Ghost-free bigravity theories and their cosmological solutions

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Theories with massive gravitons

- Deformations of GR that explain the observed universe acceleration, $m \sim 1/$(cosm. horizon size).
- Problems: do not reduce to GR in the weak field when $m \to 0$ (VdVZ discontinuity), have a ghost, no uniqueness.
- Remedies exist for the VdVZ problem – Vainstein mechanism. Very recently a class of ghost-free models has been discovered.
- We wish to study cosmologies in these models.

Contents

- Massive gravity in D=4
- Ghost-free theories
- Proportional backgrounds
- FLRW cosmologies with non-bidiagonal metrics
- FLRW cosmologies with bidiagonal metrics
- Anisotropic cosmologies

I. Massive gravity in D=4

Non-linear Pauli-Fierz

4D manifold with two metrics

$$g_{\mu\nu}(x) \qquad \text{and} \qquad f_{\mu\nu}(x) = \eta_{AB}\partial_\mu X^A(x)\partial_\nu X^B(x)$$

and the action

$$S = \frac{1}{\kappa^2}\int\left(\frac{1}{2}R + m^2\mathcal{L}_{\rm int}\right)\sqrt{-g}\,d^4x + S_{[\rm m]}$$

where $\mathcal{L}_{\rm int}$ is a scalar function of $H^\alpha_{\ \beta} = g^{\alpha\sigma}f_{\sigma\beta} - \delta^\alpha_\beta$

$$\mathcal{L}_{\rm int} = \frac{1}{8}((H^\alpha_{\ \alpha})^2 - H^\alpha_{\ \beta}H^\beta_{\ \alpha}) + O((H^\alpha_{\ \beta})^3)$$

Theory is not unique, but has a unique weak field limit.
Unitary gauge, $f_{\mu\nu} = \eta_{\mu\nu}$, $X^\mu = x^\mu$.

EOM

$$G_{\mu\nu} = m^2 T_{\mu\nu} + \kappa^2 T^{[\mathrm{m}]}_{\mu\nu}$$

vith

$$T_{\mu\nu} = 2\frac{\partial \mathcal{L}_{\mathrm{int}}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}_{\mathrm{int}}.$$

3ianchi identities $\Rightarrow$

$$\nabla^{\mu} T_{\mu\nu} = 0.$$

0-4=6=2+4 propagating DOF.

n the unitary gauge, $f_{\mu\nu} = \eta_{\mu\nu}$, and for weak fields, $_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, EOM reduce to

Pauli-Fierz equations

$$\frac{1}{2}\{\Box h_{\mu\nu} + \ldots\} = \frac{1}{2}m^2(h_{\mu\nu} - h\eta_{\mu\nu}) + \kappa^2 T^{[\mathrm{m}]}_{\mu\nu}$$

applying ∂^μ $\Rightarrow$ 4 constraints $\partial^\mu h_{\mu\nu} - \partial_\nu h = 0$

tracing $\Rightarrow$ fifth constraint $3m^2 h = 2\kappa^2 T^{[m]}$

⇒ 10-5=5 DOF = 2s+1 polarizations of massive graviton.

For generic $g_{\mu\nu}$ no scalar constraint ⇒ 6-th propagating state, has negative norm: Boulware-Deser ghost /'72/.

VdVZ : $m \to 0,\ \Box h^{(2)}_{\mu\nu} + \ldots = 2\kappa^2 T^{[\mathrm{m}]}_{\mu\nu},\quad \Box h^{(0)} = \kappa^2 T^{[\mathrm{m}]}$

Vainstein: $h^{(0)}$ is strongly coupled for $r < r_V = (r_g/m^4)^{1/5}$

II. Ghost free theories

The dRGT massive gravity

$$\gamma^{\mu}{}_{\nu} = \sqrt{g^{\mu\alpha} f_{\alpha\nu}} \;\;\Rightarrow\;\; \gamma^{\mu}{}_{\sigma}\gamma^{\sigma}{}_{\nu} = g^{\mu\alpha} f_{\alpha\nu}$$

λ_A are eigenvalues of $\gamma^{\mu}{}_{\nu}$. The action is

$$S = \frac{1}{8\pi G} \int \left(\frac{1}{2} R + m^2 \mathcal{U} \right) \sqrt{-g}\, d^4x$$

with

$$\mathcal{U} = b_0 + b_1 \sum_{A} \lambda_A + b_2 \sum_{A<B} \lambda_A \lambda_B + b_3 \sum_{A<B<C} \lambda_A \lambda_B \lambda_C + b_4 \lambda_0 \lambda_1 \lambda_2 \lambda_3$$

There is scalar constraint. If only $b_1 \neq 0$ ($b_1 = -1$) then

$$G^{\mu}{}_{\nu} + m^2 (\gamma^{\mu}{}_{\nu} - \delta^{\mu}{}_{\nu}\gamma) = 0$$

Constraints

$$g^{\mu\nu} = \eta^{AB} e^{\mu}_{A} e^{\nu}_{B}, \qquad f_{\mu\nu} = \eta_{AB}\, \omega^{A}_{\mu} \omega^{B}_{\nu}, \qquad \gamma^{\mu}_{\ \nu} = e^{\mu}_{A} \omega^{A}_{\nu}$$

$$\mathcal{E}_{\mu\nu} \equiv G_{\mu\nu} + m^2(\gamma_{\mu\nu} - g_{\mu\nu}\gamma) = 0 \ \Rightarrow \ \gamma_{[\mu\nu]} = 0 \ \Rightarrow \ \hat{\gamma}^2 = \hat{g}^{-1}\hat{f}$$

$$\mathcal{E} \equiv \mathcal{E}^{\mu}_{\ \mu} = -R - 3m^2\gamma = 0.$$

$$4: \quad 0 = \mathcal{C}^{\mu} \equiv \nabla^{\nu}\mathcal{E}^{\mu}_{\ \nu} = m^2(\nabla^{\nu}\gamma^{\mu}_{\ \nu} - \nabla^{\mu}\gamma);$$

$$1: \quad 0 = \mathcal{C} = \nabla_{\sigma}((\gamma^{-1})^{\sigma}_{\ \mu}\mathcal{C}^{\mu}) + \frac{m^2}{2}\mathcal{E} =$$

$$= m^2\left(\nabla_{\sigma}\left\{(\gamma^{-1})^{\sigma}_{\ \mu}(\nabla^{\nu}\gamma^{\mu}_{\ \nu} - \nabla^{\mu}\gamma)\right\} - \frac{R}{2} - \frac{3m^2}{2}\gamma\right)$$

