Entanglement Entropy of Black Holes

Sergey N. Solodukhin

(University of Tours)

Talk at IHES

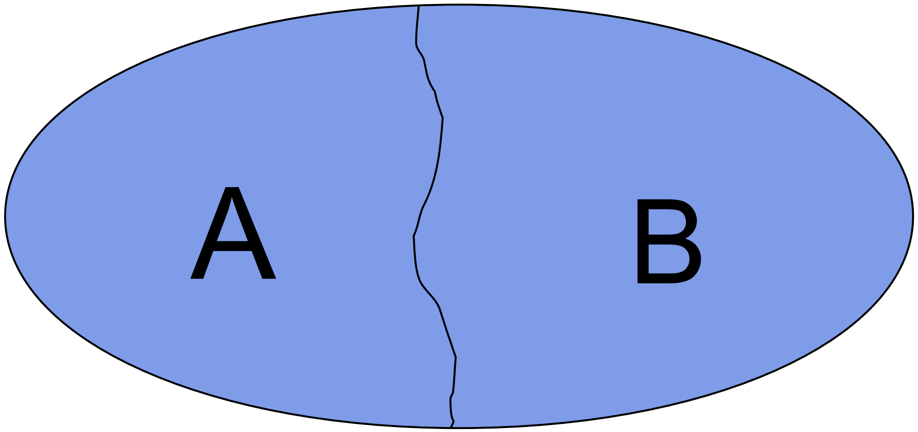
Plan of the talk:

- Entanglement entropy
  a) Replica trick
  b) Useful mathematical tools

- Entanglement entropy of black holes:
  a) UV divergences
  b) Logarithmic terms in the entropy
  c) Renormalization
  d) Non-minimal coupling

- Some other developments
  a) Holographic interpretation
  b) Entanglement entropy in IR and UV modified theories
  c) Logarithmic terms in generic 4d CFT

Entanglement entropy



Σ

$$|\psi(A,B)>$$

$$\rho(A,B) = |\psi><\psi|$$

$$\rho_A = Tr_B \rho(A,B)$$

$$S_A = -Tr_A \rho_A \ln \rho_A$$

Bombelli et el (86), Srednicki (93),
Frolov-Novikov (93)

Properties

- $S_A = S_B$ if $|\psi(A,B)\rangle$ is pure state

⇒ S_A **depends on local geometry:**

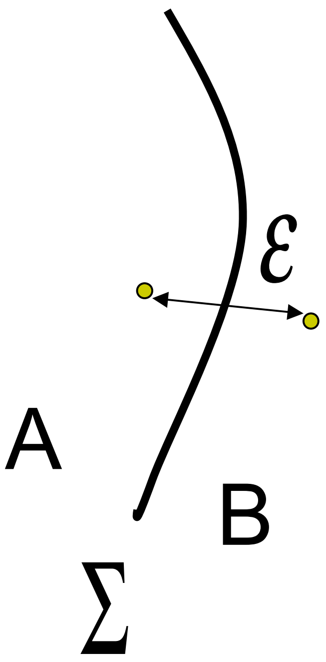
i) intrinsic or extrinsic geometry of Σ

ii) geometry of space-time near Σ

(modulo Gauss-Codazzi)

Properties

- S_A is non-zero due to short-distance correlations between A and B



$$< \phi(x), \phi(y) > \sim \frac{1}{|x-y|^{d-2}}$$

S_A depends on UV regulator ε

$$S \sim \frac{A(\Sigma)}{\epsilon^{d-2}}$$

Properties

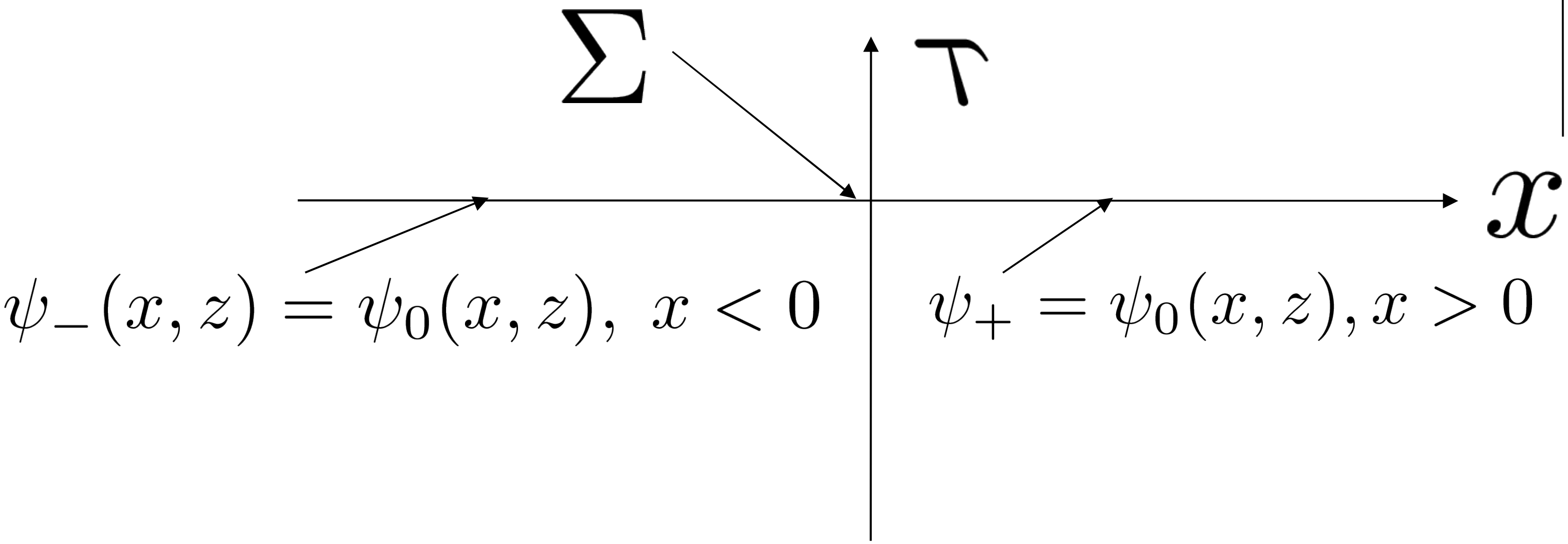
- $S_A \neq S_B$ if $T \neq 0$

in the large size limit EE approaches thermal entropy:

