\mu \mathrm{e}^{-s\Delta} \right]$$

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Heat-kernel expansion for constrained fields on spherical topology

$$\mathrm{Tr}_i\left[\mathrm{e}^{-s\Delta}\right] = \frac{1}{(4\pi s)^2}\int d^4x\sqrt{g}\left[c_1 + c_2 R + c_3 R^3 + c_4 R_{\mu\nu}R^{\mu\nu} + c_5 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right]$$

	0	1	2	1T	2T
c_1	1	4	10	3	5
c_2	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{5}{3}$	$\frac{1}{4}$	$-\frac{5}{6}$
c_3	$\frac{1}{72}$	$\frac{1}{18}$	$\frac{5}{36}$	$\frac{5}{24}$	$-\frac{137}{216}$
c_4	$-\frac{1}{180}$	$-\frac{1}{45}$	$-\frac{1}{18}$	$-\frac{39}{40}$	$-\frac{17}{108}$
c_5	$\frac{1}{180}$	$-\frac{11}{180}$	$-\frac{4}{9}$	$\frac{11}{60}$	$\frac{5}{18}$

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Straightforward to evaluate operator traces without vertices $\mathbb{M}$

Step 4b: Evaluate the operator traces including insertions $\mathbb{M}$

1. use commutators to bring trace argument into standard form:
   - contracted cov. derivatives: $\Longrightarrow$ collected into a single function $W(\Delta)$
   - remainder: $\Longrightarrow$ matrix-valued insertion $\mathcal{O}$

2. Laplace transform $W(\Delta) \to \tilde{W}(s)$

$$\mathrm{Tr}\,[\,W(\Delta)\,\mathcal{O}\,] = \int_0^\infty ds\,\tilde{W}(s)\,\langle x|\,\mathrm{e}^{-s\Delta}\,\mathcal{O}\,|x\rangle$$

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3. evaluate trace using off-diagonal Heat-kernel (act $\mathcal{O}$ on H)

$$\langle x|\,\mathcal{O}\,\mathrm{e}^{-s\Delta}|x\rangle = \langle x|\,\mathcal{O}\,|x'\rangle\langle x'|\mathrm{e}^{-s\Delta}|x\rangle = \int d^4x\sqrt{g}\,\mathrm{tr}_i\left[\mathcal{O}H(s,x,x')\right]_{x=x'}$$

$$H(s,x,x') := \langle x'|\mathrm{e}^{-s\Delta}|x\rangle = \frac{1}{(4\pi s)^2}\,\mathrm{e}^{-\frac{\sigma(x,x')}{2s}}\sum_{n=0}^{\infty} s^n A_{2n}(x,x')$$

   - $A_{2n}(x,x')$: heat-coefficients at non-coincident point
   - $2\sigma(x,x')$: geodesic distance between x, x'

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properties of $H(s, x, x')$ in the coincidence limit:

- $A_{2n}(x, x)$ $\longrightarrow$ standard heat-kernel coefficients
- derivatives of A_{2n} $\longrightarrow$ additional powers of curvatures
- $\sigma(x, x) = 0 \, , \; \sigma_{;\mu} = 0$ $\longrightarrow$ vanish in coincidence limit
- $\sigma_{;\mu\nu}(x, x) = g_{\mu\nu}(x)$ $\longrightarrow$ non-vanishing

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example: $\mathcal{O} = R^{\mu\nu}D_\mu D_\nu$

$$\mathrm{Tr}\left[W(\Delta)\,\mathcal{O}\right] = -\frac{1}{32\pi^2}\int_0^\infty ds\,\frac{1}{s^3}\tilde{W}(s)\int d^4x\sqrt{g}R + \mathcal{O}(R^2)$$

The algorithm on the computer

mathematical input

- off-diagonal heat-kernel coefficients for unconstrained fields

$\Longrightarrow$ heat-kernel coefficients for transverse fields

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manipulation of traces including vertices

- commutation rules for covariant derivatives
- canonicalize trace-arguments including $\mathbb{M}$
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implementation: in progress

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first applications to gravity:

- EH-truncation: background-independence of β-functions
- ghost-wavefunction renormalization

Confirms NGFP in full agreement with asymptotic safety