Zumino '70/

The ghost-free bigravity

$$S = \frac{1}{2\kappa_g^2}\int R\sqrt{-g}\,d^4x + \frac{1}{2\kappa_f^2}\int \mathcal{R}\sqrt{-f}d^4x - \frac{m^2}{\kappa^2}\int \mathcal{U}\sqrt{-g}\,d^4x$$
$$+ \quad S_{\rm m}[g, \text{g-matter}] + S_{\rm m}[f, \text{f-matter}]\,,$$

$$\kappa_g = \kappa\cos\eta, \quad \kappa_f = \kappa\sin\eta, \quad \gamma^\mu_{\ \nu} = \sqrt{g^{\mu\alpha}f_{\alpha\nu}}$$

$$\mathcal{U} = \sum_k b_k\mathcal{U}_k = b_0 + b_1\sum_A \lambda_A + b_2\sum_{A<B}\lambda_A\lambda_B + b_3\sum_{A<B<C}\lambda_A\lambda_B\lambda_C + b_4\,\lambda_0\lambda_1\lambda_2\lambda_3$$

$$g \leftrightarrow f, \qquad \lambda_A \leftrightarrow \frac{1}{\lambda_A}, \qquad \sum_k b_k\,\mathcal{U}_k\sqrt{-g} \leftrightarrow \sum_k b_{4-k}\,\mathcal{U}_k\sqrt{-f}$$

Flat space is the solution and m is the FP mass if only

$b_0 = 4c_3 + c_4 - 6$, $b_1 = 3 - 3c_3 - c_4$, $b_2 = 2c_3 + c_4 - 1$,

$b_3 = -(c_3 + c_4)$, $b_4 = c_4$. **/Hassan,Rosen 2011/**

Field equations

$$G^{\rho}_{\lambda} = m^2 \cos^2\eta \, T^{\rho}_{\lambda} + T^{[\mathrm{m}]\,\rho}_{\;\;\lambda}\,, \qquad \mathcal{G}^{\rho}_{\lambda} = m^2 \sin^2\eta \, \mathcal{T}^{\rho}_{\lambda} + \mathcal{T}^{[\mathrm{m}]\,\rho}_{\;\;\lambda}\,,$$

$$\begin{aligned} T^{\rho}_{\lambda} &= \tau^{\rho}_{\lambda} - \delta^{\rho}_{\lambda}\mathcal{U}\,, \qquad \mathcal{T}^{\rho}_{\lambda} = -\frac{\sqrt{-g}}{\sqrt{-f}}\,\tau^{\rho}_{\lambda}\,, \\ \tau^{\rho}_{\lambda} &= \{b_1\mathcal{U}_0 + b_2\mathcal{U}_1 + b_3\mathcal{U}_2 + b_4\mathcal{U}_3\}\gamma^{\mu}_{\;\nu} \\ &- \{b_2\mathcal{U}_0 + b_3\mathcal{U}_1 + b_4\mathcal{U}_2\}(\gamma^2)^{\mu}_{\;\nu} \\ &+ \{b_3\mathcal{U}_0 + b_4\mathcal{U}_1\}(\gamma^3)^{\mu}_{\;\nu} - b_4\mathcal{U}_0\,(\gamma^4)^{\mu}_{\;\nu} \end{aligned}$$

- Massive gravity for $\eta \to 0$ if $f_{\mu\nu}$ becomes flat.
- $g_{\mu\nu} = f_{\mu\nu} = \eta_{\mu\nu} \to g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$, $f_{\mu\nu} = \eta_{\mu\nu} + \delta f_{\mu\nu}$,

$$h^{(\mathrm{mass})}_{\mu\nu} = \cos\eta\,\delta g_{\mu\nu} + \sin\eta\,\delta f_{\mu\nu}\,, \quad h^{(0)}_{\mu\nu} = \cos\eta\,\delta f_{\mu\nu} - \sin\eta\,\delta g_{\mu\nu}$$

III. Proportional backgrounds

$$f_{\mu\nu} = C^2 g_{\mu\nu} \quad \Rightarrow \quad \gamma^{\mu}_{\ \nu} = C\delta^{\mu}_{\nu}$$

$$G^{\rho}_{\lambda} + \Lambda_g(C)\delta^{\rho}_{\lambda} = T^{[\mathrm{m}]\,\rho}_{\ \ \lambda}\,, \qquad \mathcal{G}^{\rho}_{\lambda} + \Lambda_f(C)\delta^{\rho}_{\lambda} = \mathcal{T}^{[\mathrm{m}]\,\rho}_{\ \ \lambda}\,.$$

$$\Lambda_g(C) = \cos^2\eta\left(b_0 + 3b_1\,C + 3b_2\,C^2 + b_3\,C^3\right)\,,$$
$$\Lambda_f(C) = \frac{\sin^2\eta}{C^3}\left(b_1 + 3b_2 C + 3b_3 C^2 + b_4 C^3\right)\,.$$

$\mathcal{G}^{\nu}_{\mu} = G^{\nu}_{\mu}/C^2 \Rightarrow\ \Lambda_f = \Lambda_g/C^2$, $\mathcal{T}^{[\mathrm{m}]\mu}_{\ \ \ \nu} = T^{[\mathrm{m}]\mu}_{\ \ \ \nu}/C^2$ (fine tuning)

$\Lambda_f - \Lambda_g/C^2 = 0 \Rightarrow$ 4 solutions for C,Λ.

$c_3 = 1, c_4 = 0.3, \eta = 1, \quad \Rightarrow \quad \Lambda_g = \{10.126;\ 0;\ -0.509;\ -4.505\}.$

$C = 1 \Rightarrow \Lambda_g = 0 \Rightarrow$ GR. $\Lambda_g > 0$ – self acceleration.
No massive gravity limit.

IV. FLRW cosmologies with non-bidiagonal metrics

(de Sitter sector)

Spherical symmetry

$$\begin{aligned} ds_g^2 &= -Q^2dt^2 + N^2dr^2 + R^2d\Omega^2 \\ ds_f^2 &= -(aQdt + cNdr)^2 + (cQdt - bNdr)^2 + u^2R^2d\Omega^2\,, \end{aligned}$$

Q, N, R, a, b, c, u depend on t, r,

$$\gamma^\mu{}_\nu = \sqrt{g^{\mu\alpha}f_{\alpha\nu}} = \begin{pmatrix} a & cN/Q & 0 & 0 \\ -cQ/N & b & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix}.$$

$$T_r^0 = -\frac{cN}{Q}[(c_3 + c_4)u^2 + 2(1 - 2c_3 - c_4)u + 3c_3 + c_4 - 3].$$

No radial flux

$$[(c_3 + c_4)u^2 + 2(1 - 2c_3 - c_4)u + 3c_3 + c_4 - 3] = 0$$

$$\Rightarrow \quad u = \frac{1}{c_3 + c_4}\left(2c_3 + c_4 - 1 \pm \sqrt{1 - c_3 + c_4 + c_3^2}\right)$$

$$\Rightarrow \quad T^0_0 = T^r_r = const. \quad \Rightarrow$$

$$\overset{(g)}{\nabla}_\mu T^\mu_\nu \sim T^r_r - T^\theta_\theta = (c_3 u - u - c_3 + 2)((u - a)(u - b) + c^2) = 0$$