$$d = 2: \qquad S = \frac{c\pi}{3} LT$$

$$d > 2: \qquad S \propto (LT)^{d-1}$$

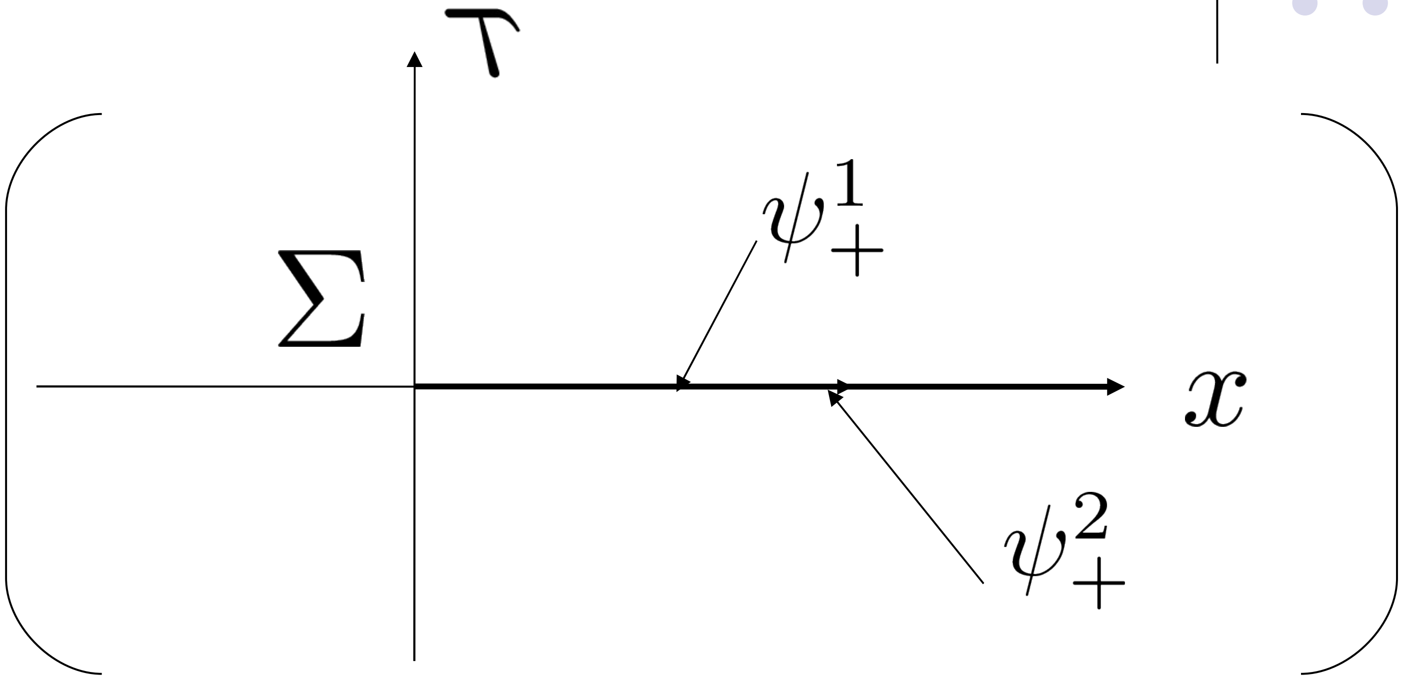
Replica method: wave function



$$\Psi[\psi_0(x,z)] = \int_{\psi(X)|_{\tau=0}=\psi_0(x,z)} \mathcal{D}\psi \; e^{-W[\psi]}$$

Replica method: a reduced density matrix

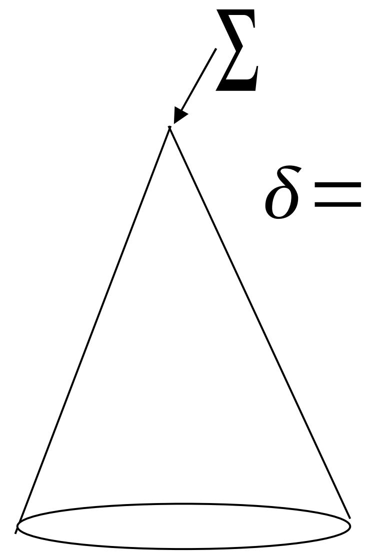
Path integral on



$$= \rho(\psi^1_+, \psi^2_+)$$

Replica Method: trace of density matrix

$$Tr\,\rho^n = \text{Path Integral on}$$



$$\delta = 2\pi(1-n)$$

$$S_{ent} = -Tr\,\hat{\rho}\ln\hat{\rho} = -\left[(n\partial_n - 1)\ln Tr\,\rho^n\right]_{n=1}$$

Susskind (93), Callan-Wilczek (94)

Uniqueness of analytic continuation

$$n=1,2,..\to\alpha \quad , \quad \Re(\alpha)>1$$

Regularized trace of renormalized density matrix $\hat{\rho}=\frac{\rho}{Tr\,\rho}$ is bounded

$$|Tr_{\varepsilon}\hat{\rho}^{\alpha}|<1 \quad if \quad \Re(\alpha>1)$$

Suppose we know $Tr_{\varepsilon}\rho^{n}=Z_{0}(n) \quad for \quad \alpha=n \quad , \quad n=1,2,3,..$

The we can represent $Z(\alpha)=Tr_{\varepsilon}\rho^{\alpha}$ in the form

$$Z(\alpha)=Z_{0}(\alpha)+\sin(\pi\alpha)\,g(\alpha)$$

where $g(\alpha)$ is analytic and $|g(\alpha=x+iy)|<e^{-\pi|y|}$

By Carlson's theorem $g(\alpha)\equiv 0$

Heat kernel and the Sommerfeld formula

$$(\partial_s + D) K(s, x, x') = 0$$
$$K(s=0, x, x') = \delta(x, x')$$

$2\pi\alpha$-periodic function from a 2π-periodic is constructed by using

$$K_\alpha(s, \phi, \phi') = K(s, \phi-\phi') + \frac{i}{4\pi\alpha}\int_\Gamma \cot\frac{w}{2\alpha} K(s, \phi-\phi'+w)\, dw$$

Sommerfeld (1897)

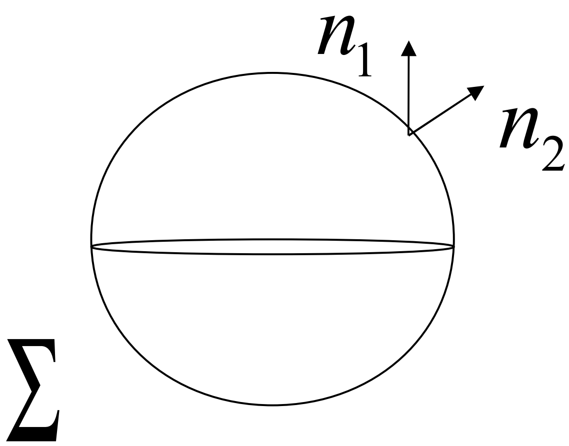
In presence of Abelian symmetry $\phi \to \phi + w$

it is still a solution to heat equation

Useful mathematical tools (A): Riemann curvature and conical singularity

$$R_{\alpha\beta}^{\mu\nu} = R_{(reg)\alpha\beta}^{\mu\nu} + R_{(sing)\alpha\beta}^{\mu\nu}$$

$$R_{(sing)\alpha\beta}^{\mu\nu} = 2\pi(1-\alpha)[(n^\mu n_\alpha)(n^\nu n_\beta) - (n^\mu n_\beta)(n^\nu n_\alpha)]\delta_\Sigma$$



$$(n^\mu n_\alpha) = n_1^\mu n_\alpha^1 + n_2^\mu n_\alpha^2$$

Fursaev, SS (94)

A consequence: the Euler number for a manifold with cone singularity

$$\chi(M_\alpha)=\frac{1}{32\pi^2}\int_{M_\alpha/\Sigma}(R^2-4R_{\mu\nu}^2+R_{\mu\nu\alpha\beta}^2)+\sum_i(1-\alpha_i)\chi(\Sigma_i)$$

Fursaev, SS (94)

(Rediscovered by Atiyah, LeBrun (2012))

A special case is when M_α possesses a continuous Abelian isometry so that Σ_i are the fixed point sets of this isometry and $\alpha_i=\alpha$.

Then we arrive at a reduction formula $\chi(M)=\sum_i \chi(\Sigma_i)$

Example: singular surface of S_α^d $(d\geqslant 3)$ is S^{d-2} so that $\chi(S^d)=\chi(S^{d-2})$

Useful mathematical tools (B): Heat kernel method

$$W(\alpha) = -\frac{1}{2}\int_{\epsilon^2}^{\infty} \frac{ds}{s}\,\mathrm{Tr}K_{E_\alpha}(s)$$

$$\mathrm{Tr}K_{E_\alpha}(s) = \frac{1}{(4\pi s)^{\frac{d}{2}}}\sum_{n=0} a_n s^n$$

Coefficients in the expansion decompose on
the bulk (regular) and the surface (singular) parts:

$$a_n = a_n^{reg} + a_n^{\Sigma}$$

Heat kernel method: regular terms in the expansion

Scalar field operator: $\mathcal{D} = -(\nabla^2 + X)$

$$a_0^{reg} = \int_{E_\alpha} 1 \ , \quad a_1^{reg} = \int_{E_\alpha} (\frac{1}{6}\bar{R} + X) \, ,$$

$$a_2^{reg} = \int_{E_\alpha} \left(\frac{1}{180}\bar{R}^2_{\mu\nu\alpha\beta} - \frac{1}{180}\bar{R}^2_{\mu\nu} + \frac{1}{6}\nabla^2(X + \frac{1}{5}\bar{R}) + \frac{1}{2}(X + \frac{1}{6}\bar{R})^2 \right)$$

These terms are proportional to α and do not contribute to the entropy

Heat kernel method: surface terms in the expansion

$$a_0^{\Sigma} = 0; \quad a_1^{\Sigma} = \frac{\pi}{3}\frac{(1-\alpha^2)}{\alpha}\int_{\Sigma} 1\,,$$

$$a_2^{\Sigma} = \frac{\pi}{3}\frac{(1-\alpha^2)}{\alpha}\int_{\Sigma}(\frac{1}{6}\bar{R}+X) - \frac{\pi}{180}\frac{(1-\alpha^4)}{\alpha^3}\int_{\Sigma}(\bar{R}_{ii}-2\bar{R}_{ijij})\,,$$

where $\bar{R}_{ii} = \bar{R}_{\mu\nu}n_i^{\mu}n_i^{\nu}$ and $\bar{R}_{ijij} = \bar{R}_{\mu\nu\lambda\rho}n_i^{\mu}n_i^{\lambda}n_j^{\nu}n_j^{\rho}$

Fursaev (94)

Important remark:

These mathematical tools work only if there is abelian isometry in subspace orthogonal to entangling surface Σ .
This is not so for a surface (sphere, cylinder..) in flat Minkowski spacetime!

However: they work perfectly for Killing horizons!