Assuming the latter is true,

$$\Rightarrow \boxed{T^\mu_\nu = const \times \delta^\mu_\nu, \;\; \mathcal{T}^\mu_\nu = const \times \delta^\mu_\nu}$$

Equations

$$(A) \qquad G^{\mu}_{\nu} + \Lambda_g \delta^{\mu}_{\nu} = T^{[\mathrm{m}]\mu}{}_{\nu}$$

$$(B) \qquad \mathcal{G}^{\mu}_{\nu} + \Lambda_f \delta^{\mu}_{\nu} = \mathcal{T}^{[\mathrm{m}]\mu}{}_{\nu}$$

$$(C) \qquad (c_3 u - u - c_3 + 2)[(u-a)(u-b) + c^2] = 0$$

$$\Lambda_g = m^2 \cos^2\eta (1-u)(c_3 u - u - c_3 + 3) > 0,$$

$$\Lambda_f = m^2 \sin^2\eta \frac{u-1}{u^2}(c_3 u - c_3 + 2) < 0$$

$$T^{[\mathrm{m}]\mu}{}_{\nu} = \mathrm{diag}[-\rho(t), P(t), P(t), P(t)], \qquad \mathcal{T}^{[\mathrm{m}]\mu}{}_{\nu} = 0$$

Chamseddine, M.S.V, '11/, /d'Amico et al. '11/, /Kobayashi et al. '12/, Gratia, Hu, Wyman '12/, /M.S.V. '12/

(A)+ (B)

$$
\begin{aligned}
ds_g^2 &= -dt^2 + \mathbf{a}^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \\
ds_f^2 &= -\Delta(U)\, dT^2 + \frac{dU^2}{\Delta(U)} + U^2 d\Omega^2 \,.
\end{aligned}
$$

where

$$
\dot{\mathbf{a}}^2 - \frac{\mathbf{a}^2}{3}(\Lambda + \rho) = -k, \qquad \Delta(U) = 1 - \frac{\Lambda_f}{3} U^2
$$

How to relate T, U to t, r ? One had

$$
ds_f^2 = -(aQdt + cNdr)^2 + (cQdt - bNdr)^2 + u^2 R^2 d\Omega^2 \,.
$$

(C)

$$ds_f^2 = -(\omega^0)^2 + (\omega^1)^2 + u^2R^2d\Omega^2 = -(\vartheta^0)^2 + (\vartheta^1)^2 + U^2d\Omega^2.$$

while

$$\theta^0 = \sqrt{\Delta}dT, \quad \theta^1 = \frac{dU}{\sqrt{\Delta}}, \quad \omega^0 = \mathbf{a}\,(adt + cdr), \quad \omega^1 = \mathbf{a}\,(bdr - cdt),$$

$U = uR = ur\mathbf{a}(t)$, while θ^0, θ^1 are boost-related to ω^0, ω^1

$$\omega^0 = \theta^0 \sec\alpha + \theta^1 \tan\alpha, \;\; \omega^1 = \theta^1 \sec\alpha + \theta^0 \tan\alpha \qquad (\star)$$

Equating coefficients in front of dt, dr in $(\star)$ gives a, b, c, α in terms of U, Δ. Inserting to $(u-a)(u-b)+c^2=0$ gives

Consistency

$$\mathbf{a}\sqrt{1-kr^2}\,(\dot{U}T'-\dot{T}U')-u^2\mathbf{a}^2+u\mathbf{a}\sqrt{\frac{A_+A_-}{\Delta}}=0\quad(\dagger)$$

$A_\pm=\mathbf{a}\,(\Delta\dot{T}\pm\dot{U})+\sqrt{1-kr^2}(U'\pm\Delta T')$, $\Delta=1-\frac{\Lambda_f}{3}U^2$, $U=ur\mathbf{a}$. Exact solutions of $(\dagger)$ are found in the massive gravity limit, when $\eta=\Lambda_f=0$, $\Delta=1$,

$$k=0:\quad T(t,r)\;=\;C\int^t\frac{dt}{\dot{\mathbf{a}}}+\left(\frac{u^2}{4C}+Cr^2\right)\mathbf{a}\,,$$

$$k=\pm1:\quad T(t,r)\;=\;\int^t\sqrt{(C^2+ku^2)(\dot{\mathbf{a}}^2+k)}\,dt+C\mathbf{a}\sqrt{1-kr^2}$$

$\Rightarrow T(t,r),U(t,r)$ are found, consistency is fulfilled.

Properties of the solutions

- g-metric is FLRW with open, closed or flat sections. Matter-dominated at early times, Λ-dominated at late time $\Rightarrow$ self-acceleration. Without matter – de Sitter or static Schwarzschild-de Sitter **/Isham,Story '78/**
- f-metric is AdS. When $\eta \to 0$, $\Lambda_f \sim \sin^2 \eta \to 0 \Rightarrow f_{\mu\nu}$ is flat, massive gravity is recovered.
- In the massive gravity limit this gives the complete FLRW solution. In the literature is called 'inhomogeneous', since the fluctuations are expected to be non-FLRW, although this effect should be suppressed by smallness of m.

V. FLRW cosmologies with diagonal metrics

(Vainstein sector)

M.S.V. ’11

M. von Strauss et al. ’11

M. Cristosomi et al. ’11

Diagonal metrics

$$
\begin{aligned}
ds_g^2 &= -dt^2 + e^{2\Omega}\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \qquad k = 0, \pm 1 \\
ds_f^2 &= -\mathcal{A}^2 dt^2 + e^{2\mathcal{W}}\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right).
\end{aligned}
$$

Equations (here $\Lambda_g(\xi), \Lambda_f(\xi)$ are polynomials in $\xi = e^{\mathcal{W}-\Omega}$)

$$
\dot{\Omega}^2 = \frac{\Lambda_g(\xi) + \rho_g}{3} - \frac{k}{4}\, e^{-2\Omega}, \qquad \frac{\dot{\mathcal{W}}^2}{\mathcal{A}^2} = \frac{\Lambda_f(\xi) + \rho_f}{3} - \frac{k}{4}\, e^{-2\mathcal{W}}, \qquad (\bullet)
$$

and the conservation condition

$$
\left[\left(e^{\mathcal{W}}\right)^{\cdot} - \mathcal{A}\left(e^{\Omega}\right)^{\cdot}\right]\left(b_1 + 2b_2\xi + b_3\xi^2\right) = 0.
$$