Entanglement entropy of black holes

Wave function of black hole $\psi(\varphi_+, \varphi_-)$ is functional of modes

inside (φ_-) and modes outside (φ_+) black hole horizon

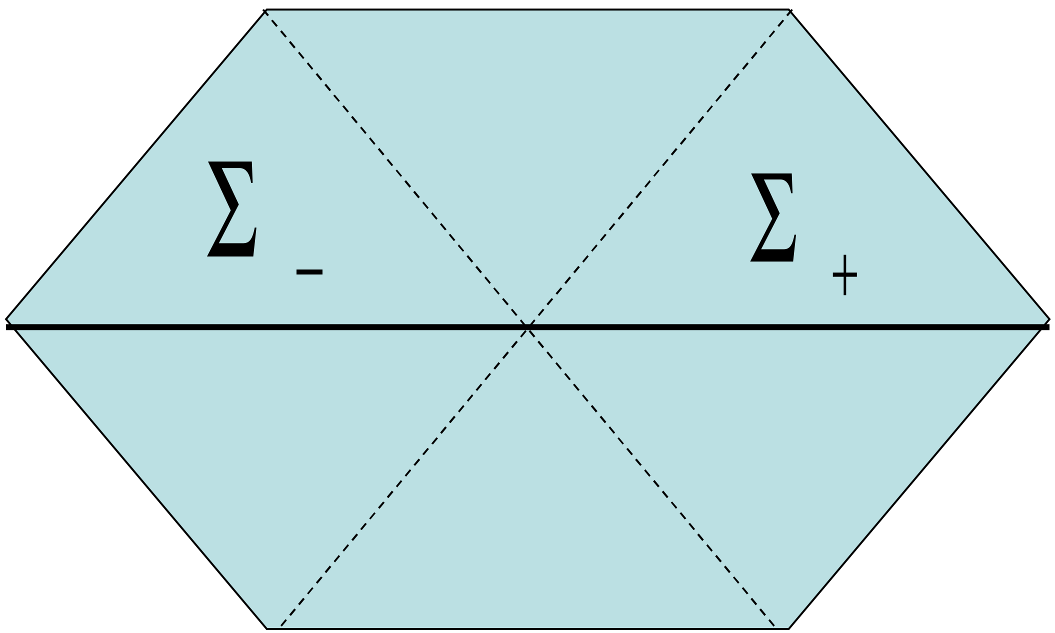
Barvinsky, Frolov and Zelnikov (94)

Partition function

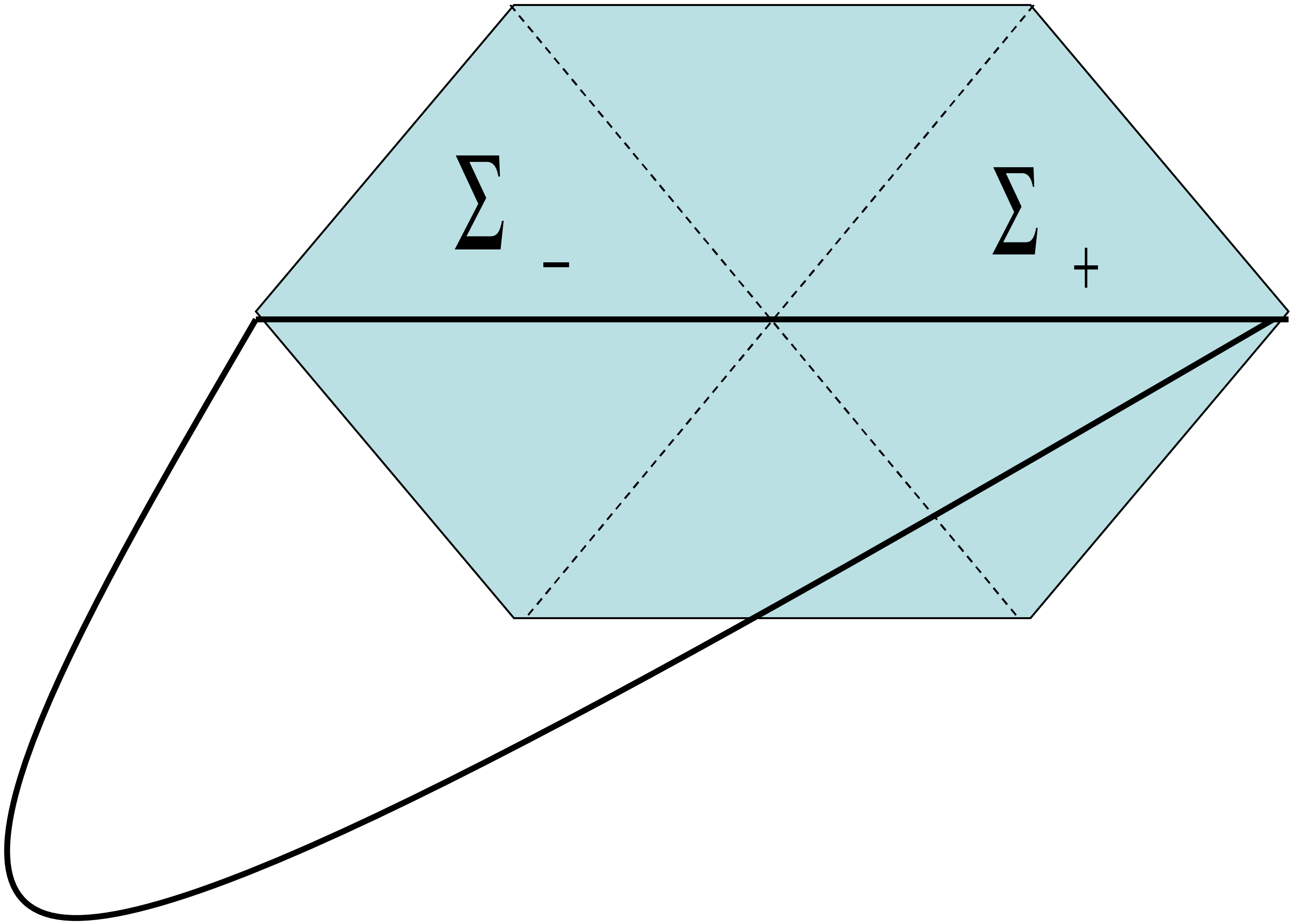
$\mathrm{Tr}\rho^\alpha = e^{-W(\alpha)}$ is given by functional integral