Generic solutions

$$\left[\left(e^{\mathcal{W}}\right)^{\cdot}-\mathcal{A}\left(e^{\Omega}\right)^{\cdot}\right]=0 \;\;\Rightarrow\;\; \dot{\mathcal{W}}\xi=\dot{\Omega}\mathcal{A} \;\;\Rightarrow$$

he difference of 2 ODE's ($\bullet$) gives algebraic relation

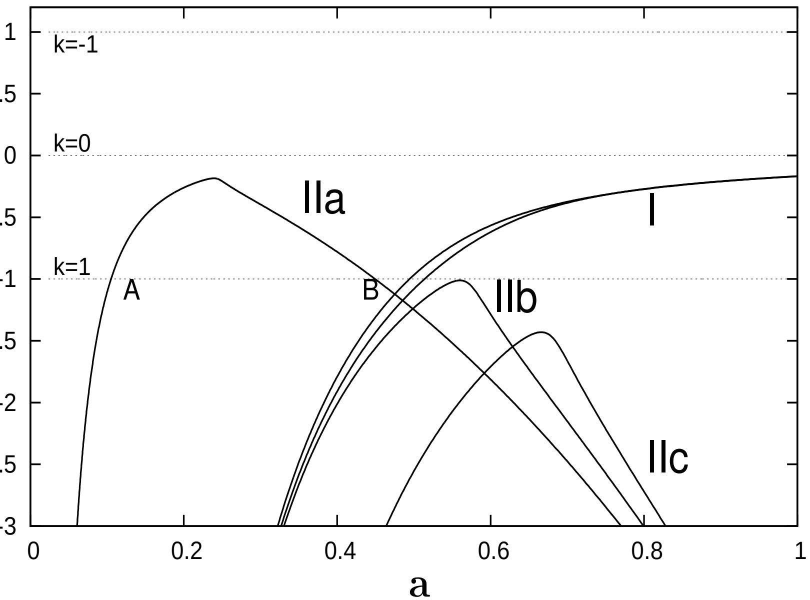
$$\Lambda_g(\xi)+\rho_g(\Omega)=\xi^2(\Lambda_f(\xi)+\rho_f(\Omega,\xi)) \;\;\Rightarrow\;\; \xi=\xi(\Omega). \;\; (\S)$$

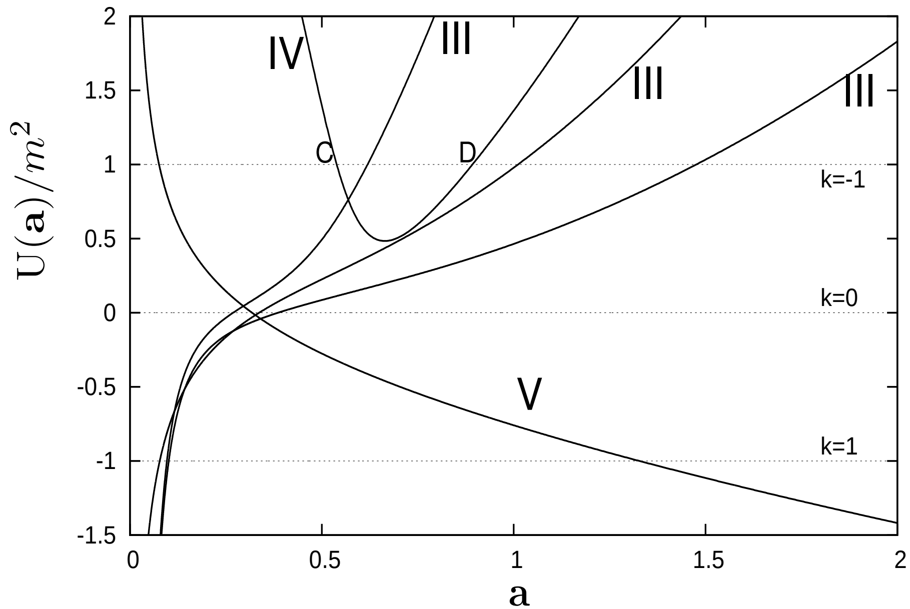
njecting $\xi(\Omega)$ back to ($\bullet$) gives the Friedmann equation

$$\dot{\mathbf{a}}^2+\mathrm{U}(\mathbf{a})=-k$$

vhere $\mathbf{a}=2e^{\Omega}$ and $\mathrm{U}=-(\mathbf{a}^2/3)(\Lambda_g+\rho_g)$.
There are several roots of ($\S$) $\Rightarrow$ several types of $\mathrm{U}(\mathbf{a})$.

Physical and exotic cosmologies





- physical: $\rho \gg m^2 T_0^0$ for small $\mathbf{a}$, $\rho \ll m^2 T_0^0$ for large $\mathbf{a}$
- exotic: $\rho \ll m^2 T_0^0$ for any $\mathbf{a}$.
- $f_{\mu\nu}$ is not flat for $\eta \to 0 \Rightarrow$ no massive gravity limit

Special solutions

$= e^{\mathcal{W}-\Omega}$ is constant and satisfies $b_1 + 2b_2\xi + b_3\xi^2 = 0$,
$= 2e^{\Omega}$ fulfills

$$\dot{\mathbf{a}}^2 - (\mathbf{a}^2/3)(\Lambda_g(\xi) + \rho_g) = -k$$

$\Rightarrow$ cosmology with constant $\Lambda_g(\xi)$, also

$$\mathcal{A}^2 = -f_{00} = \frac{(\Lambda_g + \rho_g)\mathbf{a}^2 - 3k}{(\Lambda_f + \rho_f)\mathbf{a}^2 - 3k/\xi^2}$$

For $k = -1$ (open universe) admits the massive gravity limit
$, \Lambda_f, \rho_f \to 0$, $f_{\mu\nu} \to \eta_{\mu\nu} \Rightarrow$ the only ‘truly homogeneous and sotropic’ massive gravity cosmology, although unstable at non-linear level. **/Mikohyama et al./**

VI. Anisotropic cosmologies with diagonal metrics

Kei-ichi Maeda, M.S.V. arXiv:1302.6198

Bianchi class A types

$$ds_g^2 = -\alpha(t)^2 dt^2 + h_{ab}(t)\,\omega^a \otimes \omega^b, \quad ds_f^2 = -\mathcal{A}^2(t) dt^2 + \mathcal{H}_{ab}(t)\,\omega^a \otimes \omega^b\,.$$

$$[e_a, e_b] = C^c{}_{ab} e_c, \;\; C^c{}_{ab} = n^{cd}\epsilon_{dab}, \quad n^{ab} = \mathrm{diag}[n^{(1)}, n^{(2)}, n^{(3)}]$$

	I	II	VI$_0$	VII$_0$	VIII	IX
$n^{(1)}$	0	1	1	1	1	1
$n^{(2)}$	0	0	-1	1	1	1
$n^{(3)}$	0	0	0	0	-1	1

If $h_{ab}, \mathcal{H}_{ab}$ are diagonal $\Rightarrow G^0_r = \mathcal{G}^0_r = 0 \Rightarrow$ no radial fluxes.

$$h_{ab} = \mathrm{diag}[\alpha_1^2, \alpha_2^2, \alpha_3^2], \quad \mathcal{H}_{ab} = \mathrm{diag}[\mathcal{A}_1^2, \mathcal{A}_2^2, \mathcal{A}_3^2].$$

3-curvature and $\mathcal{U}$

$$[\alpha_1, \alpha_2, \alpha_3] = e^{\Omega} \times \left[e^{\beta_+ + \sqrt{3}\beta_-}, e^{\beta_+ - \sqrt{3}\beta_-}, e^{-2\beta_+}\right]$$

$$[\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3] = e^{\mathcal{W}} \times \left[e^{\mathcal{B}_+ + \sqrt{3}\mathcal{B}_-}, e^{\mathcal{B}_+ - \sqrt{3}\mathcal{B}_-}, e^{-2\mathcal{B}_+}\right]$$

$$\overset{(3)}{R} = \frac{2n^{(1)}n^{(3)}}{\alpha_2^2} - \frac{1}{2}\, e^{-4\Omega} \left\{n^{(1)}\alpha_1^2 - n^{(2)}\alpha_2^2 + n^{(3)}\alpha_3^2\right\}$$

$$\begin{aligned} U_g &= b_0 e^{3\Omega} + b_3 e^{3\mathcal{W}} \\ &+ b_1 e^{\mathcal{W}+2\Omega} \left(e^{-2(\mathcal{B}_+ - \beta_+)} + 2e^{\mathcal{B}_+ - \beta_+} \cosh[\sqrt{3}(\mathcal{B}_- - \beta_-)]\right) \\ &+ b_2 e^{2\mathcal{W}+\Omega} \left(e^{2(\mathcal{B}_+ - \beta_+)} + 2e^{-(\mathcal{B}_+ - \beta_+)} \cosh[\sqrt{3}(\mathcal{B}_- - \beta_-)]\right). \end{aligned}$$

$$\overset{(3)}{\mathcal{R}} : \alpha_a \to \mathcal{A}_a, \quad U_f : b_k \to b_{k+1}, \qquad \mathcal{U}\sqrt{-g} = \alpha U_g + \mathcal{A} U_f .$$

Equations

$$\left(e^{3\Omega}\,\frac{\dot\Omega}{\alpha}\right)^2 = \left(e^{3\Omega}\,\frac{\dot\beta_+}{\alpha}\right)^2 + \left(e^{3\Omega}\,\frac{\dot\beta_-}{\alpha}\right)^2 + \frac{1}{6}\left[2\cos^2\eta\, e^{3\Omega}U_g - e^{6\Omega}\overset{(3)}{R} + 2e^{6\Omega}\rho_g\right],$$

$$\left(e^{3\Omega}\,\frac{\dot\Omega}{\alpha}\right)^{\cdot} = \frac{1}{6}\left[\cos^2\eta\left(\frac{\partial U}{\partial\Omega} + 3\alpha\,\mathcal{U}_g\right) - 2\alpha\, e^{3\Omega}\overset{(3)}{R} + 3\alpha\, e^{3\Omega}(\rho_g - P_g)\right],$$

$$\left(e^{3\Omega}\,\frac{\dot\beta_\pm}{\alpha}\right)^{\cdot} = -\frac{1}{12}\,\frac{\partial}{\partial\beta_\pm}\left(2\cos^2\eta\, U - \alpha\, e^{3\Omega}\overset{(3)}{R}\right),$$

$$\left(e^{3\mathcal{W}}\,\frac{\dot{\mathcal{W}}}{\mathcal{A}}\right)^2 = \left(e^{3\mathcal{W}}\,\frac{\dot{\mathcal{B}}_+}{\mathcal{A}}\right)^2 + \left(e^{3\mathcal{W}}\,\frac{\dot{\mathcal{B}}_-}{\mathcal{A}}\right)^2 + \frac{1}{6}\left[2\sin^2\eta\, e^{3\mathcal{W}}\mathcal{U}_f - e^{6\mathcal{W}}\overset{(3)}{\mathcal{R}} + 2e^{6\mathcal{W}}\rho_f\right],$$

$$\left(e^{3\mathcal{W}}\,\frac{\dot{\mathcal{W}}}{\mathcal{A}}\right)^{\cdot} = \frac{1}{6}\left[\sin^2\eta\left(\frac{\partial U}{\partial\mathcal{W}} + 3\mathcal{A}\mathcal{U}_f\right) - 2\mathcal{A}\, e^{3\mathcal{W}}\overset{(3)}{\mathcal{R}} + 3\mathcal{A}\, e^{3\mathcal{W}}\,(\rho_f - P_f)\right],$$

$$\left(e^{3\mathcal{W}}\,\frac{\dot B_\pm}{\mathcal{A}}\right)^{\cdot} = -\frac{1}{12}\,\frac{\partial}{\partial\mathcal{B}_\pm}\left(2\sin^2\eta\, U - \mathcal{A}\, e^{3\mathcal{W}}\overset{(3)}{\mathcal{R}}\right).$$

$$\alpha\left(\dot W\frac{\partial}{\partial\mathcal{W}} + \dot{\mathcal{B}}_+\frac{\partial}{\partial\mathcal{B}_+} + \dot{\mathcal{B}}_-\frac{\partial}{\partial\mathcal{B}_-}\right)U_g = \mathcal{A}\left(\dot\Omega\frac{\partial}{\partial\Omega} + \dot\beta_+\frac{\partial}{\partial\beta_+} + \dot\beta_-\frac{\partial}{\partial\beta_-}\right)\mathcal{U}_f\,.$$

$f_{\mu\nu} = C^2 g_{\mu\nu} \Rightarrow$ **equal anisotropies**

$$ds_g^2 = -dt^2 + e^{2\Omega}\left(e^{2\beta_+ + \sqrt{3}\beta_-} dx_1^2 + e^{2\beta_+ - \sqrt{3}\beta_-} dx_2^2 + e^{-4\beta_+} dx_3^2\right)$$

$\Rightarrow$ Bianchi I, equations

$$\left(e^{3\Omega}\,\dot{\Omega}\right)^2 = \sigma_+^2 + \sigma_-^2 + \frac{1}{3}\left(\Lambda_g + \rho_g\right)e^{6\Omega}\,, \qquad e^{3\Omega}\,\dot{\beta}_\pm = \sigma_\pm$$

where $\Lambda_g = \cos^2\eta\left(b_0 + 3b_1\,C + 3b_2\,C^2 + b_3\,C^3\right) \equiv 3H^2\,,$

$$\left(b_0 + 3b_1\,C + 3b_2\,C^2 + b_3\,C^3\right) = \frac{\tan^2\eta}{C}\left(b_1 + 3b_2 C + 3b_3 C^2 + b_4 C^3\right)$$

$$\Omega = Ht + O(e^{-3Ht}), \qquad \boxed{\beta_\pm = \mathcal{B}_\pm} = \beta_\pm(\infty) + O(e^{-3Ht})\,.$$