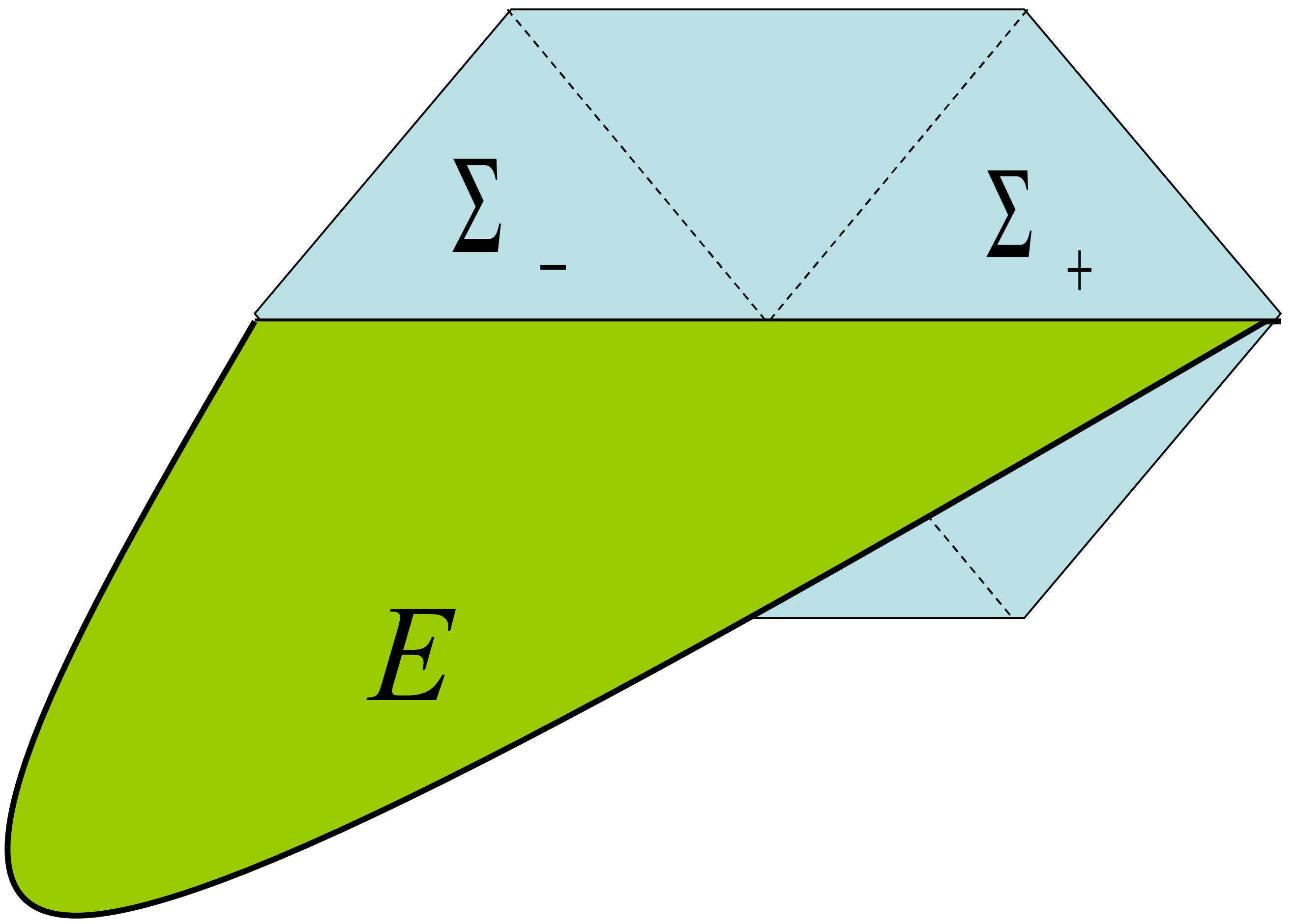
over E_α , α – fold cover of Euclidean black hole instanton

(manifold with conical singularity at horizon)



Σ_-
Σ_+


Σ_-
Σ_+



Σ_-
Σ_+
E

Entanglement entropy of 4d black hole

Scalar field operator: $\mathcal{D} = -(\nabla^2 + X)\,, \;\; X = -\xi \bar{R}$

$$S_{d=4} = \frac{A(\Sigma)}{48\pi\epsilon^2} - \frac{1}{144\pi}\int_\Sigma \left(\bar{R}(1+6\xi) - \frac{1}{5}(\bar{R}_{ii} - 2\bar{R}_{ijij})\right)\ln\epsilon$$

S.S. (94)

Kerr-Newman black hole (m,a,q)

Entropy of a minimal scalar field, $\xi=0$

$$S_{KN} = \frac{A(\Sigma)}{48\pi\epsilon^2} + \frac{1}{45}\left(1 - \frac{3q^2}{4r_+^2}(1 + \frac{(r_+^2+a^2)}{ar_+}\tan^{-1}(\frac{a}{r_+}))\right)\ln\frac{r_+}{\epsilon}$$

Mann, SS (96)

Horizon area $A(\Sigma) = 4\pi(r_+^2 + a^2)$

$$r_+ = m + \sqrt{m^2 - a^2 - q^2}$$

Interesting limits:

- Schwarzschild black hole (q=a=0)

$$S_{Sch} = \frac{A(\Sigma)}{48\pi\epsilon^2} + \frac{1}{45}\ln\frac{r_+}{\epsilon}$$

- Extreme charged black hole (a=0, q=m)

$$S_{Ext} = \frac{A(\Sigma)}{48\pi\epsilon^2} - \frac{1}{90}\ln\frac{r_+}{\epsilon}$$

- Extreme Kerr black hole (q=0, a=m)

$$S_{Ext-Kerr} = \frac{A(\Sigma)}{48\pi\epsilon^2} + \frac{1}{45}\ln\frac{r_+}{\epsilon}$$

Renormalization

Bare gravitational action

$$W_{bare} = \int \left(-\frac{1}{16\pi G}(R + 2\Lambda) + c_1 R^2 + c_2 R^2_{\mu\nu} + c_3 R^2_{\alpha\beta\mu\nu} \right)$$

Black hole entropy $$S_{BH} = \frac{A_\Sigma}{4G} - 4\pi \int_\Sigma (2c_1 R + c_2 R_{ii} + 2c_3 R_{ijij})$$

Renormalization of entropy: $$S_{BH}(G, c_i) + S_{div}(\epsilon) = S_{BH}(G^{ren}, c_i^{ren})$$

Renormalization of action: $$\frac{1}{4G} + \frac{1}{48\pi\epsilon^2} = \frac{1}{4G^{ren}}$$

Susskind and Uglum (94), Jacobson (94), Fursaev and SS (94)