Dynamical system formulation

$$\dot{y}_N = F_N(\alpha, \mathcal{A}, y_M)\,,$$

with

$$\begin{aligned} y_0 &= e^{\Omega}, \qquad y_1 = e^{\beta_+}, \qquad y_2 = e^{\sqrt{3}\beta_-}, \\ y_3 &= e^{\mathcal{W}}, \qquad y_4 = e^{\mathcal{B}_+}, \qquad y_5 = e^{\sqrt{3}\mathcal{B}_-}, \\ y_6 &= \frac{e^{3\Omega}}{\alpha}\,\dot{\Omega}, \quad y_7 = \frac{e^{3\Omega}}{\alpha}\,\dot{\beta}_+, \quad y_8 = \frac{e^{3\Omega}}{\alpha}\,\dot{\beta}_-, \\ y_9 &= \frac{e^{3\mathcal{W}}}{\mathcal{A}}\,\dot{\mathcal{W}}, \quad y_{10} = \frac{e^{3\mathcal{W}}}{\mathcal{A}}\,\dot{\mathcal{B}}_+, \quad y_{11} = \frac{e^{3\mathcal{W}}}{\mathcal{A}}\,\dot{\mathcal{B}}_-. \end{aligned}$$

plus three constraints

$$\mathcal{C}_1(y_N) = 0, \quad \mathcal{C}_2(y_N) = 0, \quad \mathcal{C}_3(y_N) = 0.$$

Constraints

$$\dot{\mathcal{C}}_1 = \sum_{N=0}^{11} \frac{\partial \mathcal{C}_1}{\partial y_N} F_N \sim \dot{\mathcal{C}}_2 = \sum_{N=0}^{11} \frac{\partial \mathcal{C}_2}{\partial y_N} F_N \sim \mathcal{C}_3 \approx 0$$

f $\mathcal{C}_3 = 0 \Rightarrow \mathcal{C}_1, \mathcal{C}_2$ propagate. Does $\mathcal{C}_3$ propagate itself ?

$$\dot{\mathcal{C}}_3 = \sum_{N=0}^{11} \frac{\partial \mathcal{C}_3}{\partial y_N} F_N = \alpha X_\alpha(y_M) + \mathcal{A} X_{\mathcal{A}}(y_M) \approx 0$$

$\Rightarrow$ condition of propagation of all constraints

$$\mathcal{A} = -\frac{X_\alpha}{X_{\mathcal{A}}} \alpha$$

$\Rightarrow$ it is enough to impose the constraints only at $t = t_0$.

Strategy

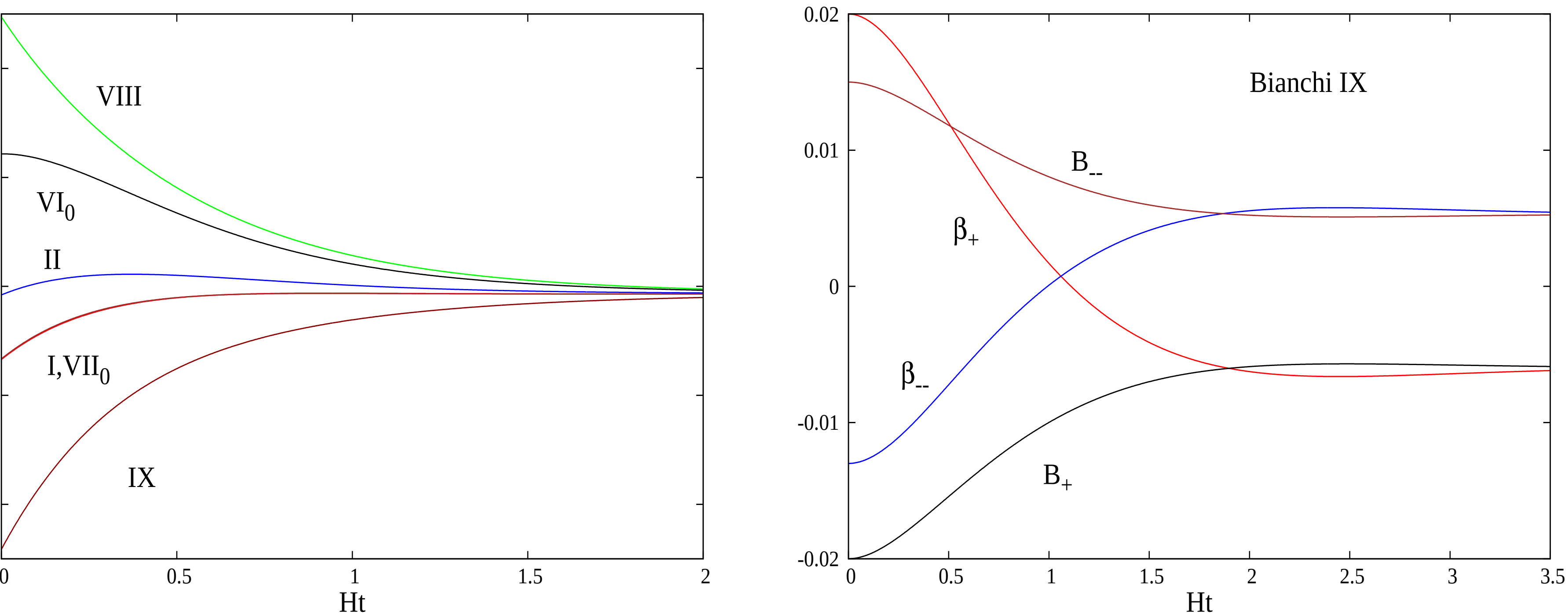
At the initial moment $t = 0$ the universe is an anisotropic
deformation of a finite size FLRW. One chooses $\Omega(0) = 0 \Rightarrow$
he initial universe size $e^{\Omega} \sim 1$ in 1/m units. The initial
anisotropies $\beta_{\pm}$, $\mathcal{B}_{\pm}$, $\dot{\beta}_{\pm}$, $\dot{\mathcal{B}}_{\pm} \sim 10^{-2}$. The f-sector is empty,
$\rho_f = 0$. The g-sector contains radiation + dust,

$$\rho_g = 0.25 \times e^{-4\Omega} + 0.25 \times e^{-3\Omega}$$

The dimensionful energy $m^2 M_{\rm pl}^2 \rho_g \sim 10^{-10} ({\rm eV})^4$, assuming
hat $m \sim 10^{-33} {\rm eV}$.

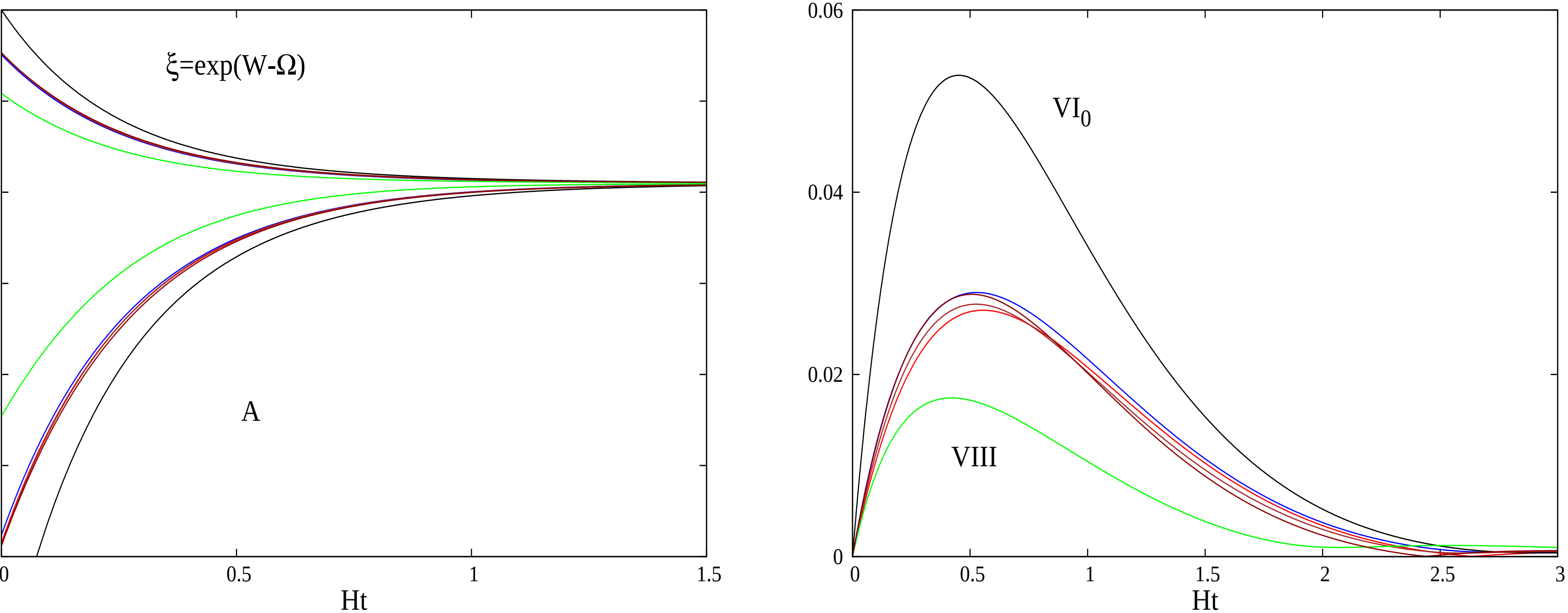
For all Bianchi types, the solutions rapidly approach a state
with a constant expansion rate and constant and non-zero
anisotropies = late time attractor.

Expansion rate and anisotropies



For Bianchi I one can scale away the constant values of $\beta_{\pm} = \mathcal{B}_{\pm}$, but not for other Bianchi types $\Rightarrow$ universe generically approaches an anisotropic state, although it expands with a constant rate.

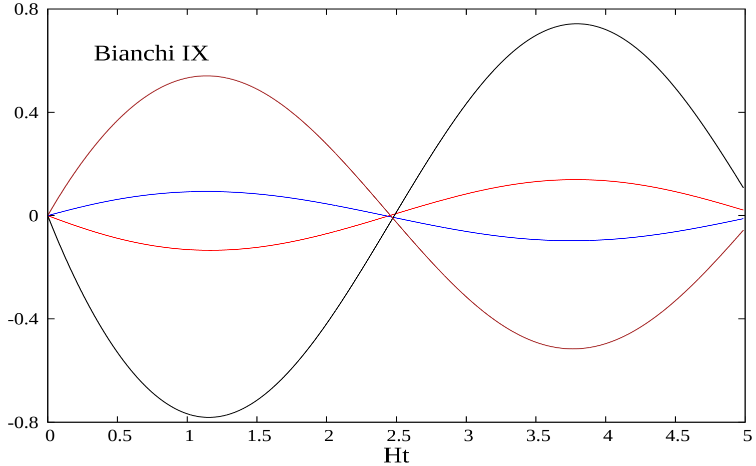
f-metric and shears



Both $\mathcal{A}$ and $e^{\mathcal{W}-\Omega}$ approach the same value $\Rightarrow f_{\mu\nu} = C^2 g_{\mu\nu}$.

Right: $\Sigma = \sqrt{\dot\beta_+^2 + \dot\beta_-^2}/\dot\Omega$, the relative contribution of shears to the total energy. If only one or two Hubble times have elapsed since the acceleration started, then Σ is not small.