The statement is valid for any field (fermionic and bosonic) except gauge fields (s=2 and s=1)

Puzzle of non-minimal coupling

Non-minimal field operator $\mathcal{D} = -(\nabla^2 + X)\,,\ X = -\xi \bar{R}$

Renormalization of Newton constant

$$G^{-1}_{ren} = G^{-1} + \frac{1}{2\pi}(\frac{1}{6} - \xi)\frac{1}{\epsilon^2}$$

Entanglement entropy on Ricci flat metrics $\bar{R} = 0$
does not depend on ξ

$$S = \frac{A(\Sigma)}{48\pi\epsilon^2}$$

Gauge fields: s=1 and s=2

$$\frac{1}{4G_{ren}}=\frac{1}{4G}+\frac{1}{(4\pi)^{\frac{d-2}{2}}(d-2)}\left(\frac{D_s(d)}{6}-c_s(d)\right)\frac{1}{\varepsilon^{d-2}}$$

Spin s=1: $D_1(d)=d-2, \qquad c_1(d)=1$

Spin s=2: $D_2(d)=\frac{d(d-3)}{2}, \qquad c_2(d)=\frac{(d^2-d+4)}{2}$

Entanglement Entropy: $S=\frac{D_s(d)}{6(d-2)(4\pi)^{\frac{d-2}{2}}}\frac{A(\Sigma)}{\varepsilon^{d-2}}$

Most intriguing question: can entanglement entropy account for entire BH entropy?

- A natural identification: UV cut-off at Planck scale

$$1/\epsilon = \Lambda \sim M_{PL} \quad \text{then} \quad S_{ent} \sim S_{BH}$$

- Do coefficients precisely agree?

- Entanglement entropy and induced gravity, problem of non-minimal coupling

Jacobson(94), Frolov et al. (96),
Hawking, Maldacena, Strominger (2000)

SOME OTHER DEVELOPMENTS

The AdS/CFT correspondence

Holographic picture: physics in the bulk has equivalent description on the boundary

’t Hooft (93), Susskind (94)

A concrete realization: duality between string theory on anti-de Sitter (AdS) space-time and CFT living on the boundary of AdS

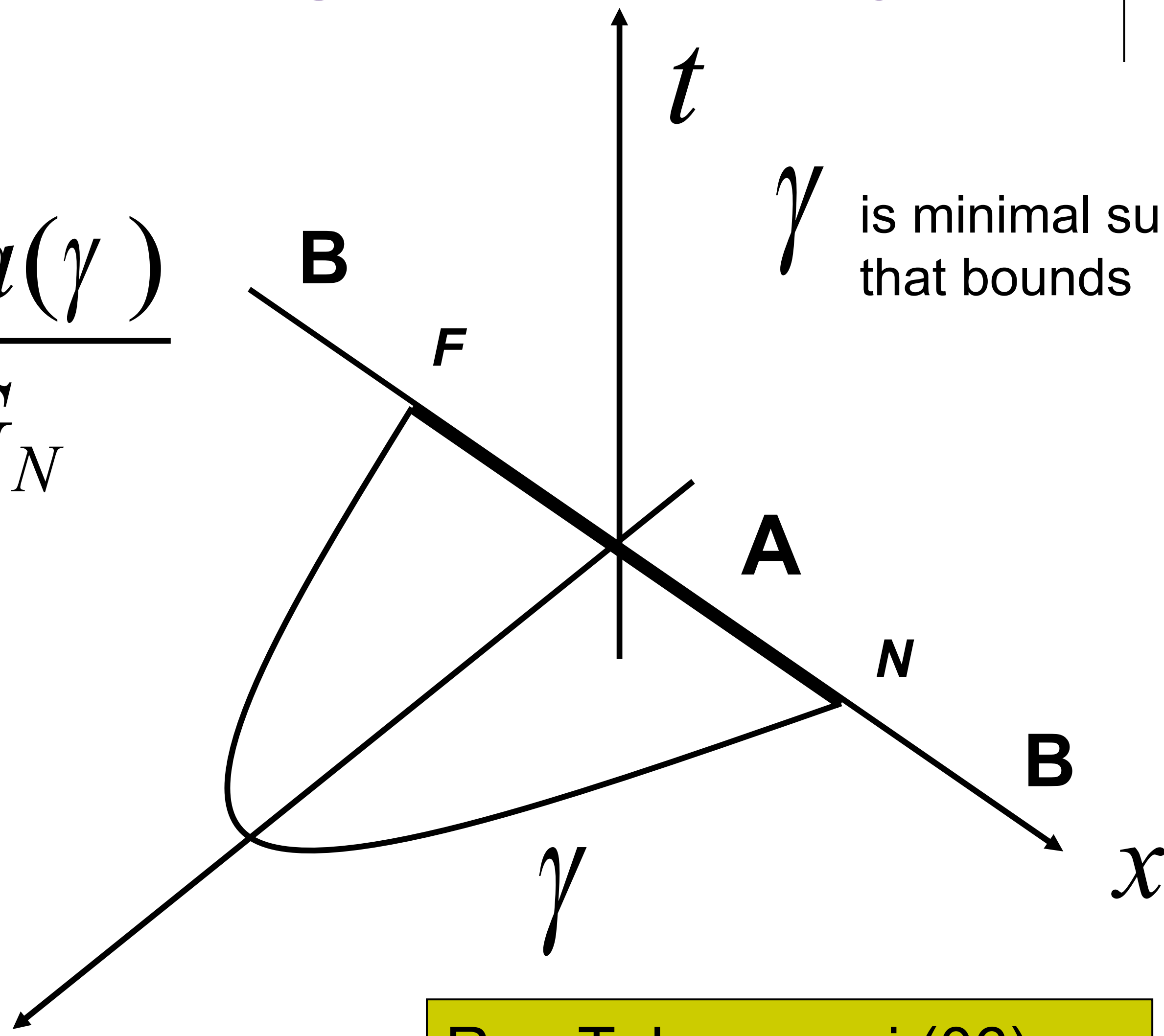
Maldacena (97), Witten (98), Polyakov, Gubser, Klebanov (98)

Boundary dimension d=4: boundary CFT is N=4 SU(N) super-Yang-Mills

Is there a holographic interpretation of entanglement entropy?