Late time anisotropies

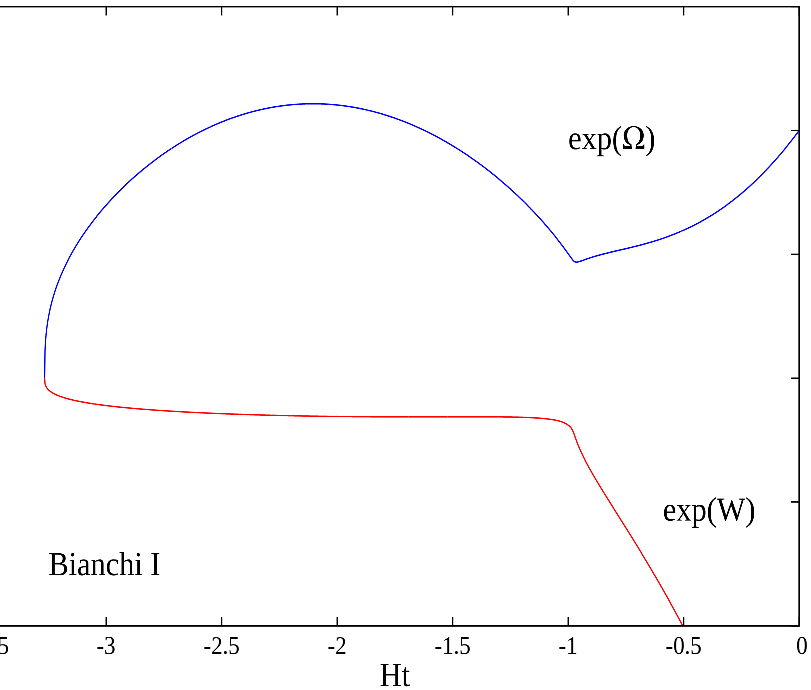


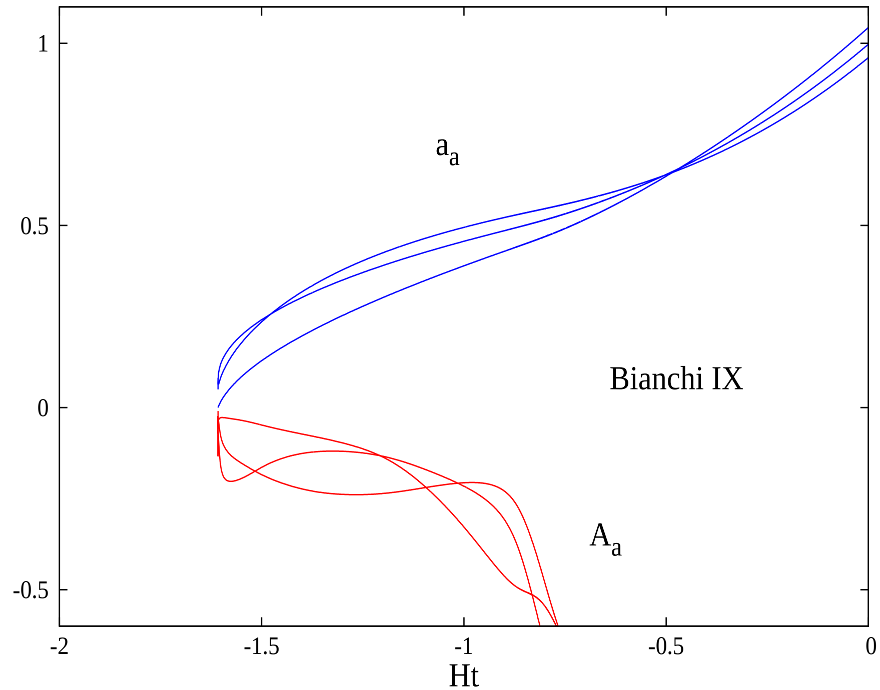
The rescaled shears $e^{3\Omega/2}\dot{\beta}_{\pm}$ and $e^{3\Omega/2}\dot{\mathcal{B}}_{\pm}$ oscillate with constant amplitudes. Linearizing around $f_{\mu\nu} = C^2 g_{\mu\nu}$,

$$\dot{\beta}_{\pm} \sim \dot{\mathcal{B}}_{\pm} \sim e^{-3Ht/2}\cos(H\omega t) \quad \text{with} \quad \omega = \omega(C, b_k, \eta, H)$$

The shear energy $\boxed{\dot{\beta}_{+}^2 + \dot{\beta}_{-}^2 \sim e^{-3\Omega} \sim 1/\mathbf{a}^3}$ falls off as the energy of a non-relativistic (dark ?) matter. In GR $\sim 1/\mathbf{a}^6$.

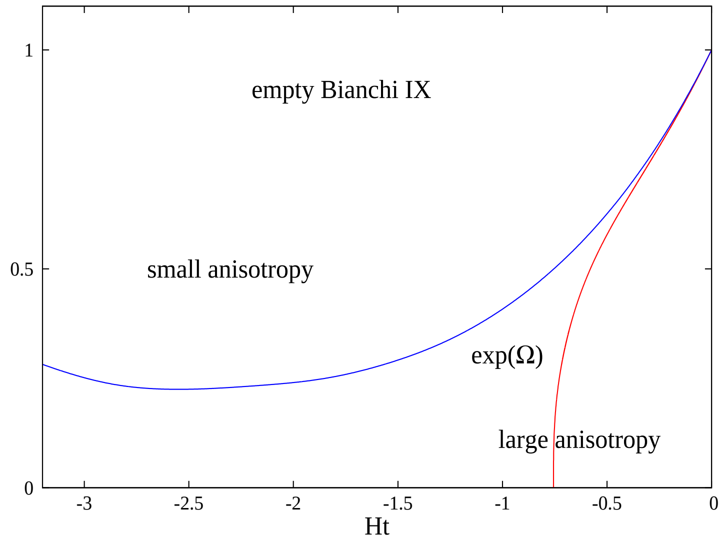
Near singularity behaviour





When continued to the past, the solutions show a singularity where both e^{Ω} and $e^{\mathcal{W}}$ vanish. For Bianchi IX anisotropies start fluctuating near singularity.

Empty Bainchi IX

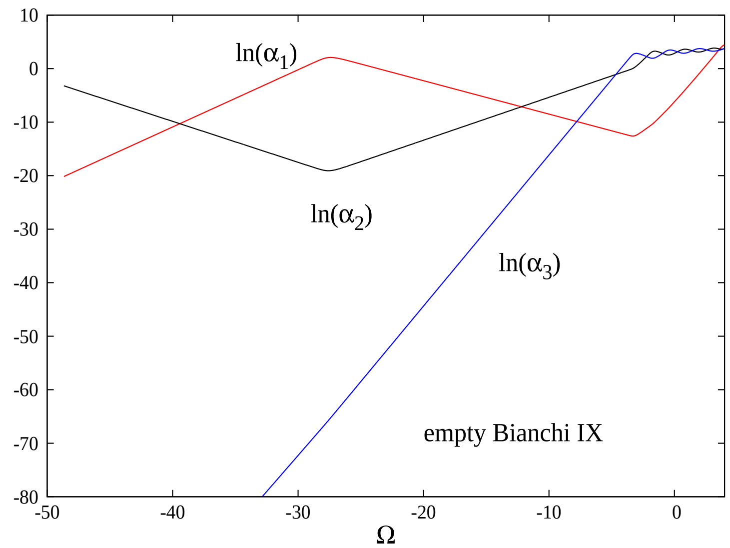


f $\rho_g = \rho_f = 0$ and anisotropies vanish $\Rightarrow$ de Sitter with a
ounce in the past,

$$2He^{\Omega} = \cosh(t - t_0).$$

For small anisotropies it is still a bounce, while for larger
anisotropies a singularity appears.

Empty Bainchi IX – chaos



Near singularity – a sequence of Kasner-like periods with

$$\alpha_a \propto t^{p_a} \quad \text{with} \quad p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1.$$

Matter cannot change this, as ρ grows slower than shears,

$$1/\mathbf{a}^6 \leftarrow \quad \text{shear energy} = \dot{\beta}_+^2 + \dot{\beta}_-^2 \quad \rightarrow 1/\mathbf{a}^3$$

Conclusions

3 types of FLRW self-accelerating cosmologies in bigravity:

[a] $f_{\mu\nu} = C^2 g_{\mu\nu}$, require source fine-tuning, $\rho_f = \rho_g / C^2$
[b] bidiagonal, approach [a] at late times when $\rho_f = \rho_g \to 0$
[c] non-bidiagonal, admit the limit of flat f-metric

Anisotropic cosmologies approach anisotropic versions of [a]. In GR shear energy $\sim 1/\mathbf{a}^6$, while in bigravity it is $\sim 1/\mathbf{a}^3$, which could perhaps mimic dark matter. The Bianchi IX bigravity cosmology is chaotic near singularity.

It is unclear if there exist non-bidiagonal anisotropic cosmologies.