Holographic Entanglement Entropy

$$S = \frac{Area(\gamma\ )}{4G_N}$$



γ is minimal surface that bounds Σ

Ryu-Takayanagi (06)

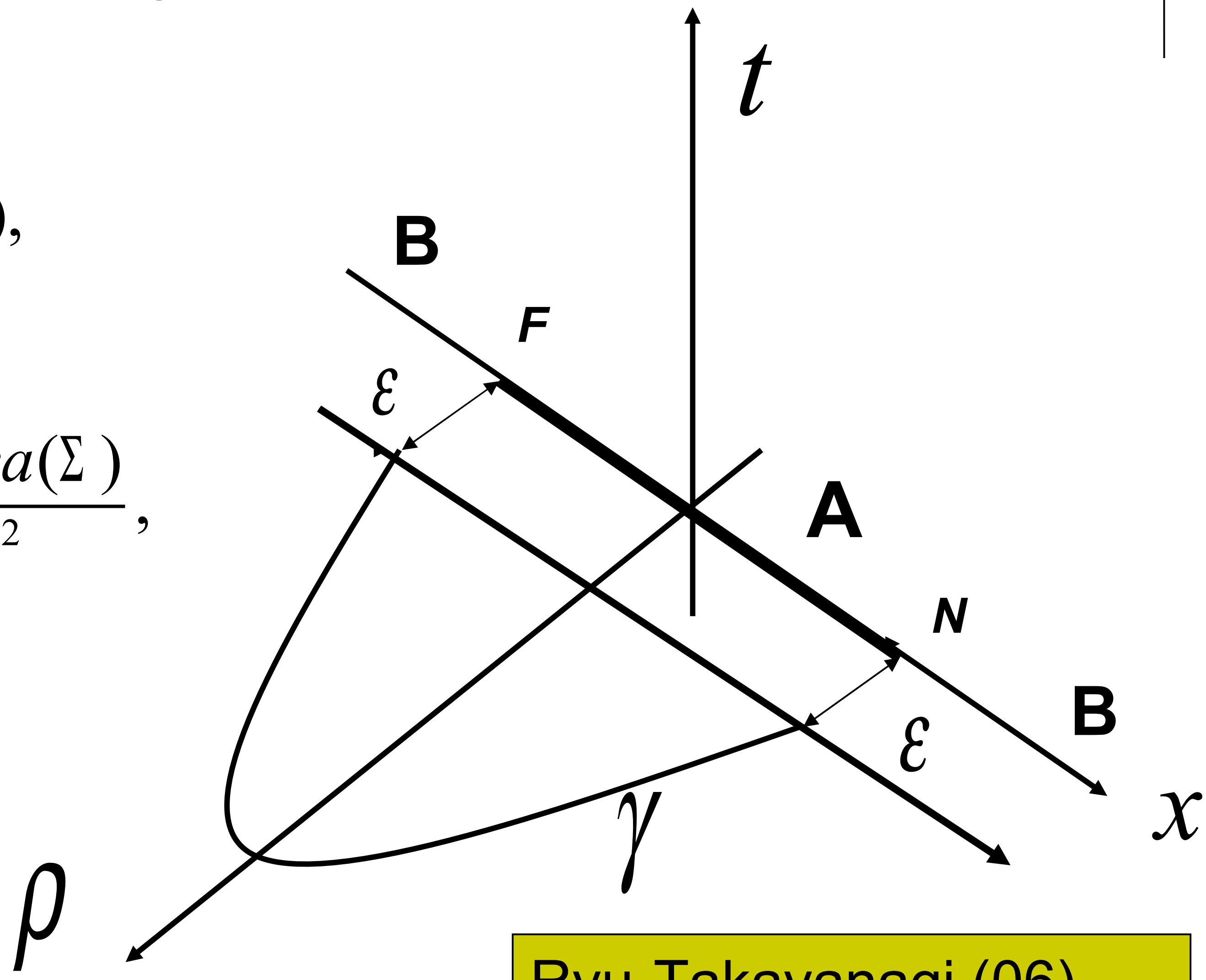
UV/IR duality

$$S = \frac{c}{6}\ln(\frac{1}{\varepsilon}),$$

$$d = 2$$

$$S \sim \frac{N^2 Area(\Sigma)}{\varepsilon^{d-2}},$$

$$d > 2$$



Ryu-Takayanagi (06)

UV and IR modified theories

More general Lorentz invariant field operator $\mathcal{D} = F(\nabla^2)$

Examples:

(i) 4d brane in spacetime with compact fifth dimension

$$F(p^2) = \frac{p}{L}\tanh(Lp)$$

(ii) DGP model $F(p^2) = p^2 + m\sqrt{p^2}$

(iii) Non-commutative field theory $F(p^2) = p^2 + \frac{1}{\theta^2 p^2}$

(iv) UV modified theory $F(p^2) = p^2\, e^{p^2/\Lambda^2}$

Entropy in UV(IR) modified theories

Heat kernel on space with conical singularity

$$\mathrm{Tr}K_\alpha(s)=\frac{1}{(4\pi)^{d/2}}\left(\alpha V P_d(s)\,+\,\frac{\pi}{3\alpha^2}(1-\alpha^2)A(\Sigma)P_{d-2}(s)+..\right)$$

Entanglement entropy

$$S=\frac{A(\Sigma)}{12\cdot(4\pi)^{(d-2)/2}}\int_{\epsilon^2}^\infty\frac{ds}{s}P_{d-2}(s)\,,$$

where

$$P_n(s)=\frac{2}{\Gamma(\frac{n}{2})}\int_0^\infty dp\,p^{n-1}\,e^{-sF(p^2)}$$

Nesterov, SS (2010)

Entropy in UV modified theories

(i) No matter how fast function $F(p^2)$ grows for large p entanglement entropy is always UV divergent

(ii) The area law and the statement on renormalization of entropy are valid for any $F(p^2)$

(iii) Example: $F(p^2) \simeq m^2 e^{\frac{p^2}{\Lambda^2}}$

$$\boxed{S \simeq \frac{A(\Sigma)}{48\pi}\Lambda^2 \ln^2(\epsilon m)}$$

Nesterov, SS (2010)

Entropy in non-Lorentz invariant theories

$$D = -\partial_t^2 + F(-\vec{\nabla}^2)$$

- there is no rotational symmetry in (r,t) plane

- only $2\pi n$ periodicity is allowed

- it is enough to compute entropy

$$S = \frac{A(\Sigma)}{12(4\pi)^{(d-2)/2}} \int_{\varepsilon^2}^{\infty} \frac{ds}{s} P_{d-2}(s)$$

$P_n(s)$ is the same as in Lorentz invariant case

Entropy in non-Lorentz invariant theories

Polynomial field operators: $F(-\vec{\nabla}^2) = m^{2(1-n)}(-\vec{\nabla}^2)^n$

Heat operator $\exp(-s\,D)$ is invariant under rescaling

$\vec{x} \rightarrow \lambda\,\vec{x}\ ,\ t \rightarrow \lambda^n t\ ,\ s \rightarrow \lambda^{2\mathrm{n}} s$ and $\vec{x} \rightarrow \beta\,\vec{x}\ ,\ m \rightarrow \beta^{n/(1-n)} m$

Structure of entanglement entropy is fixed by this invariance

$$S \sim \left(\frac{m^{n-1}}{\varepsilon}\right)^{\frac{d-2}{n}} A(\Sigma)$$

Logarithmic term in entropy of generic 4d CFT

Effective action

$$W_{CFT} = \frac{a_0}{\epsilon^4} + \frac{a_1}{\epsilon^2} + a_2 \ln \epsilon + w(g)\,, \;\; w(\lambda^2 g) = w(g) - a_2 \ln \lambda$$

A and B type conformal anomaly

$$a_2^{\rm bulk} = A E_{(4)} + B I_{(4)} \quad ,$$

$$E_{(4)} = \frac{1}{64} \int_E (R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 4 R_{\mu\nu} R^{\mu\nu} + R^2) \quad ,$$

$$I_{(4)} = -\frac{1}{64} \int_E (R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2)$$

Duff (77)
Christensen, Duff (78)

Logarithmic term in entropy of generic 4d CFT

Entanglement entropy of arbitrary surface Σ

$$S_{(A,B)} = \frac{a_1^\Sigma}{\epsilon^2} + a_2^\Sigma \ln \epsilon + s(g) \ , \quad s(\lambda^2 g) = s(g) - a_2^\Sigma \ln \lambda$$

Surface anomaly
(combination of conformal symmetry and holographic interpretation)

$$a_2^\Sigma = A a_A^\Sigma + B a_B^\Sigma \ ,$$
$$a_A^\Sigma = \frac{\pi}{8} \int_\Sigma (R_{abab} - 2R_{aa} + R - \mathrm{Tr} k^2 + k_a k_a) = \frac{\pi}{8} \int_\Sigma R_\Sigma \ ,$$
$$a_B^\Sigma = -\frac{\pi}{8} \int_\Sigma (R_{abab} - R_{aa} + \frac{1}{3} R - (\mathrm{Tr} k^2 - \frac{1}{2} k_a k_a)) \ ,$$

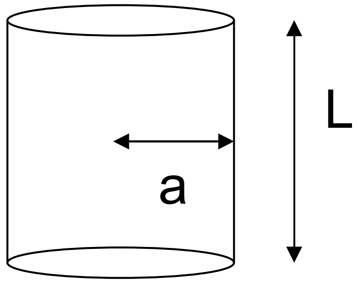
SS (2008)

where k^a is extrinsic curvature of Σ (vanishes for black hole horizon)

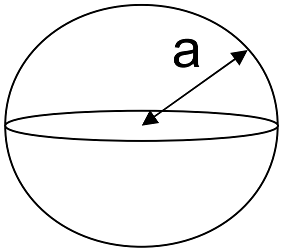
EE in flat 4d space-time

$$S_{(A,B)} = \frac{N \cdot A(\Sigma)}{4\pi \varepsilon^2} + \frac{\pi}{8} \int_{\Sigma} [AR_{\Sigma} + B(trK^2 - \frac{1}{2} K^i K^i)] \ln \varepsilon + s(A,B)(g)$$

$$S_{(A,B)}^{cylinder} = \frac{2\pi La}{4\pi \varepsilon^2} + B \frac{\pi^2}{8} (\frac{L}{a}) \ln \frac{\varepsilon}{a}$$



$$S_{(A,B)}^{sphere} = \frac{N \cdot a^2}{\varepsilon^2} + A\pi^2 \ln \frac{\varepsilon}{a}$$



SS(08)

Logarithmic term in entropy of generic 4d CFT: flat spacetime

Round sphere in Minkowski spacetime:

- generalization to higher dimensions: Cassini-Huerta (2010), Dowker (2010), Myers et al (2010)

- The logarithmic term is the same as for extreme black hole SS(2010)

since near-horizon region $H_2 \times S_2$ and Minkowski spacetime

are conformally related

Logarithmic term in entropy of generic 4d CFT: black holes

Extreme charged black hole $s_{log}^{ext} = A\pi^2$

The Schwarzschild black hole $s_{log}^{sch} = (A - B)\pi^2$

Extreme Kerr black hole $s_{log}^{Ext-Kerr} = (A - B)\pi^2$

For a generic 4d CFT

$$A = \frac{1}{90\pi^2}(n_0 + 11n_{1/2} + 62n_1 + 0n_{3/2} + 0n_2) \ ,$$
$$B = \frac{1}{30\pi^2}(n_0 + 6n_{1/2} + 12n_1 - \frac{233}{6}n_{3/2} + \frac{424}{3}n_2)$$

Why log corrections are interesting?

- they are important at the final stage of evaporation

$$S = 4\pi G M^2 - \sigma \ln M$$

$$T^{-1} = 8\pi G M - \frac{\sigma}{M}$$

- consistency with microscopic calculation for extreme black holes

Banerjee, Gupta, Sen (2010)

Some open questions

- entanglement entropy in string theory

- non-minimal coupling (gauge fields)

- dynamical entangling surface (a brane?)

- ...

More work has to be done..